A NONLINEAR ANALYSIS OF THE MOON'S PHYSICAL LIBRATION IN LONGITUDE

DONALD H. ECKHARDT

Terrestrial Sciences Laboratory, Air Force Cambridge Research Laboratories, L. G. Hanscom Field, Bedford, Mass., U.S.A.

and

KENNETH DIETER

Dabcovich and Co., Inc., Lexington, Mass., U.S.A.

(Received 4 September, 1970)

Abstract. The Euler equations for the forced physical librations of the Moon have already been solved by using a digital computer to perform the semi-literal mathematical manipulations. Very near resonance, the computer solution for the physical libration in longitude is complemented by the solution of the appropriate Duffing equation with a dissipation term. Because of its apparent proximity to a resonant frequency, the term whose argument is 2ω – twice the mean angular distance of the Moon's perigee from the ascending node of its orbit – is especially important. Its phase, which soon should be measurable, is related to the Moon's anelasticity. The term's frequency, in units of the sidereal month, increases as the semi-major axis of the Moon's orbit about the Earth increases. Using the Moon's mechanical ellipticity of Koziel and the rate of increase of the semi-major axis of MacDonald, it is estimated that the 2ω term will cross the resonant frequency in 130 million years and, if the rate of energy dissipation is sufficiently low, a transient libration will be induced.

1. Introduction

The Euler equations for the forced physical librations of a perfectly elastic Moon have been solved by using a digital computer to perform the semi-literal mathematical manipulations (Eckhardt, 1970). Very near a three year resonance, the computer solution for the physical libration in longitude is inadequate because it neglects nonlinear and dissipation effects. To complement the computer solution, we have investigated the physical libration in longitude near this resonance, taking into account nonlinear and dissipation terms. We have considered also the secular increase in the period of the relevant torque term and its effect, as the period passes through resonance, on the libration. Our method of investigation has been analytic, but since our goal has been to achieve a qualitative insight into libration near resonance, we have ignored many small effects and we have made some broad extrapolations. If our theory were more complete, our results would be more exact, but we believe that our understanding of the problem would not be significantly improved.

2. The Euler Equation for Libration in Longitude

The Euler dynamical equation for the physical libration in longitude, τ, may be

The Moon **2** (1971) 309–319. *All Rights Reserved*

approximated by

$$\frac{1}{n^2}\frac{d^2\tau}{dt^2} = 2.97\gamma\left[\tfrac{1}{2}\sin 2(s-\tau) + u\right], \tag{1}$$

where n is the mean motion of the Moon; γ has its usual connotation as the relevant moment of inertia ratio (approximately 2.3×10^{-4}); $s=s(t)$ is the center equation and inequalities in the lunar longitude; and $u=u(t)$ is composed of cross terms involving the center equation and inequalities in the lunar longitude and lunar parallax (Eckhardt, 1970).

According to lunar theory s is, at most, approximately 0.1 radians; let us suppose that τ is small enough that the following approximation is valid,

$$\begin{aligned}\tfrac{1}{2}\sin 2(s-\tau) &= \tfrac{1}{2}\sin 2s\cos 2\tau - \tfrac{1}{2}\cos 2s\sin 2\tau \\ &\approx \tfrac{1}{2}(1-2\tau^2)\sin 2s - \overline{\cos 2s}\,(\tau - \tfrac{2}{3}\tau^3),\end{aligned} \tag{2}$$

where $\overline{\cos 2s} = 1 - 2\overline{\sin^2 s} = 0.985$ is the mean value of $\cos 2s$. It is known by observation that the amplitude of τ is now less than 10^{-3} radians, so there is little loss in precision in the linearization $\frac{1}{2}\sin 2(s-\tau) = \frac{1}{2}\sin 2s - 0.985\tau$ which, inserted into (1), gives the linear differential equation

$$\frac{1}{n^2}\frac{d^2\tau}{dt^2} + 2.93\gamma\tau = 2.93\gamma\left[\frac{\frac{1}{2}\sin 2s + u}{0.985}\right]. \tag{3}$$

The term in brackets on the RHS of (3) can be expanded into a Fourier sine series, $\Sigma_i H_i \sin\xi_i$, whose arguments are elements of the additive group generated by the Hansen or Delaunay arguments. The particular solution to (3) is then given by

$$\tau = \Sigma_i \tau_i = \Sigma_i \frac{h_i H_i}{h_i - 1}\sin\xi_i\,,$$

where

$$h_i = 2.93\gamma \bigg/ \left[\frac{1}{n}\frac{d\xi_i}{dt}\right]^2. \tag{4}$$

For the coefficient of the τ term whose argument is $2\omega = 2F - 2l$ (twice the mean distance of the Moon's perigee from the ascending node of the Moon's orbit) there is a singularity at $\gamma = \bar{\gamma} = 2.126\times 10^{-4}$. Because γ is actually very close to $\bar{\gamma}$, the coefficient of this three year libration term is considerably amplified over the one arc-second amplitude of the corresponding forcing term, and the sign of the coefficient depends on the sign of $\gamma - \bar{\gamma}$; that is, the sign is ambiguous. We shall examine this term by dropping the subscript i, letting $\xi = 2\omega$, and considering the equation

$$\frac{d^2\tau}{d\xi^2} + h\tau = hH\sin\xi\,.$$

3. The Linear Equation with Dissipation

Let us first consider the effect of a rate dependent dissipation term in this differential equation. The equation may be modified as

$$\frac{d^2\tau}{d\xi^2} + 2\alpha\frac{d\tau}{d\xi} + h\tau = hH\sin\xi, \tag{5}$$

where, for *unit* angular velocity, $2\alpha = 1/Q$ is the dimensionless specific dissipation function (Kaula, 1968). In the mantle of the Earth, Q is observed to be essentially independent of frequency for frequencies between 10^{-4} and 10^6 Hz. It increases with depth from about 80 in the upper mantle to 2000 in the lower mantle. For frequencies less than 10^{-4} Hz and almost as low as $(1/2\pi)(d\xi/n\,dt) \approx 10^{-8}$ Hz, the Q of the Earth is observed to be somewhere between 10 and 100. This lower Q is likely due to high dissipation at the core-mantle and ocean-crust interfaces and is probably not applicable to the mantle. At 10^{-8} Hz, the Earth's mantle *per se* likely has a Q between 80 and 2000 and, by analogy, we may expect that the interior of the Moon has approximately the same Q. A conservative guestimate for α, then, is somewhere between 10^{-2} and 10^{-4} per unit time ($\frac{1}{2}\pi$ cycles).

The particular solution to (5) may be written in the form

$$\tau = x\sin\xi + y\cos\xi, \tag{6}$$

where x and y are constants given by

$$x = -(h-1)\,r^2/hH, \tag{7a}$$

$$y = -2\alpha r^2/hH, \tag{7b}$$

with

$$x^2 + y^2 = r^2 = h^2H^2/[(h-1)^2 + 4\alpha^2]. \tag{7c}$$

Because of the dissipation term, the homogeneous solution of (3) includes the factor $\exp(-\alpha\xi)$ and it will decay with time. The behavior of τ is adequately described by the particular solution alone.

A credible estimate of γ, which is derived from Koziel's (1967) mechanical ellipticity, $f=0.633$, is $\gamma = 2.30\times10^{-4}$; using this in (4), we calculate $h=1.082$. Using $\gamma=2.30\times$ $\times10^{-4}$ in a semi-literal computer solution of the coupled nonlinear Euler dynamical equations for the rotation of a perfectly elastic Moon gives $x=-15''$ and, of course, $y=0''$ (Eckhardt, 1970). Using (7a, b, c) to extrapolate from the $1/Q=0$ case gives

$$x = -\frac{15''}{1+(12/Q)^2}$$

and

$$y = 12x/Q.$$

The shift in phase between the forcing terms on the RHS of (5) and the equation's solution is $\tan^{-1}(y/x) = \tan^{-1}(12/Q)$. For example, for a Q of 90, x barely changes from $-15''$, but y becomes $-2''$ and the phase shift is 0.13 radians. When $\sin\xi = 0$, $|\tau| = |y| = 2''$; this is a displacement along the lunar equator of 17 meters from the non-dissipative libration. Such a displacement has become quite significant with the advent of lunar laser observatories which have range resolutions better than one meter. Conceivably, then, we may be able to learn a little about the anelasticity of the Moon from the analysis of lunar laser ranging data.

4. The Nonlinear Equation with Dissipation

Now let us modify (5) by adding the τ^3 term in the expansion (2). The τ^2 term still may be neglected because, for the ξ harmonic, $|\frac{2}{3}\tau^3| \gg |\tau^2 \sin 2s|$. We get

$$\frac{d^2\tau}{d\xi^2} + 2\alpha \frac{d\tau}{d\xi} + h(\tau - \tfrac{2}{3}\tau^3) = hH \sin\xi, \tag{8}$$

which is Duffing's equation with a dissipation term, a familiar equation in the study of nonlinear oscillations.

To solve this equation, we use the method of Hayashi (1964). The approximate solution to (8) may be written in the form used above

$$\tau = x \sin\xi + y \cos\xi, \tag{6}$$

but now we assume $x = x(\xi)$ and $y = y(\xi)$ to be functions which vary with ξ, but slowly enough so that $d^2x/d\xi^2$ and $d^2y/d\xi^2$ may be neglected. Inserting (6) into (8) and equating to zero the coefficients of the $\sin\xi$ terms and then of the $\cos\xi$ terms gives, respectively,

$$-2\frac{dy}{d\xi} - x + 2\alpha\left[\frac{dx}{d\xi} - y\right] = hx\left[\tfrac{1}{2}r^2 - 1\right] + hH,$$

$$2\frac{dx}{d\xi} - y + 2\alpha\left[\frac{dy}{d\xi} + x\right] = hy\left[\tfrac{1}{2}r^2 - 1\right].$$

Rearranging, we get

$$(1 + \alpha^2)\frac{dx}{d\xi} = N(x, y), \tag{9a}$$

$$(1 + \alpha^2)\frac{dy}{d\xi} = -M(x, y), \tag{9b}$$

where

$$M(x, y) = [x - \alpha y]f(r) + \alpha y + \alpha^2 x + \tfrac{1}{2}hH, \tag{10a}$$

$$N(x, y) = [y + \alpha x]f(r) - \alpha x + \alpha^2 y + \tfrac{1}{2}\alpha hH, \tag{10b}$$

and

$$f(r) = \tfrac{1}{2}(1 - h + \tfrac{1}{2}hr^2).$$

The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact if, and only if, $\alpha = 0$. Its integral is

$$r^4 + 4pr^2 + 4q = 0, \tag{11}$$

where

$$\begin{aligned} p &= (1 - h)/h = (\bar{\gamma} - \gamma)/\gamma, \\ q &= 2(Hx + C), \end{aligned} \tag{12}$$

and C is an integration constant. An example of a set of solutions to (11) for $h = 1.00094$ is given in Figure 1. Note that the eccentric circles represent a beat phenomenon between forced and free librations of nearly the same frequency.

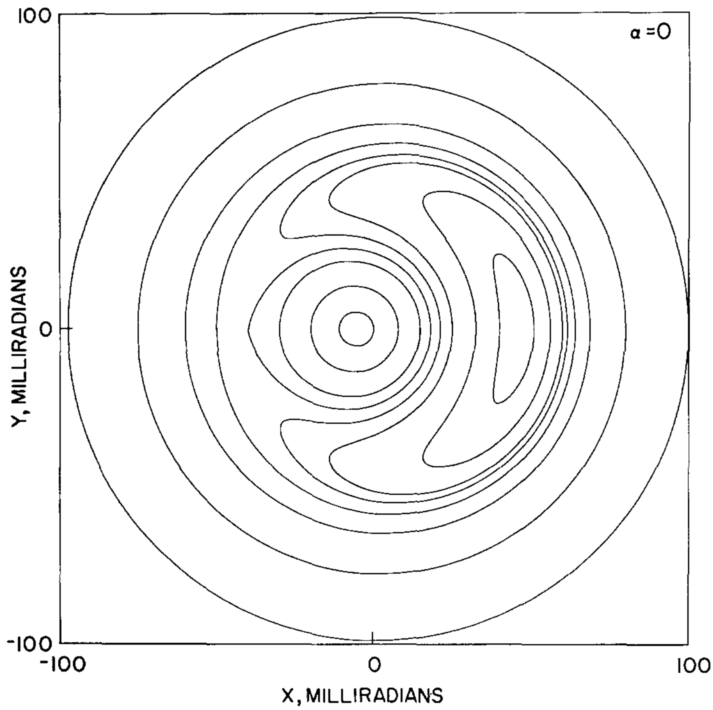

Fig. 1. For a perfectly elastic Moon the integral curves for libration are closed. This set of curves was generated for $\gamma = 2.128 \times 10^{-4}$. From left to right along the x-axis occur one unstable saddle point, then two neutral centers.

5. The Steady State Solution

By the method of averaging, Habibullin (1966) arrived at equations (Habibullin's 113) analogous to (9a, b). He then considered the steady state conditions $dx/d\xi = dy/d\xi = 0$ which would describe stable points toward which the solutions of (9a, b) would approach because of dissipation; but he assumed that the dissipation was small enough to neglect in setting

$$N(x, y, \alpha = 0) = yf = 0,$$

and

$$M(x, y, \alpha = 0) = xf + \tfrac{1}{2}hH = 0\,.$$

The solution to this pair of equations is $y=0$ and $x=a$, where a is the root of $M(a, 0)=0$; this is the Duffing relation

$$a^3 + 2pa + 2H = 0\,.$$

The real roots of this cubic equation locate singular points, stable and unstable, of the integral curves (11). (For an elaboration of this relationship, see Hayashi, 1964.) Using (12), the Duffing relation may be written

$$\gamma = \bar{\gamma}/(1 - a^2/2 - H/a)\,.$$

Near resonance, this equation supplements the detailed computer solution for $\tau(\xi)=a \sin\xi$. An appropriate fit for the solutions $a=a(\gamma)$ in Table III of Eckhardt (1970) is given with $\bar{\gamma}=2.126\times10^{-4}$ and $H=-1''\!.07$. This is plotted in Figure 2. For $\gamma>2.127\times10^{-4}$ where the Duffing relation has three roots the intermediate root, in absolute magnitude, corresponds with an unstable solution; the other roots are stable.

If $\alpha\neq0$, the singular points which are located by the requirements that $M=N=0$ move away from the x (zero phase) axis. Algebraically juggling these requirements, we

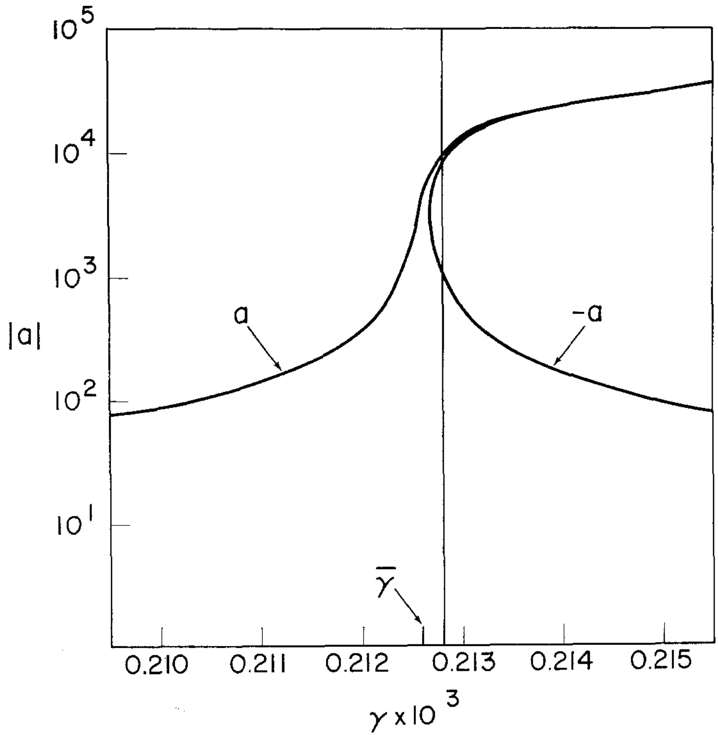

Fig. 2. This plot of the Duffing relation gives the amplitude, in arc-seconds, of a $\sin 2\omega$ forced physical libration term for a perfectly elastic Moon. The vertical line at $\gamma=2.128\times10^{-4}$ intersects the curve at the three singular points of Figure 1.

find the solution

$$x = -(pr^2 + \tfrac{1}{2}r^4)/H, \tag{13a}$$

$$y = -2\alpha r^2/hH, \tag{13b}$$

where r^2 is a real root of the cubic equation

$$r^6 + 4pr^4 + 4(p^2 + 4\alpha^2/h^2)\,r^2 - 4H^2 = 0. \tag{13c}$$

We assume that $H = -5\times10^{-6} \approx -1''.07$; then for $\alpha=0$, the solution is identical with that of the Duffing relation; for $10^{-4} \geqslant \alpha > 0$, there is a finite range for h over which (13c) has three real positive roots for r^2, the intermediate root corresponding with an unstable solution; and for $\alpha > 10^{-4}$, over the entire range of h there is only one real root to (13c), so the steady state solution is unique and practically the same as the particular solution of the original linear differential equation. Note that if terms in r^4 and r^6 can be neglected, the solution (13a, b, c) is identical with the solution (7a, b, c).

6. Secular Transition Through Resonance

Concerning ourselves with the details of the solution near a resonance may appear pedantic because currently $\tau(\xi)$ is far enough from its resonance to safely ignore the nonlinear effects. There exists, however, a slow secular decrease in h which effects a parallel decrease in the resonant frequency in units of ξ. (If uniform time were the independent variable, the resonant frequency would remain constant, and ξ would grow.) If $\gamma > \bar{\gamma}$, resonance will be attained in the future; if $\gamma < \bar{\gamma}$, resonance has already been attained.

Referring to the definition of h_i in (4), we write

$$h = 2.93\gamma \Big/ \left[\frac{2}{n}\frac{\mathrm{d}\omega}{\mathrm{d}t}\right]^2. \tag{14}$$

By lunar theory

$$\frac{1}{n}\frac{\mathrm{d}\omega}{\mathrm{d}t} = g - c \tag{15}$$

for

$$g = 1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4 - \frac{9797}{2048}m^5 - \cdots, \tag{16}$$

$$c = 1 - \frac{3}{4}m^2 - \frac{225}{32}m^3 - \frac{4071}{128}m^4 - \frac{265493}{2048}m^5 - \cdots, \tag{17}$$

and

$$m = n'/n = 0.0748 \textit{ currently},$$

where n' is the mean motion of the Sun (Brouwer and Clemence, 1961). Because of tidal friction, the semi-major axis of the Moon's orbit about the Earth is increasing at a rate of about 3.2 cm/yr (MacDonald, 1964); that is, the semi-major axis is increasing at the relative rate $8.3\%/10^9$ yr. By Kepler's third law $1/n$ is therefore increasing at the relative rate $\frac{3}{2} \times 8.3\%/10^9$ yr $= 12.5\%/10^9$ yr; so, for $n' =$ constant $(1/m)(\mathrm{d}m/\mathrm{d}t) = 0.125 \times 10^{-9}$/yr. Using (14)–(17)

$$\begin{aligned} \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{-2}{g-c}\frac{\mathrm{d}}{\mathrm{d}t}(g-c) \\ &= \frac{-2}{g-c}\left[3m^2 + \frac{81}{4}m^3 + \frac{1899}{16}m^4 + \frac{79905}{128}m^5 + \cdots\right]\frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t} \\ &= -0.61 \times 10^{-9}/\mathrm{yr}. \end{aligned}$$

If $h = 1.082$ now, resonance will occur in about 130×10^6 yr.

It is of interest to see what will happen (or has already happened if $h < 1$) as h decreases and passes through $h = 1$. Over a short span of time encompassing this

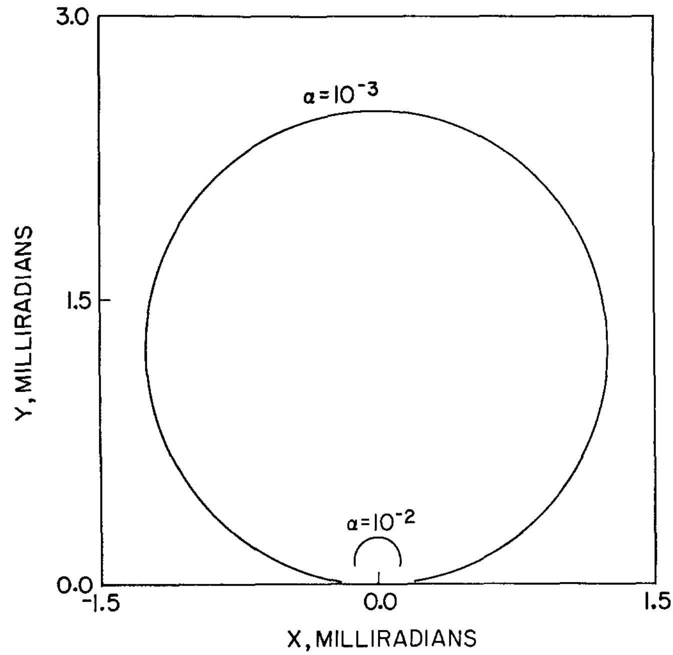

Fig. 3. With $\alpha = 10^{-2}$ and 10^{-3}, the integral curves about resonance are virtually identical with plots of the particular solutions of the linear problem.

phenomenon, we assume that $h(\xi)$ varies linearly with ξ such that $h(0) = 1.025$ and $\mathrm{d}h/\mathrm{d}\xi = -0.3 \times 10^{-9}$ (currently $\mathrm{d}\xi/\mathrm{d}t = 2.1$ radians/yr). We take $H = -5 \times 10^{-6}$ and, for various values of α solve (9a, b) by numerical integration. Because $h(0)$ is sufficiently removed from unity, the linear solution (7a, b, c) provides suitable initial conditions for x and y. The integration is carried through resonance until $h(\frac{1}{6} \times 10^9) = 0.975$. The integration was accomplished using a standard fourth order Runge-Kutta scheme with a step size $\Delta\xi = 833$. The computations were done on an IBM 7094 and the results were plotted as phase plane diagrams by a Calcomp 780 plotter.

If $\alpha=0$ and h is fixed the numerical integration of (9a, b) should yield the same family of closed trajectories as the analytic integral, (11). For $\gamma=2.128\times10^{-4}$, $h=1.0094$ and Duffing's relation has three roots (shown in Figure 2) which correspond with singular points of the corresponding phase plane diagram. This diagram, presented in Figure 1, was plotted using the numerical integration solution with initial conditions selected along the x-axis such that a set of suitably spaced curves is generated. This set of closed curves serves as a check on the accuracy of our numerical integration technique.

The phase plane plots for varying h are presented in Figures 3–6. In Figure 3, the cases $\alpha=10^{-2}$ and $\alpha=10^{-3}$ are shown superposed at the same scale. The curves are

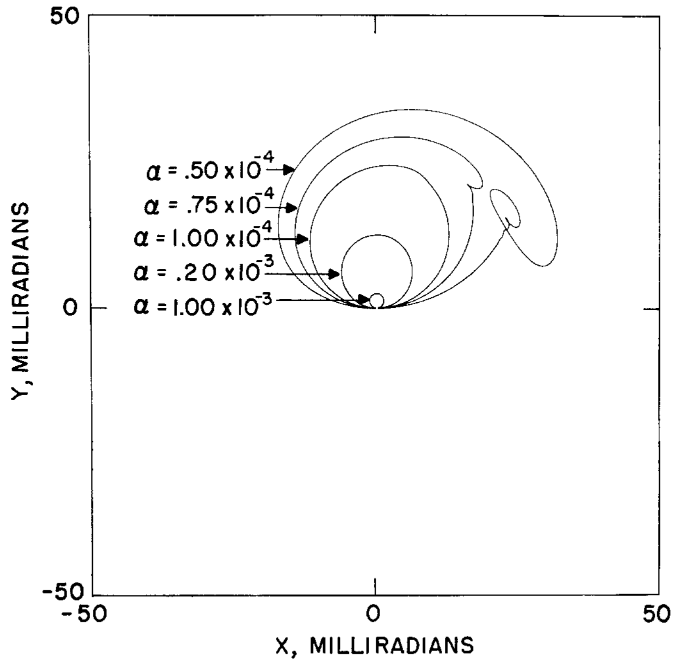

Fig. 4. The libration solutions move generally clockwise in the phase plane as h decreases through unity. For $\alpha\leqslant10^{-4}$ the steady state solution for the libration has a discontinuity and a transient is introduced.

approximately the arcs of circles of radius $H/4\alpha$ and origin $x=0$, $y=H/4\alpha$. They are virtually identical with the plots of $x(h)$, $y(h)$ which are generated by the particular solutions (7a, b, c) of the linear problem.

In Figure 4, the cases $\alpha=10^{-3}$, $\alpha=0.2\times10^{-3}$, $\alpha=10^{-4}$, $\alpha=0.75\times10^{-4}$ and $\alpha=$ $=0.50\times10^{-4}$ are shown superposed at the same scale. The steady state solution (13a, b, c) for monotonically decreasing h always corresponds with the minimum real root r^2 of (13c). For $\alpha>10^{-4}$ there is only one real root and the steady state solution is continuous; but for $\alpha\leqslant10^{-4}$ the steady state solution has a discontinuity where the number of real roots drops from three to one. The complete solution must be continuous, so a compensating transient must be introduced where the steady state solution jumps. As α becomes smaller, the jump becomes larger as does the transient, and the transient decays more slowly. This is apparent in Figure 4.

The cases $\alpha = 10^{-5}$ and $\alpha = 10^{-6}$ are presented in Figures 5 and 6. In both cases the transient introduced has approximately the same magnitude, but the rate of decay of the transient differs between the two cases by a factor of ten. In Figure 5 one can clearly see that the transient solution tends to decay in a spiral, so the total solution approaches the moving steady state solution in a moving spiral. For Figure 6, the plotter was stopped to avoid wearing through the paper where the curves become very dense. The central portion was filled in by a draftsman.

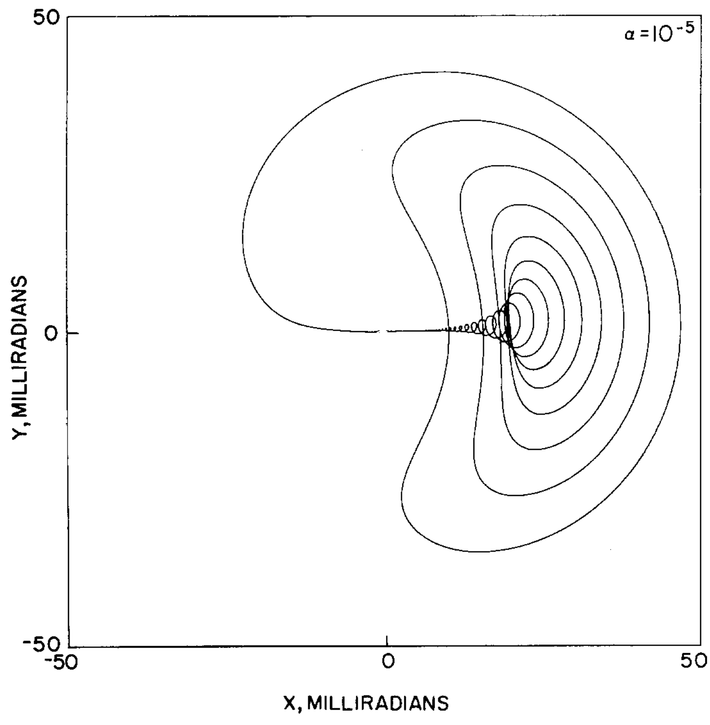

Fig. 5. The case $\alpha = 10^{-5}$ shows that the transient solution decays in a spiral about a moving steady state solution.

Acknowledgements

Work Performed by Dabcovich and Co., Inc., was under Contract F19628-70-C0237 for the Analysis and Simulation Branch, AFCRL. We ackowledge the excellent support and cooperation of the Analysis and Simulation Branch and the Computer Processing Branch of AFCRL.

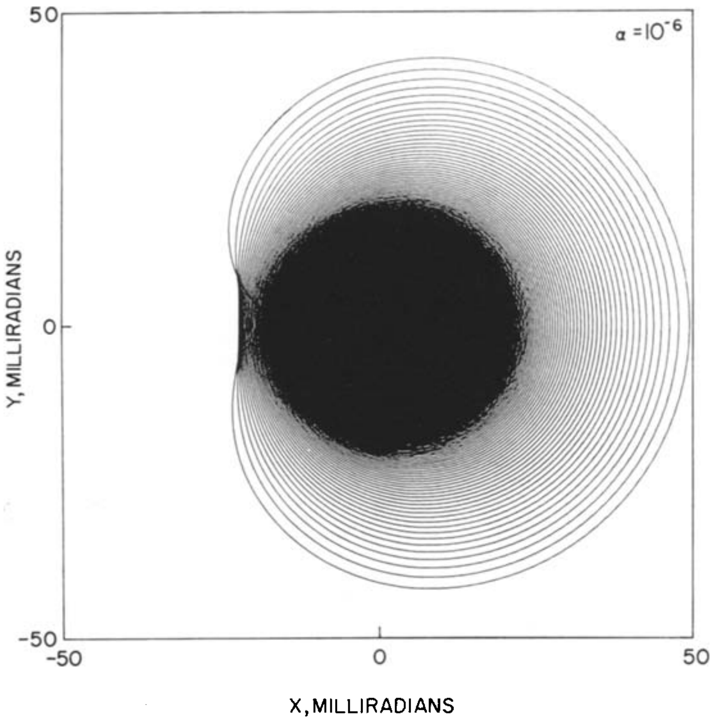

Figure 6. The case $\alpha = 10^{-6}$ shows a slowly decaying transient.

References

Brouwer, D. and Clemence, G. M.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York.

Eckhardt, D. H.: 1970, 'Lunar Libration Tables', *The Moon* **1**, 264.

Habibullin, S. T.: 1966, 'A Nonlinear Theory of the Physical Libration of the Moon', *Tr. Astron. Inst. Univ. Kazan*, No. 34.

Hayashi, C.: 1964, *Nonlinear Oscillations in Physical Systems*, McGraw-Hill, New York.

Kaula, W. M.: 1968, *An Introduction to Planetary Physics*, Wiley, New York.

Koziel, K.: 1967, 'Recent Researches on the Determination of the Moon's Physical Libration Constants, with Special Consideration of Cracow Investigations', in *Measure of the Moon* (ed. by Z. Kopal and C. L. Goudas), D. Reidel, Dordrecht, The Netherlands.

MacDonald, G. J. F.: 1964, 'Tidal Friction', *Rev. Geophys.* **2**, 467.