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A Generalization of the Iterated Logarithm Law for Weighted Sums with Infinite Variance

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Summary. A generalization of the classical Law of the Iterated Logarithm (LIL) is obtained for the weighted i.i.d. case consisting of sequences $\{\sigma_n Y_n\}$ where the weights $\{\sigma_n\}$ are nonzero constants and $\{Y_n\}$ are i.i.d. random variables. If Y is symmetric but not necessarily square integrable and if the weights satisfy a certain growth rate, conditions are given which guarantee that $\{\sigma_n Y_n\}$ obey a Generalized Law of the Iterated Logarithm (GLIL) in the sense that $\limsup_{n\to\infty} \sum_1^n \sigma_j Y_j/a_n = 1$ almost certainly for some positive constants a_n. Teicher has shown that such weights entail the classical LIL when $EY^2 < \infty$ and Feller has treated the GLIL when $\sigma_n = 1$ and $EY^2 = \infty$. The main finding here asserts that if $\{q_n\}$ satisfies $q_n^2 = nG(q_n) \log\log q_n$ where G is a specified slowly varying function, asymptotically equivalent to the truncated second moment of Y, and if a certain series converges, then the GLIL obtains with $a_n = (2/n)^{\frac{1}{2}} s_n q_n$ where $s_n^2 = \sum_1^n \sigma_j^2$.

1. Introduction

Independent random variables $\{X_n\}$, each with mean zero and finite variance, obey the Law of the Iterated Logarithm (LIL) if

$$\limsup_{n\to\infty} \sum_1^n X_j/(2d_n^2 \log_2 d_n^2)^{\frac{1}{2}} = 1 \tag{1}$$

almost certainly (a.c.) where $d_n^2 = \operatorname{Var} \sum_1^n X_j$, $n \geqq 1$. (Throughout, in order to avoid minor complications, it proves convenient to define for $x > 0$

$$\log x = \log_1 x = \log \max\{e, x\}; \qquad \log_r x = \log_1 \log_{r-1} x, \qquad r \geqq 2$$

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where $\log x$ (when $x \geqq e$) denotes the natural logarithm.) The celebrated theorem of Kolmogorov [6] asserts that (1) holds provided each X_n has mean 0 and is bounded (say $|X_n| \leqq B_n$, a.c.), $d_n \to \infty$, and $B_n^2 = o(d_n^2/\log_2 d_n^2)$.

Attention here is restricted to the so-called *weighted i.i.d. case* consisting of X_n's of the form $\sigma_n Y_n$ where $\{Y_n\}$ are independent, identically distributed (i.i.d.) random variables and $\{\sigma_n\}$ are nonzero constants. The $\{Y_n\}$ are referred to as the *underlying random variables* and the $\{\sigma_n\}$ as *weights.* Clearly the weighted i.i.d. case contains the i.i.d. case via $\sigma_n \equiv 1$ and then by the theorem of Hartman and Wintner [3] the conditions $EY=0$, $0<EY^2<\infty$ entail (1). Setting $s_n^2 = \sum_1^n \sigma_j^2$, $n \geqq 1$, and noting $EY=0$ and $EY^2 < \infty$ entail $\operatorname{Var} \sum_1^n \sigma_j Y_j = s_n^2 EY^2$, $\{\sigma_n Y_n\}$ obeying the LIL (1) is tantamount to

$$\limsup_{n\to\infty} \sum_1^n \sigma_j Y_j/(2EY^2 s_n^2 \log_2 s_n^2)^{\frac{1}{2}} = 1, \text{ a.c.} \tag{2}$$

Conditions which guarantee the LIL (2) in the weighted i.i.d. case have been obtained by Teicher [13, 14] when $EY=0$, $EY^2<\infty$. These conditions are of a moment nature on Y and on the growth rate of $n\sigma_n^2/s_n^2$. Moreover, Teicher [13], generalizing the result of Strassen [12] in the i.i.d. case, showed that

$$EY^2 = \infty, \qquad \sigma_n^2 = o(s_n^2/\log_2 s_n^2), \qquad s_n^2 \to \infty \tag{3}$$

implies

$$\limsup_{n\to\infty} \left|\sum_1^n \sigma_j Y_j\right| \Big/ (s_n^2 \log_2 s_n^2)^{\frac{1}{2}} = \infty, \text{ a.c.} \tag{4}$$

The absolute value can of course be removed in (4) if Y is symmetric. It is natural to search for "normalizing" constants $a_n > 0$ for which $\limsup_{n\to\infty} \sum_1^n \sigma_j Y_j/a_n = 1$, a.c. Then, $\{\sigma_n Y_n\}$ is said to obey a *Generalized Law of the Iterated Logarithm* (GLIL).

Feller [2] considered the GLIL question for symmetric i.i.d. random variables with infinite variance and obtained a sufficient condition in the form of convergence of a certain integral. Despite the brilliance and originality of Feller's work, the specific assumptions being made and the proofs are not always clear. Kesten [5] has also questioned certain arguments.

Here, conditions are given for a GLIL to obtain in the weighted i.i.d. case where Y is symmetric with infinite variance and the weights satisfy a certain growth rate of Teicher [13]. The function $G(y) \equiv 2\int_0^y tP\{|Y|>t\}\,dt$ is supposed *slowly varying at infinity.* Under the *latter supposition*, Theorem 1 includes the i.i.d. case of Feller via $\sigma_n \equiv 1$. The proof (Lemmas 12, 13, 14) follows the general pattern outlined by Feller who, however, dealt exclusively with $H(y) \equiv E\{Y^2 I_{[|Y| \leqq y]}\}$ rather than with $G(y)$. The advantage of working with G rather than H has been stressed by Professor Teicher.

According to Lemma 3, slow variation of G is equivalent to the distribution of Y belonging to the *domain of attraction of the normal law.* Heyde [4] has shown that for symmetric i.i.d. random variables $\{Y_n\}$ to obey a GLIL it is necessary that the distribution of Y belong to the *domain of partial attraction of the normal law* and Kesten [5] has proved this to be sufficient (identifying the normalizing sequence to within a constant). Heyde's necessity result has been generalized to the weighted i.i.d. case by Rosalsky and Teicher [9].

2. Preliminaries

The main results of this paper, namely Theorem 1 and Corollary 1 thereof, establish a GLIL. Some preliminaries are needed before even stating these results.

Lemma 1. *If L is positive and slowly varying (at infinity), then* $\log L(y) = o(\log y)$ *as $y \to \infty$ and hence $L(y) = o(y^\alpha)$, all $\alpha > 0$.*

Proof. The second statement is an immediate consequence of the first which, in turn, follows directly from the following well-known representation of a positive slowly varying function:

$$L(y) = b(y) \exp\left\{\int_1^y \varepsilon(t)/t\, dt\right\} \tag{5}$$

where $b(y) \to b$ in $(0, \infty)$ and $\varepsilon(y) = o(1)$ as $y \to \infty$. (For a proof see [1, p. 274] or [10, p. 2].) □

Lemma 2. *Let Y be a nondegenerate random variable and define*

$$G(y) \equiv 2\int_0^y tP\{|Y| > t\}\, dt, \qquad y \geqq 0, \tag{6}$$

$$H(y) \equiv E\{Y^2 I_{[|Y| \leqq y]}\}, \qquad y \geqq 0. \tag{7}$$

Then G is nondecreasing (strictly increasing if Y is unbounded), continuous, and

$$G(y) = H(y) + y^2 P\{|Y| > y\}, \qquad y \geqq 0, \tag{8}$$

$$G(y)/y^2 \downarrow 0 \quad \text{as } 0 < y \uparrow \infty. \tag{9}$$

Proof. Integration by parts yields (8), and the rest is straightforward. □

Condition (14) of the next lemma is the assertion that Y belongs to the *domain of attraction of the normal law.* Moreover, for all p in $(0, 2)$, (14) and Lemma 1 entail that $y^{p-1} P\{|Y| > y\}$ is Lebesgue integrable over $[0, \infty)$ which is tantamount to $Y \in \mathscr{L}_p$.

Lemma 3. *The following are equivalent:*

$$G \text{ is slowly varying,} \tag{10}$$

$$H \text{ is slowly varying,} \tag{11}$$

$$G(y) \sim H(y), \tag{12}$$

$$\lim_{y \to \infty} y^2 P\{|Y| > y\}/G(y) = 0, \tag{13}$$

$$\lim_{y \to \infty} y^2 P\{|Y| > y\}/H(y) = 0. \tag{14}$$

Proof. To show that (10) implies (11), note that for arbitrary δ in (0, 1), via (8)

$$G(y)-G(\delta y)=\int_{\delta y}^{y} 2t\,P\{|Y|>t\}\,dt \geqq (1-\delta^2)\,y^2\,P\{|Y|>y\}=(1-\delta^2)(G(y)-H(y)).$$

Thus,

$$0\leqq 1-\frac{H(y)}{G(y)}\leqq (1-\delta^2)^{-1}\left(1-\frac{G(\delta y)}{G(y)}\right)=o(1),$$

whence for arbitrary $t>0$

$$H(ty)/H(y)=(1+o(1))\,G(ty)/G(y)=1+o(1).$$

It is proved in [1] that (11) implies (14). (See the *proof* of Theorem 2, p. 275.)

That (14) implies (13) and (13) implies (12) follow directly from (8).

To show that (12) implies (10), note that for arbitrary δ in (0, 1)

$$\begin{aligned}0\leqq G(y)-G(\delta y)&=\int_{\delta y}^{y} 2t\,P\{|Y|>t\}\,dt\leqq (1-\delta^2)\,y^2\,P\{|Y|>\delta y\}\\ &=(1-\delta^2)\,\delta^{-2}(G(\delta y)-H(\delta y))\end{aligned}$$

yielding

$$0\leqq \frac{G(y)}{G(\delta y)}-1\leqq (1-\delta^2)\,\delta^{-2}\left(1-\frac{H(\delta y)}{G(\delta y)}\right)=o(1),$$

and it only need be shown that $G(y)\sim G(ty)$ for $t>1$. Setting $\delta=t^{-1}$, by the portion already proved,

$$G(ty)/G(y)=G(ty)/G(\delta t y)=1+o(1). \quad \square$$

The function Q of the next lemma or more specifically, it's inverse, plays a crucial role in determining the normalizing constants for a GLIL. Thus it is only the tail behavior of Q that is important and the definition of $Q(y)$ for small y is irrelevant.

Lemma 4. *If G is slowly varying, then for some $y_0\geqq e^e$*

$$Q(y)\equiv\begin{cases} y^2/(G(y)\log_2 y), & y\geqq y_0\\ y_0\,y/(G(y_0)\log_2 y_0), & 0\leqq y\leqq y_0\end{cases} \tag{15}$$

is continuous and strictly increasing over $[0,\infty)$ with $\lim_{y\to\infty} Q(y)=\infty$.

Proof. The continuity of Q follows from that of G and since the denominator of Q is slowly varying, Lemma 1 ensures that $Q(y)\to\infty$. It suffices to show that there exists $y_0\geqq e^e$ such that $y_0\leqq x<z\leqq x+1$ implies $Q(z)>Q(x)$. Let $\delta>0$ be such that $1-2\delta-\delta^2>0$. Now Lemma 3 ensures for $e^e\leqq y_0$ sufficiently large that

$$\begin{gathered} y^2\,P\{|Y|>y\}/G(y)\leqq\delta, \qquad \log_2 y\geqq (1+\delta)^{-1}\log_2(y+1),\\ (\log_2 y-(2\log y)^{-1})/(\log_2 y)^2\geqq (1-\delta)/\log_2 y \end{gathered} \tag{16}$$

for $y \geqq y_0$. Then for $y_0 \leqq x < z \leqq x+1$,

$$G(x)\,G(z)(Q(z)-Q(x)) = G(x)\left(\frac{z^2}{\log_2 z} - \frac{x^2}{\log_2 x}\right) - \frac{x^2}{\log_2 x}\int_x^z 2t\,P\{|Y|>t\}\,dt$$

$$\geqq G(x)\left(\frac{z^2}{\log_2 z} - \frac{x^2}{\log_2 x} - \frac{x^2 P\{|Y|>x\}(z^2-x^2)}{G(x)\log_2 x}\right). \tag{17}$$

Consequently, via (17), (16), the choice of δ, and the Mean Value Theorem, for some θ in (x^2, z^2)

$$Q(z)-Q(x) \geqq \frac{1}{G(z)}\left(\frac{\log_2 \theta^{\frac{1}{2}} - (2\log\theta^{\frac{1}{2}})^{-1}}{(\log_2 \theta^{\frac{1}{2}})^2}(z^2-x^2) - \frac{x^2 P\{|Y|>x\}(z^2-x^2)}{G(x)\log_2 x}\right)$$

$$\geqq \frac{z^2-x^2}{G(z)}\left(\frac{1-\delta}{\log_2 z} - \frac{\delta(1+\delta)}{\log_2 z}\right) > 0. \quad \square$$

Let q_y, $y \geqq 0$, denote the inverse function of Q where G is slowly varying. Clearly $0 \leqq q_y \uparrow \infty$.

Lemma 5. *The function q_y satisfies*

$$q_y^2 = y\,G(q_y)\log_2 q_y, \qquad \textit{all large } y, \tag{18}$$

$$\log_2 q_y \sim \log_2 y. \tag{19}$$

Proof. Assertion (18) follows by definition of inverse function, whence for all large y

$$\log q_y^2 = \log y + \log G(q_y) + \log_3 q_y$$

implying

$$\log q_y^2 \sim \log y \tag{20}$$

via Lemma 1 and yielding (19). $\square$

Lemma 6. *If $l(y)$ is the inverse of the continuous increasing function $y^\gamma L(y)$, $y>0$, where L is slowly varying and $\gamma>0$, then $y^{-1/\gamma} l(y)$ is slowly varying.*

Proof. If $R(y) = y^\gamma L(y)$, then for any $t>0$

$$R(t^{1/\gamma} l(y)) = t\,l^\gamma(y)\,L(t^{1/\gamma} l(y)) \sim t\,l^\gamma(y)\,L(l(y)) = t\,y.$$

Thus, for all δ in $(0,1)$ and large y

$$(1-\delta)\,t\,y \leqq R(t^{1/\gamma} l(y)) \leqq (1+\delta)\,t\,y$$

or since l is increasing

$$l((1-\delta)\,t\,y) \leqq t^{1/\gamma} l(y) \leqq l((1+\delta)\,t\,y).$$

Taking $u=(1-\delta)t$ and then $u=(1+\delta)t$, it follows that for all $u>0$

$$\left(\frac{u}{1+\delta}\right)^{1/\gamma} l(y) \leqq l(u\,y) \leqq \left(\frac{u}{1-\delta}\right)^{1/\gamma} l(y).$$

Thus, $l(u\,y)/l(y) \sim u^{1/\gamma}$ which is tantamount to slow variation of $y^{-1/\gamma} l(y)$. $\square$

Corollary. The function q_y satisfies for all $t>0$

$$q_{ty} \sim t^{\frac{1}{2}} q_y \text{ as } y \to \infty; \quad u_n \sim t v_n \to \infty \text{ implies } q_{u_n} \sim t^{\frac{1}{2}} q_{v_n} \text{ as } n \to \infty. \tag{21}$$

Proof. The first assertion of (21), which is tantamount to slow variation of $y^{-\frac{1}{2}} q_y$, follows from Lemma 6. Moreover, this assertion and slow variation and monotonicity of $y^{-\frac{1}{2}} q_y$ (see (18)) entail

$$u_n^{-\frac{1}{2}} q_{u_n} \sim (t v_n)^{-\frac{1}{2}} q_{t v_n} \sim u_n^{-\frac{1}{2}} t^{\frac{1}{2}} q_{v_n}$$

yielding the second portion of (21). □

Lemma 7. *If* $0<u(y)\to\infty$, $0<v(y)\to\infty$, *and* L *is positive, nondecreasing, and slowly varying, then*

$$\log \frac{L(u(y)\,v(y))}{L(u(y))} = o(\log v(y)) \quad \text{as} \quad y \to \infty. \tag{22}$$

Proof. The lemma follows from the representation (5). □

Corollary. *The function* q_y *satisfies for arbitrary* δ, $r>0$ *and all large* n

$$G(q_{n([(\log_2 n)^r]+1)}) \leqq (\log_2 n)^{\delta}\, G(q_n) \tag{23}$$

where $[y]$ *denotes the greatest integer in* y.

Proof. For all large n, (22) entails

$$G(q_n (\log_2 q_n)^{\frac{1}{2}(r+\delta)}) / G(q_n) \leqq (\log_2 q_n)^{\frac{1}{2}\delta}. \tag{24}$$

Hence, by definition of Q, (19), (24), and (18), for all large n

$$Q(q_n (\log_2 q_n)^{\frac{1}{2}(r+\delta)}) \geqq n([(\log_2 n)^r]+1),$$

whence

$$q_{n([(\log_2 n)^r]+1)} \leqq q_n (\log_2 q_n)^{\frac{1}{2}(r+\delta)}$$

implying (23) via (24) and (19). □

Lemma 8. *If* $s_n^2 = \sum_1^n \sigma_j^2$, $n \geqq 1$, *where* $\{\sigma_n\}$ *are constants satisfying* $0<\sigma_n^2 \uparrow$ *and* $\gamma_n \equiv n\sigma_n^2/s_n^2 = O((\log_2 n)^{\beta})$, *some* β *in* $[0, 1)$, *then*

$$s_n^2 \sim s_{n+1}^2, \tag{25}$$

$$\log_2 s_n^2 \sim \log_2 n, \tag{26}$$

$$\sigma_n^2 = o(s_n^2 / \log_2 s_n^2). \tag{27}$$

Proof. Note that $\sigma_n^2/s_n^2 = \gamma_n/n = o(1)$ which is tantamount to (25). Hence, for all large j,

$$\frac{s_j^2}{s_{j-1}^2} = 1 + \frac{s_j^2\, j \sigma_j^2}{s_{j-1}^2\, j s_j^2} \leqq 1 + 2\,\frac{\gamma_j}{j}.$$

Thus, for all large m and $n>m$,

$$s_n^2/s_m^2 \leqq \prod_{m+1}^{n} (1+2\gamma_j j^{-1}) \leqq \exp\left\{\sum_{m+1}^{n} 2\gamma_j j^{-1}\right\} \leqq \exp\{(\log_2 n)\log n\}. \tag{28}$$

Hence, for all large n, $\log s_n^2 \leqq (\log_2 n)^2 \log n$ implying

$$\log_2 s_n^2 \leqq 2 \log_3 n + \log_2 n \sim \log_2 n,$$

and (26) obtains since $s_n^2 \geqq n\sigma_1^2$. Assertion (27) is a direct consequence of (26) and $\sigma_n^2/s_n^2 = O((\log_2 n)^\beta)/n$. □

Define monotone sequences $\{e_n\}$ and $\{c_n\}$ by

$$e_0 = 0, \quad e_n = \inf_{i \geqq n} \gamma_i^{-1} G(q_i) \log_2 i, \ n \geqq 1; \quad c_n = (n e_n)^{\frac{1}{2}}, \ n \geqq 0.$$

Lemma 9. *If G is slowly varying and $\{\sigma_n\}$ satisfies the assumptions of Lemma 8, then $\{c_n\}$ satisfies*

$$c_{n+1} = O(c_n); \quad c_n^2/n \uparrow \infty \tag{29}$$

$$(1+o(1))\, q_n^2/(K(\log_2 q_n)^\beta) \leqq c_n^2 \leqq (1+o(1))\, q_n^2 \quad \textit{where } K \textit{ is such that } \gamma_n \leqq K(\log_2 n)^\beta, \tag{30}$$

$$\log_2 c_n \sim \log_2 q_n \sim \log_2 n; \quad \sigma_n^2 c_n^2 \leqq (1+o(1))\, s_n^2 q_n^2/n. \tag{31}$$

Proof. When $e_n = e_{n+1}$, clearly $c_{n+1} \leqq 2c_n$. Alternatively, if $e_n < e_{n+1}$, then $e_n = \gamma_n^{-1} G(q_n) \log_2 n$ and so

$$\frac{c_{n+1}^2}{c_n^2} \leqq \frac{(n+1)\sigma_n^2 G(q_{n+1}) \log_2(n+1)}{\gamma_{n+1} s_n^2 G(q_n) \log_2 n} = O(1)$$

by (25), slow variation of G, and (21). Moreover, $c_n^2/n \geqq K^{-1} G(q_n) (\log_2 n)^{1-\beta} \to \infty$ and clearly $c_n^2/n \uparrow$, establishing (29). Next, recalling (18) and (19)

$$c_n^2 \leqq (1+o(1))\, s_n^2 q_n^2/(n\sigma_n^2) \leqq (1+o(1))\, q_n^2$$

yielding the last half of (31) and the second inequality of (30). On the other hand, $c_n^2 \geqq nK^{-1} G(q_n)(\log_2 n)^{1-\beta}$, and (30) follows via (19) and (18). The first half of (31) follows immediately from (30) and (19). □

Lemma 10. *If $0 < t_n/n^\alpha \uparrow$ for some $\alpha > 0$ and if Y is any random variable, then $\sum_1^\infty P\{|Y| > \lambda t_n\}$ either converges for all $\lambda > 0$ or else it diverges for all $\lambda > 0$.*

Proof. This result is well known. (See, for example, [11, p. 131].) □

Lemma 11. *If $C(y) \equiv y^2/c^{-1}(y)$, $y > 0$, where $c(y) = (y e(y))^{\frac{1}{2}}$ for $y \geqq 0$ and $e(y)$ is the continuous extension of $\{e_n\}$ defined by linear interpolation between integers, that is,*

$$e(y) = (e_{n+1} - e_n)(y - n) + e_n \quad \textit{for } 0 \leqq n \leqq y \leqq n+1,$$

then

$$C(y)\uparrow; \ c(y) \textit{ is a continuous strictly increasing extension of } \{c_n\} \textit{ with } c(y) \to \infty \tag{32}$$

and moreover

$$\int_{[|y|>0]} y^2/C(|y|)\, dF(y) < \infty \tag{33}$$

(where F is the distribution of Y) iff

$$\sum_1^\infty P\{|Y| > \lambda c_n\} < \infty \tag{34}$$

for some (and hence all) $\lambda > 0$.

Proof. The proof of (32) is straightforward. Next, convergence in (34) for some $\lambda > 0$ is equivalent to convergence for all $\lambda > 0$ by Lemma 10. Moreover, convergence in (34) with $\lambda = 1$ is tantamount to $\sum_1^\infty P\{c^{-1}(|Y|) > n\} < \infty$ which is equivalent to $E\{c^{-1}(|Y|)\} < \infty$. □

For $0 \leqq u < v < \infty$, define $J(u, v] = \int_{[u<|y|\leqq v]} y^2/C(|y|)\, dF(y)$.

3. Mainstream

With these preliminaries accounted for, it becomes possible to discuss and establish a GLIL for weighted sums of nondegenerate symmetric i.i.d. random variables with $\sigma^2 \equiv EY^2 \leqq \infty$. Recall Teicher's theorem that (3) implies (4). Consequently if $\sigma^2 = \infty$, then under the proviso (3) a GLIL with normalizing constants $\{a_n\}$ requires $\limsup_{n\to\infty} a_n/(s_n^2 \log_2 s_n^2)^{\frac{1}{2}} = \infty$.

Note that if $a_n \equiv (2/n)^{\frac{1}{2}} s_n q_n$ and $\sigma^2 = \infty$ where G is slowly varying and $\{\sigma_n\}$ satisfy the conditions of Lemma 8, then by Lemmas 8, 5, and 3, (3) holds and

$$a_n^2/(s_n^2 \log_2 s_n^2) = (2G(q_n) \log_2 q_n)/\log_2 s_n^2 \sim 2H(q_n) \to \infty.$$

Moreover, $\sigma_n^2 \uparrow$ entails $a_n \uparrow \infty$. It will be demonstrated that, under appropriate conditions, $\{a_n\}$ is a proper choice of normalizing constants for a GLIL.

An alternative approach is to define

$$a_n' = 2^{\frac{1}{2}} \sup\left\{y \geqq e^e \colon y^2 \Big/ \left(\sum_1^n \sigma_j^2\, G(y/|\sigma_j|) \log_2 y\right) \leqq 1\right\}$$

and this is closer in spirit to that of Feller [2] (in the special case $\sigma_n \equiv 1$) and to condition (4) of Theorem 1 of [8]. When $\gamma_n = O(1)$, the author has shown that $a_n \sim a_n'$. Another contrast with [2] is the emphasis now on G rather then H, the former having advantages such as continuity, (9), etc.

Of course, if $\sigma^2 < \infty$, then slow variation of H and G is automatic and under the assumptions of Lemma 8,

$$a_n^2 = 2s_n^2 q_n^2/n = 2s_n^2\, G(q_n) \log_2 q_n \sim 2\sigma^2 s_n^2 \log_2 s_n^2$$

yields the normalization for the classical LIL (2) (Theorem 3 of [13]).

Theorem 1. *Let* $s_n^2 = \sum_1^n \sigma_j^2$, $n \geqq 1$, *where* $\{\sigma_n\}$ *are constants satisfying* $0 < \sigma_n^2 \uparrow$ *and* $\gamma_n \equiv n\sigma_n^2/s_n^2 = O((\log_2 n)^\beta)$ *for some* β *in* $[0, 1)$. *If* $\{Y, Y_n\}$ *are nondegenerate, symmetric i.i.d. random variables with* $EY^2 \leqq \infty$ *such that the function* G *of* (6) *is slowly varying and if the series of* (34) *converges for some* $\lambda > 0$, *then*

$$\limsup_{n\to\infty} \frac{\sum_1^n \sigma_j Y_j}{(2/n)^{\frac{1}{2}} s_n q_n} = 1, \quad a.c. \tag{35}$$

Proof. Let $p > 1$ and $0 < \eta < (1-\beta)/8$. Define for $n \geqq 1$,

$$X_n = Y_n I_{[|Y_n| \leqq c_n/(\log_2 c_n)^p]}, \qquad Z_n = Y_n I_{[|Y_n| > c_n/(\log_2 c_n)^\eta]}, \qquad W_n = Y_n - X_n - Z_n.$$

Then, it suffices to show

$$\lim_{n\to\infty} \frac{\sum_1^n \sigma_j Z_j}{s_n q_n/n^{\frac{1}{2}}} = 0, \quad \text{a.c.} \tag{36}$$

$$\lim_{n\to\infty} \frac{\sum_1^n \sigma_j W_j}{s_n q_n/n^{\frac{1}{2}}} = 0, \quad \text{a.c.} \tag{37}$$

$$\limsup_{n\to\infty} \frac{\sum_1^n \sigma_j X_j}{s_n q_n/n^{\frac{1}{2}}} = 2^{\frac{1}{2}}, \quad \text{a.c.} \tag{38}$$

and this will be done in the ensuing three lemmas.

Lemma 12. (36) *holds.*

Proof. Set $b_n = c_n/(\log_2 c_n)^\eta$, $n \geqq 1$. It will first be shown that there exists a constant $C < \infty$ such that if $0 < \delta < 1$ and

$$Y_n' \equiv Y_n I_{[b_n < |Y_n| \leqq \delta c_n]}, \qquad S_n' \equiv \sum_1^n \sigma_j Y_j', \qquad n \geqq 1,$$

then

$$\limsup_{n\to\infty} \frac{|S_n'|}{s_n q_n/n^{\frac{1}{2}}} \leqq C\delta, \quad \text{a.c.} \tag{39}$$

Set $n_0 = 0$. Recalling (29), a constant $M > 1$ and an integer n_1 may be chosen such that $n \geqq n_1$ implies

$$c_{n+1}/c_n < M \leqq (\log_2 c_{n+1})^\eta. \tag{40}$$

Given n_{k-1} for some $k \geqq 2$, let $n_k = \max\{n\colon b_n < c_{n_{k-1}}\}$. Then $\{n_k, k \geqq 0\}$ is strictly increasing and

$$b_{n_k} < c_{n_{k-1}} \leqq b_{n_k+1} \tag{41}$$

for $k \geqq 2$. If

$$A_k = \bigcup_{\substack{n_{k-1} < i, j \leqq n_k \\ i \neq j}} [Y_i' \neq 0 \text{ and } Y_j' \neq 0], \qquad k \geqq 1$$

then

$$P\{A_k\} \leqq \Big(\sum_{n_{k-1}<n\leqq n_k} P\{Y_n' \neq 0\}\Big)^2. \tag{42}$$

Now, for all large k, using Lemma 11, (41), and (31)

$$\sum_{n_{k-1}<n\leqq n_k} P\{Y_n' \neq 0\} \leqq \sum_{n_{k-1}<n\leqq n_k} b_n^{-2} E\{Y^2 I_{[b_n<|Y|\leqq c_n]}\}$$

$$\leqq \frac{n_k}{b_{n_{k-1}}^2} \int_{[b_{n_{k-1}}<|y|\leqq c_{n_k}]} y^2\, dF(y) \tag{43}$$

$$\leqq \frac{n_k}{b_{n_{k-1}}^2} C(c_{n_k})\, J(b_{n_{k-1}}, c_{n_k}]$$

$$= O(1)\,(\log_2 q_{n_k})^{4\eta} J(c_{n_{k-3}}, c_{n_k}]. \tag{44}$$

Next, (30) and slow variation of G ensure for all large n that $H(c_n) \leqq (1+o(1))\, G(q_n)$. Hence, from (43), (18), (30), (41), and (31), for all large k

$$\sum_{n_{k-1}<n\leqq n_k} P\{Y_n' \neq 0\} \leqq 2n_k G(q_{n_k})/b_{n_{k-1}}^2 = O\left(\frac{c_{n_k}^2}{b_{n_{k-1}}^2 (\log_2 q_{n_k})^{1-\beta}}\right)$$

$$= O\left(\frac{(\log_2 c_{n_k})^{2\eta}\, c_{n_{k-1}}^2}{b_{n_{k-1}}^2 (\log_2 q_{n_k})^{1-\beta}}\right) = O((\log_2 q_{n_k})^{\beta+4\eta-1}). \tag{45}$$

Thus, from (42), (44), (45), the choice of η, (34), and Lemma 11

$$\sum_1^\infty P\{A_k\} < \infty. \tag{46}$$

Define the random integer κ by $\kappa = \min\{k \geqq 2\colon (n_{k^*-1}, n_{k^*}]$ contains at most one nonzero term $\sigma_n Y_n'$ for all $k^* \geqq k\}$ ($=\infty$, otherwise). According to (46) and the Borel-Cantelli Lemma, $\kappa < \infty$, a.c. Thus, with probability one, each block $(n_{k-1}, n_k]$ with $k \geqq \kappa$ contains at most one nonzero term $\sigma_n Y_n'$ whose absolute value is at most $|\sigma_n|\,\delta c_n \leqq \delta |\sigma_{n_k}|\, c_{n_k}$. Then, with probability one, for all $n_\kappa \leqq n_{k-1} < n \leqq n_k$

$$|S_n'| \leqq |S_{n_{\kappa-1}}'| + \delta \sum_{r=\kappa}^{k-1} |\sigma_{n_r}|\, c_{n_r} + \delta |\sigma_n|\, c_n \leqq |S_{n_{\kappa-1}}'| + \delta |\sigma_n| \left(\sum_1^{k-1} c_{n_r} + c_n\right). \tag{47}$$

But

$$\sum_1^{k-1} c_{n_r} \leqq 2(c_{n_{k-1}} + c_{n_{k-3}} + \ldots + c_{n_1} \text{ or } c_{n_2}) \tag{48}$$

according to whether k is even or odd. Now (41) and (40) entail $c_{n_{k-1}} \leqq M^{-1} c_{n_{k+1}}$, $k \geqq 2$, and applying this repeatedly j times to $c_{n_{k-(2j+1)}}$ in (48) yields $c_{n_{k-(2j+1)}} \leqq M^{-j} c_{n_{k-1}} \leqq M^{-j} c_n$ and consequently $\sum_1^{k-1} c_{n_r} \leqq 2Mc_n/(M-1)$. Hence, from (47) and (31), with probability one for all large k and all n in $(n_{k-1}, n_k]$

$$|S_n'| \leqq |S_{n_{\kappa-1}}'| + \delta |\sigma_n|\, c_n (3M-1)/(M-1) \leqq |S_{n_{\kappa-1}}'| + C\delta s_n q_n / n^{\frac{1}{2}}$$

where $C = (6M-2)/(M-1)$, and (39) follows.

Next, by (34) and the Borel-Cantelli Lemma for all $\delta>0$, with probability one,

$$\sum_1^n \sigma_j Y_j I_{[|Y_j|>\delta c_j]} = O(1). \tag{49}$$

Thus, from (39) and (49), $\limsup_{n\to\infty} \left|\sum_1^n \sigma_j Z_j\right| \Big/ (s_n q_n/n^{\frac{1}{2}}) \leqq C\delta$, a.c. and (36) then follows from the arbitrariness of δ. □

Lemma 13. (37) *holds.*

Proof. Set $n_0=0$ and define n_k to be the smallest integer n such that $s_n^2>e^k$, $k\geqq 1$. Then, via (25), $s_{n_k}^2 \sim e^k$ and $n_k<n_{k+1}$ for all large k. Since $\sigma_n^2\uparrow$, $s_{3n_k}^2 \geqq 3s_{n_k}^2 > e^{k+1}$ implying

$$n_k < n_{k+1} \leqq 3n_k \tag{50}$$

for all large k. Thus, via (18), (21), and slow variation of G

$$n_k s_{n_{k+1}}^2 q_{n_{k+1}}^2 / (n_{k+1} s_{n_k}^2 q_{n_k}^2) \sim e.$$

In view of the Almost Sure Stability Criterion [7, p. 252], setting $U_k = \sum_{n_{k-1}<j\leqq n_k} \sigma_j W_j$, $k\geqq 1$, it suffices to show that $\lim_{k\to\infty} U_k/(s_{n_k} q_{n_k}/n_k^{\frac{1}{2}})=0$, a.c. and so it suffices to verify that

$$\sum_1^\infty n_k^2 E\{U_k^4\}/(s_{n_k}^4 q_{n_k}^4) < \infty. \tag{51}$$

Set $V_k = \operatorname{Var}\{U_k\} = \sum_{n_{k-1}<j\leqq n_k} E\{\sigma_j^2 W_j^2\}$, $k\geqq 1$. By the multinomial expansion, independence, and symmetry, for some positive integer A and all large k

$$E\{U_k^4\} \leqq E\{\sum_{n_{k-1}<j\leqq n_k} \sigma_j^4 W_j^4\} + A V_k^2 \leqq (\sigma_{n_k}^2 c_{n_k}^2 (\log_2 c_{n_k})^{-2\eta} + A V_k) V_k. \tag{52}$$

Now, employing (30), (12), (18), and (19)

$$V_k \leqq s_{n_k}^2 H(q_{n_k}) = (1+o(1))\, s_{n_k}^2 q_{n_k}^2/(n_k \log_2 n_k),$$

whence from (52) and (31) for all large k

$$E\{U_k^4\} \leqq 2 s_{n_k}^2 q_{n_k}^2 V_k/(n_k (\log_2 n_k)^{2\eta}). \tag{53}$$

An alternative upper bound for V_k is obtained as follows: Now $1\leqq \gamma_n \leqq K(\log_2 n)^\beta$ yields $K^{-1} G(q_n)(\log_2 n)^{1-\beta} \leqq e_n \leqq G(q_n)\log_2 n$ and so by monotonicity and the definition of c_n and employing (50), (21), slow variation of G, and (31), it follows for all large k and $n_{k-1}<j\leqq n_k$ that

$$\begin{aligned}\frac{c_j^2}{(\log_2 c_j)^{2p}} \cdot \frac{(\log_2 c_{n_k})^{2p+1}}{c_{n_k}^2} &\geqq \frac{c_{n_{k-1}}^2 \log_2 c_{n_k}}{c_{n_k}^2}\\ &\geqq \frac{n_{k-1} G(q_{n_{k-1}})(\log_2 n_{k-1})^{1-\beta} \log_2 c_{n_k}}{K n_k G(q_{n_k}) \log_2 n_k} \to \infty\end{aligned}$$

implying for $u=p+\frac{1}{2}$ and $n_{k-1}<j\leqq n_k$ that $c_j/(\log_2 c_j)^p \geqq c_{n_k}/(\log_2 c_{n_k})^u$. Thus, for all large k

$$V_k \leqq s_{n_k}^2 \int_{[c_{n_k}/(\log_2 c_{n_k})^u<|y|\leqq c_{n_k}]} y^2\, dF(y). \tag{54}$$

Hence, to prove (51), it suffices to show, in view of (53) and (54), that

$$\sum_1^\infty \frac{n_k}{q_{n_k}^2 (\log_2 n_k)^{2\eta}} \int_{[c_{n_k}/(\log_2 c_{n_k})^u<|y|\leqq c_{n_k}]} y^2\, dF(y) < \infty. \tag{55}$$

For given $k\geqq 1$, let $j(k)$ be the smallest integer $j\geqq 1$ for which

$$c_{n_k}/(\log_2 c_{n_k})^u \leqq c_{n_j}. \tag{56}$$

For given $j\geqq 1$, let $k(j)$ be the largest integer $k\geqq 1$ for which $j\geqq j(k)$ or, equivalently, for which (56) holds. Via (32), (18), and (19), the series of (55) is dominated by

$$\begin{aligned}
&\sum_{k=1}^\infty n_k/(q_{n_k}^2(\log_2 n_k)^{2\eta}) \sum_{j=j(k)}^{k} \int_{[c_{n_{j-1}}<|y|\leqq c_{n_j}]} y^2\, dF(y) \\
&= \sum_{j=1}^\infty \int_{[c_{n_{j-1}}<|y|\leqq c_{n_j}]} y^2\, dF(y) \sum_{k=j}^{k(j)} n_k/(q_{n_k}^2(\log_2 n_k)^{2\eta}) \\
&\leqq O(1) \sum_1^\infty J(c_{n_{j-1}}, c_{n_j}]\, \frac{e_{n_j}(k(j)-j+1)}{G(q_{n_j})(\log_2 n_j)^{1+2\eta}}.
\end{aligned} \tag{57}$$

Claim. *For all $\delta>0$, if j is sufficiently large*

$$k(j) < j+2+[G(q_{n_j})(\log_2 n_j)^{1+\delta}/e_{n_j}] \equiv \kappa(j,\delta) \equiv \kappa. \tag{58}$$

Proof of Claim. Via (26),

$$\log_2 n_j \sim \log_2 s_{n_j}^2 \sim \log_2 e^j = \log j.$$

Thus, using the definition of κ and e_n, $\log j \sim \log \kappa$ implying

$$\log_2 n_\kappa \sim \log_2 n_j. \tag{59}$$

To substantiate (58), it suffices by the definition of $k(j)$ and the monotonicity of $c_n/(\log_2 c_n)^u$ to show that

$$c_{n_\kappa}/(\log_2 c_{n_\kappa})^u > c_{n_j}$$

for all large j. Now, by (29), (31), and (59)

$$c_{n_\kappa}^2/((\log_2 c_{n_\kappa})^{2u} c_{n_j}^2) \geqq (1+o(1))\, n_\kappa/(n_j(\log_2 n_j)^{2u})$$

and thus it suffices to show that

$$n_\kappa > n_j([(\log_2 n_j)^{3u}]+1) \equiv N_j$$

for all large j. Recalling the definition of n_k, it is enough to verify for all large j that $s_{N_j}^2 \leqq e^\kappa$ and this will be true if it can be shown that

$$s_{N_j}^2/s_{n_j}^2 \leqq \exp\{G(q_{n_j})(\log_2 n_j)^{1+\delta}/e_{n_j}\}. \tag{60}$$

To this end, note that by (23) eventually

$$G(q_{N_j}) \leqq G(q_{n_j}) (\log_2 n_j)^{\frac{1}{2}\delta}. \tag{61}$$

Let $v(j)$ be the largest value of v, $n_j+1 \leqq v \leqq N_j$, which maximizes γ_v in that interval. Then, noting that $\log_2 N_j \sim \log_2 n_j$ and employing (61)

$$e_{n_j} \leqq G(q_{v(j)}) (\log_2 v(j))/\gamma_{v(j)} \leqq (1+o(1))\, G(q_{n_j}) (\log_2 n_j)^{1+\frac{1}{2}\delta}/\gamma_{v(j)}. \tag{62}$$

Then, recalling (28), for all large j

$$s_{N_j}^2/s_{n_j}^2 \leqq \exp\left\{2\gamma_{v(j)} \sum_{n_j+1}^{N_j} v^{-1}\right\} \leqq \exp\{7u\gamma_{v(j)} \log_3 n_j\}$$

and (60) follows via (62) establishing the claim.

Now $\gamma_n \geqq 1$ ensures that $e_n \leqq G(q_n) \log_2 n$ and so, in view of (58),

$$\frac{e_{n_j}(k(j)-j+1)}{G(q_{n_j}) (\log_2 n_j)^{1+2\eta}} < \frac{3}{(\log_2 n_j)^{2\eta}} + \frac{1}{(\log_2 n_j)^{2\eta-\delta}} \to 0$$

as $j \to \infty$ provided $0 < \delta < 2\eta$. Thus, from (57) the series of (55) is dominated by $O(1) \sum_1^\infty J(c_{n_{j-1}}, c_{n_j}]$ which converges by (34) and Lemma 11. □

Lemma 14. (38) *holds.*

Proof. Let δ be an arbitrary number in (0, 1). Now $1 \leqq \gamma_n \leqq K(\log_2 n)^\beta$ entails $K^{-1} G(q_n)(\log_2 n)^{1-\beta} \leqq e_n \leqq G(q_n) \log_2 n$ and so by definition of c_n, (21), slow variation of G, and (31)

$$\frac{c_n^2}{c_{[\delta n]}^2} \leqq \frac{KnG(q_n) \log_2 n}{[\delta n]\, G[q_{[\delta n]}) (\log_2 [\delta n])^{1-\beta}} = O(1) (\log_2 c_n)^\beta,$$

whence $c_n/c_{[\delta n]} \leqq \log_2 c_n$ for all large n. Setting $D_n = \operatorname{Var} \sum_1^n \sigma_j X_j$, $n \geqq 1$, it follows that for all large n

$$D_n \geqq \sum_{[\delta n]+1}^{n} \sigma_j^2 H(c_{[\delta n]}/(\log_2 c_n)^p) \geqq \sum_{[\delta n]+1}^{n} \sigma_j^2 H(c_n/(\log_2 c_n)^{p+1}).$$

But

$$s_n^2 \Big/ \sum_{[\delta n]+1}^{n} \sigma_j^2 \leqq (1-\delta)^{-1}.$$

Thus via Lemmas 3 and 9, for all large n

$$D_n \geqq (1-\delta)\, s_n^2 H(c_n/(\log_2 c_n)^{p+1}) \geqq (1-\delta)^2 s_n^2 G(q_n/(\log_2 q_n)^{p+2}). \tag{63}$$

On the other hand, by Lemmas 9 and 2, for all large n

$$D_n \leqq s_n^2 H(q_n) \leqq s_n^2 G(q_n). \tag{64}$$

Next, recalling Lemma 7,

$$G(q_n)/G(q_n/(\log_2 q_n)^{p+2}) = (\log_2 q_n)^{o(1)}. \tag{65}$$

Now for all large n, Lemma 1 ensures $G(n) \leqq n^{\frac{1}{2}}$ and so $Q(n) \geqq n$ implying $q_n \leqq n$ whence employing (64) and $n\sigma_1^2 \leqq s_n^2$,

$$\log_2 D_n \leqq \log_2(s_n^2 G(q_n)) \leqq \log_2(s_n^2 G(n)) \leqq \log_2(s_n^2 n^{\frac{1}{2}}) \leqq \log_2(s_n^3/|\sigma_1|) \sim \log_2 s_n^2.$$

On the other hand from (63), $\log_2 D_n \geqq (1+o(1)) \log_2 s_n^2$. Thus, recalling (26) and (19)

$$\log_2 D_n \sim \log_2 s_n^2 \sim \log_2 n \sim \log_2 q_n. \tag{66}$$

Now, via (31), (63), (18), (65), and (66)

$$\sigma_n^2 c_n^2/(\log_2 c_n)^{2p} = o(D_n/\log_2 D_n) \tag{67}$$

and so by the Kolmogorov LIL [6]

$$\limsup_{n\to\infty} \frac{\sum_1^n \sigma_j X_j}{(D_n \log_2 D_n)^{\frac{1}{2}}} = \limsup_{n\to\infty} \frac{\left|\sum_1^n \sigma_j X_j\right|}{(D_n \log_2 D_n)^{\frac{1}{2}}} = 2^{\frac{1}{2}}, \quad \text{a.c.} \tag{68}$$

But from (66), (64), and (18), $D_n \log_2 D_n \leqq (1+o(1))\, s_n^2 q_n^2/n$ whence (68) entails (38) with equality replaced by $\leqq$ and it remains to prove that (38) holds with equality replaced by $\geqq$.

Let δ' be an arbitrary number in $(0, \delta)$ and choose $\gamma > 0$ so that $(1+\gamma)(1-\delta') = 1$. Select $C > 1$ large enough so that $(1-\delta')(1-C^{-2})^{\frac{1}{2}} - 2C^{-1} > 1-\delta$. Let $n_0 = 0$ and let n_k be the smallest integer n such that $D_n > C^{2k}$, $k \geqq 1$. Now (67) entails $E(\sigma_n X_n)^2 = o(D_n)$ or, equivalently, $D_n \sim D_{n-1}$ implying

$$D_{n_k} \sim C^{2k}. \tag{69}$$

Suppose, initially, that

$$\sum_1^\infty (k G(q_{n_k}))^{-1} \int_{[c_{n_k}/(\log_2 c_{n_k})^{p+1} < |y| \leqq q_{n_k}]} y^2\, dF(y) < \infty. \tag{70}$$

Then for k' sufficiently large recalling (12)

$$\sum_{k=k'}^\infty k^{-1} \left(1 - \frac{H(c_{n_k}/(\log_2 c_{n_k})^{p+1})}{H(q_{n_k})}\right) < \infty. \tag{71}$$

Let

$$\mathscr{K} = \mathscr{K}(\delta) \equiv \{k\colon H(c_{n_k}/(\log_2 c_{n_k})^{p+1}) < (1-\delta)\, H(q_{n_k})\}.$$

Thus (71) ensures $\sum_{k\in\mathscr{K}} k^{-1} < \infty$ and consequently

$$\sum_{k\in\mathscr{K}'} k^{-1} = \infty \tag{72}$$

where $\mathscr{K}'$ denotes the complement of $\mathscr{K}$. Define

$$A_k = \Big[\sum_{n_{k-1} < j \leqq n_k} \sigma_j X_j > (1-\delta')\, g_k h_k\Big], \qquad k \geqq 1,$$

where recalling (69)

$$g_k^2 \equiv \operatorname{Var} \sum_{n_{k-1}<j\leqq n_k} \sigma_j X_j = D_{n_k} - D_{n_{k-1}} \sim (1-C^{-2})\, D_{n_k}, \tag{73}$$

$$h_k^2 \equiv 2 \log_2 g_k^2 \sim 2 \log_2 D_{n_k}. \tag{74}$$

Thus, via (69), $h_k^2 \leqq 2(1+\gamma) \log k$ for all large k.

Now, recalling (67), by the Exponential Bounds Lemma [11, p. 262], for all large k

$$P\{A_k\} \geqq \exp\{-\tfrac{1}{2}(1+\gamma)(1-\delta')^2 h_k^2\} \geqq k^{-1}.$$

Thus, via (72) and the Borel-Cantelli Lemma

$$P\{A_k \text{ i.o. } (k\in\mathscr{K}')\} = 1. \tag{75}$$

Set

$$B_k = \left[\left|\sum_1^{n_{k-1}} \sigma_j X_j\right| \leqq 2(2D_{n_{k-1}} \log_2 D_{n_{k-1}})^{\frac{1}{2}}\right], \qquad k \geqq 2.$$

Now, via (73), (74), (69), and the choice of C

$$(1-\delta')\, g_k h_k - 2(2D_{n_{k-1}} \log_2 D_{n_{k-1}})^{\frac{1}{2}} > (1-\delta)(2D_{n_k} \log_2 D_{n_k})^{\frac{1}{2}}$$

for all large k. Consequently

$$P\{A_k B_k \text{ i.o. } (k\in\mathscr{K}')\} \leqq P\left\{\sum_1^{n_k} \sigma_j X_j > (1-\delta)(2D_{n_k} \log_2 D_{n_k})^{\frac{1}{2}} \text{ i.o. } (k\in\mathscr{K}')\right\}. \tag{76}$$

Now, via (68), $P\{B_k$ occurs for all large $k\in\mathscr{K}'\} = 1$ and so, recalling (75), both sides of (76) are equal to 1. Thus,

$$\limsup_{\substack{k\to\infty\\ k\in\mathscr{K}'}} \frac{\sum_1^{n_k} \sigma_j X_j}{(D_{n_k} \log_2 D_{n_k})^{\frac{1}{2}}} \geqq (1-\delta)\, 2^{\frac{1}{2}}, \quad \text{a.c.} \tag{77}$$

However, for all large k in $\mathscr{K}'$, recalling (63), (66), and Lemmas 3 and 5

$$\begin{aligned} D_{n_k} \log_2 D_{n_k} &\geqq (1-\delta)^2 s_{n_k}^2 H(c_{n_k}/(\log_2 c_{n_k})^{p+1}) \log_2 q_{n_k} \\ &\geqq (1-\delta)^3 s_{n_k}^2 H(q_{n_k}) \log_2 q_{n_k} \geqq (1-\delta)^4 s_{n_k}^2 q_{n_k}^2/n_k \end{aligned}$$

whence (77) entails

$$\limsup_{n\to\infty} \frac{\sum_1^n \sigma_j X_j}{s_n q_n/n^{\frac{1}{2}}} \geqq \limsup_{\substack{k\to\infty\\ k\in\mathscr{K}'}} \frac{\sum_1^{n_k} \sigma_j X_j}{s_{n_k} q_{n_k}/n_k^{\frac{1}{2}}} \geqq (1-\delta)^3\, 2^{\frac{1}{2}}, \quad \text{a.c.}$$

and (38), with equality replaced by $\geqq$, follows from the arbitrariness of δ.

The prior argument was predicated on (70) which will now be verified. By (30) and (31) letting $v = p+2$, for all large n, $c_n/(\log_2 c_n)^{p+1} \geqq q_n/(\log_2 q_n)^v$ and so the series of (70) is dominated by

$$O(1)\sum_{1}^{\infty}(kG(q_{n_k}))^{-1}\int_{[q_{n_k}/(\log_2 q_{n_k})^v<|y|\leqq q_{n_k}]} y^2\,dF(y). \tag{78}$$

Now for all large y, choosing n such that $n-1<\frac{1}{2}Q(y)\leqq n$, by (21) and (30)

$$y=q_{Q(y)}\geqq(1+o(1))\,2^{\frac{1}{2}}q_n\geqq(1+o(1))\,2^{\frac{1}{2}}c_n\geqq c(\tfrac{1}{2}Q(y)),$$

whence

$$Q(y)\leqq 2c^{-1}(y)=2y^2/C(y). \tag{79}$$

For given $k\geqq 1$, let $j(k)$ be the smallest integer $j\geqq 1$ for which

$$q_{n_k}/(\log_2 q_{n_k})^v\leqq q_{n_j}. \tag{80}$$

For given $j\geqq 1$, let $k(j)$ be the largest integer $k\geqq 1$ for which $j\geqq j(k)$ or, equivalently, for which (80) holds. Then by (78), the definition of Q, (79), (66), and (69), the series of (70) is dominated by

$$\begin{aligned}
&O(1)\sum_{j=1}^{\infty}\int_{[q_{n_{j-1}}<|y|\leqq q_{n_j}]} y^2\,dF(y)\sum_{k=j}^{k(j)}(kG(q_{n_k}))^{-1}\\
&\leqq O(1)\sum_{1}^{\infty}\int_{[q_{n_{j-1}}<|y|\leqq q_{n_j}]} Q(|y|)\,G(|y|)\log_2|y|\,dF(y)\,\frac{k(j)-j+1}{jG(q_{n_j})}\\
&\leqq O(1)\sum_{1}^{\infty}J(q_{n_{j-1}},\,q_{n_j}]\,j^{-1}(k(j)-j+1)\log j.
\end{aligned} \tag{81}$$

Claim. *For all* ε *in* (0, 1), *if* j *is sufficiently large*

$$k(j)<j+2+[j^{\varepsilon}]\equiv\kappa(j,\varepsilon)\equiv\kappa. \tag{82}$$

Proof of Claim. Using (66), (69), and the definition of κ

$$\log_2 n_\kappa\sim\log\kappa\sim\log j\sim\log_2 n_j. \tag{83}$$

To substantiate (82), it suffices by the definition of $k(j)$ and the monotonicity of $q_n/(\log_2 q_n)^v$ to show that $q_{n_\kappa}/(\log_2 q_{n_\kappa})^v>q_{n_j}$ for all large j. Since $y^{-1}q_y^2=G(q_y)\log_2 q_y$ is increasing, by (66) and (83)

$$q_{n_\kappa}^2/((\log_2 q_{n_\kappa})^{2v}q_{n_j}^2)\geqq(1+o(1))\,n_\kappa/(n_j(\log_2 n_j)^{2v})$$

and thus it suffices to show that

$$n_\kappa>n_j([(\log_2 n_j)^{3v}]+1)\equiv N_j$$

for all large j. Recalling the definition of n_k, it is enough to verify for all large j that $D_{N_j}\leqq C^{2\kappa}$ and this will be true if it can be shown that

$$D_{N_j}/D_{n_j}\leqq C^{2j^{\varepsilon}}. \tag{84}$$

To this end, via (9), (66), and (69)

$$G(q_{n_j})/G(q_{n_j}/(\log_2 q_{n_j})^{p+2})\leqq(\log_2 q_{n_j})^{2p+4}=(1+o(1))\,(\log j)^{2p+4}. \tag{85}$$

Moreover by (23), (66), and (69) for all large j

$$G(q_{N_j})/G(q_{n_j}) \leqq (\log j)^{\varepsilon}. \tag{86}$$

Invoking (28), (66), and (69), for all large j

$$s_{N_j}^2/s_{n_j}^2 \leqq \exp\left\{2K(\log_2 N_j)^{\beta} \sum_{n_j+1}^{N_j} 1/i\right\} \leqq j. \tag{87}$$

Now (63), (64), (85), (86), and (87) entail (84) for all large j, establishing the claim.

Finally, in view of (82), $j^{-1}(k(j)-j+1)\log j \to 0$ as $j\to\infty$ provided $0<\varepsilon<1$. Thus, from (81) the series of (70) is dominated by $O(1)\sum_1^{\infty} J(q_{n_{j-1}}, q_{n_j}]$ which converges by (34) and Lemma 11. □

The series of (34) can be replaced by an integral according to

Corollary 1. Under the hypotheses of the theorem, if (34) is replaced by

$$\int_{[|y|\geqq e^e]} y^2/(G(|y|)(\log_2|y|)^{1-\beta})\,dF(y)<\infty, \tag{88}$$

then (35) holds.

Proof. Let $Q_{\beta}(y)=(\log_2 y)^{\beta} Q(y) I_{[y\geqq e^e]}(y)$, $y\geqq 0$. Then by hypothesis $E\{Q_{\beta}(|Y|)\}<\infty$ implying

$$\sum_1^{\infty} P\{2Q_{\beta}(|Y|)>n\}<\infty. \tag{89}$$

Now via (19), (18), slow variation of G, and $Q_{\beta}(y)=y^2/(G(y)(\log_2 y)^{1-\beta})$

$$Q_{\beta}((n(\log_2 n)^{1-\beta} G(q_n))^{\frac{1}{2}}) \sim Q_{\beta}(q_n/(\log_2 q_n)^{\frac{1}{2}\beta}) \geqq n$$

and it follows for all large n that

$$\{|Y|>(n(\log_2 n)^{1-\beta} G(q_n))^{\frac{1}{2}}\} \subseteq \{2Q_{\beta}(|Y|)>n\}. \tag{90}$$

Next, noting that $Kc_n^2 \geqq n(\log_2 n)^{1-\beta} G(q_n)$, it follows from (90) and (89) that (34) holds with $\lambda=K^{\frac{1}{2}}$. □

Remark 1. Since $H(y)\sim G(y)$, G may be replaced by H in (88) (where e^e is replaced by any constant $a\geqq e^e$ such that $H(a)>0$).

Remark 2. When $\sigma_n\equiv 1$, β may be taken as 0 and Corollary 1 is essentially the theorem of Feller [2].

Remark 3. Convergence of the integral of (88) is not necessary for convergence of the series of (34) as will be seen via an example.

Remark 4. As already noted when $\sigma^2\equiv EY^2<\infty$, slow variation of G is automatic and $s_n^2 q_n^2/n \sim \sigma^2 s_n^2 \log_2 s_n^2$. Since $EY^2<\infty$ ensures (88), $\{\sigma_n Y_n\}$ obeys the classical LIL (2). This is a special case of Theorem 3 of Teicher [13].

Corollary 2. *Under the hypotheses of the theorem (or Corollary 1)*

$$\liminf_{n\to\infty} \frac{\sum_1^n \sigma_j Y_j}{(2/n)^{\frac{1}{2}} s_n q_n} = -1, \text{ a.c.} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\left|\sum_1^n \sigma_j Y_j\right|}{(2/n)^{\frac{1}{2}} s_n q_n} = 1, \quad \text{a.c.}$$

Corollary 3. *Under the hypotheses of the theorem (or Corollary 1)*

$$\frac{\sum_1^n \sigma_j Y_j}{s_n q_n/n^{\frac{1}{2}}} \xrightarrow{P} 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{\left|\sum_1^n \sigma_j Y_j\right|}{s_n q_n/n^{\frac{1}{2}}} = 0, \quad \text{a.c.}$$

Proof. Since (25) is tantamount to $\sigma_n^2 = o(s_n^2)$, the first statement follows from the GLIL of Theorem 1 (or Corollary 1) and Theorem 2 of [9]. The second statement is then a consequence of the fact that convergence in probability implies some subsequence converges almost certainly. □

Remark. The first half of Corollary 3 also follows readily from Theorem 1 without reference to [9]. For by (64) and (18),

$$nD_n/(s_n^2 q_n^2) \leq nG(q_n)/q_n^2 = (\log_2 q_n)^{-1} = o(1),$$

so by Čebyšev's inequality

$$\sum_1^n \sigma_j X_j/(s_n q_n/n^{\frac{1}{2}}) \xrightarrow{P} 0.$$

The Corollary is evident in view of (36) and (37).

Theorem 2. *Under the hypotheses of Theorem 1, if (34) is replaced by*

$$\sum_1^\infty P\{|Y| > \lambda q_n/\gamma_n^{\frac{1}{2}}\} = \infty, \quad \text{all } \lambda > 0 \tag{91}$$

or by

$$\int_{[|y| \geq e^e]} y^2/(G(|y|) \log_2 |y|)\, dF(y) = \infty \tag{92}$$

then

$$\limsup_{n\to\infty} \frac{\sum_1^n \sigma_j Y_j}{s_n q_n/n^{\frac{1}{2}}} = \infty, \quad \text{a.c.} \tag{93}$$

Proof. Since (91) entails $\limsup_{n\to\infty} n^{\frac{1}{2}} |\sigma_n Y_n|/(s_n q_n) = \infty$, a.c., (93) follows via $|\sigma_n Y_n| \leq \left|\sum_1^n \sigma_j Y_j\right| + \left|\sum_1^{n-1} \sigma_j Y_j\right|$, the monotonicity of $s_n q_n/n^{\frac{1}{2}}$, and symmetry.

Next, (92) is tantamount to $E\{Q(|Y|)\} = \infty$ and so

$$\sum_1^\infty P\{|Y| > q_n\} = \sum_1^\infty P\{Q(|Y|) > n\} = \infty.$$

However, $q_n/n^{\frac{1}{2}} \uparrow$, whence for all $\lambda > 0$, recalling that $\gamma_n \geq 1$, (91) holds via Lemma 10. □

4. An Interesting Example

Let $0 \leqq \alpha(1) < 1$, $\alpha(3) - 1 \geqq \alpha(2) \geqq 0$,

$$\begin{aligned}\sigma_n^2 &= (\log_2 n)^{\alpha(1)} (\log_3 n)^{\alpha(2)} n^{(\log_2 n)^{\alpha(1)} (\log_3 n)^{\alpha(2)} - 1} \\ &= n^{-1} (\log_2 n)^{\alpha(1)} (\log_3 n)^{\alpha(2)} \exp\{(\log n)(\log_2 n)^{\alpha(1)} (\log_3 n)^{\alpha(2)}\}\end{aligned}$$

for $n \geqq N \equiv [\exp\{e^e\}] + 1$ and let $\sigma_n^2 = 1$ for $1 \leqq n < N$. Let Y have density

$$\begin{aligned}f(y) = c(2|y|^3 (\log|y|)(\log_2|y|)^{\alpha(1)} (\log_3|y|)^{\alpha(3)})^{-1} \\ \cdot \exp\{(\log_2|y|)^{1-\alpha(1)} / (\log_3|y|)^{\alpha(3)}\} I_{[|y| \geqq N]}(y)\end{aligned} \tag{94}$$

where c is some constant. It can easily be verified that

$$s_n^2 \sim \exp\{(\log n)(\log_2 n)^{\alpha(1)} (\log_3 n)^{\alpha(2)}\}. \tag{95}$$

Hence,

$$B \equiv \{\beta \in [0, 1): \gamma_n = O((\log_2 n)^\beta)\} = \begin{cases} [\alpha(1), 1), & \text{if } \alpha(2) = 0 \\ (\alpha(1), 1), & \text{if } \alpha(2) > 0. \end{cases}$$

It is also easy to verify that

$$H(y) \sim c(1 - \alpha(1))^{-1} \exp\{(\log_2 y)^{1-\alpha(1)} / (\log_3 y)^{\alpha(3)}\} \tag{96}$$

and that H is slowly varying. Now $H(q_n) \sim H(n^{\frac{1}{2}})$ by (20) and (96) and so recalling (18), (12), (19), and (95)

$$\begin{aligned}2 s_n^2 q_n^2 / n \sim 2c(1 - \alpha(1))^{-1} \exp\{(\log n)(\log_2 n)^{\alpha(1)} (\log_3 n)^{\alpha(2)} \\ + (\log_2 n^{\frac{1}{2}})^{1-\alpha(1)} (\log_3 n^{\frac{1}{2}})^{-\alpha(3)}\} \log_2 n \\ \equiv a_n^2.\end{aligned} \tag{97}$$

Now for any β in B, it follows from (12), (96), and (94) that convergence of the integral of (88) is equivalent to

$$\int_N^\infty (y (\log y)(\log_2 y)^{1-(\beta-\alpha(1))} (\log_3 y)^{\alpha(3)})^{-1} \, dy < \infty. \tag{98}$$

Hence if $\alpha(2) > 0$, then $1 - (\beta - \alpha(1)) < 1$ for β in B and so the integral of (88) diverges.

Nevertheless, it will now be shown that $\{\sigma_n Y_n\}$ obeys the GLIL of Theorem 1 if and only if $\alpha(3) - 1 > \alpha(2)$ regardless of whether $\alpha(2) > 0$ or $\alpha(2) = 0$. Now convergence of the series of (34) for some $\lambda > 0$ is equivalent to that of the series of (91) for some $\lambda > 0$ which, in turn, is equivalent to

$$\int_N^\infty (y (\log y)(\log_2 y)(\log_3 y)^{\alpha(3)-\alpha(2)})^{-1} \, dy < \infty. \tag{99}$$

These equivalences are detailed in [8], but need not be reproduced here. Then by Theorems 1 and 2

$$\limsup_{n \to \infty} a_n^{-1} \sum_1^n \sigma_j Y_j = \begin{cases} 1, \text{ a.c.} & \text{if } \alpha(3) - 1 > \alpha(2) \\ \infty, \text{ a.c.} & \text{if } \alpha(3) - 1 = \alpha(2) \end{cases}$$

where $\{a_n\}$ is as in (97). Hence if $\alpha(3)-1>\alpha(2)>0$, then Theorem 1 yields a GLIL while nothing can be concluded from Corollary 1.

Finally, if $\alpha(2)=0$, then by choosing $\beta=\alpha(1)$ in B it follows from (98) and (99) that the integral of (88) and the series of (34) converge or diverge together according as $\alpha(3)>1$ or $\alpha(3)=1$.

5. Final Remarks

Remark 1. A version of Theorem 1 obtains without the proviso of symmetry provided $\sum_1^n \sigma_j Y_j$ is centered at a median. Specifically, if $\{Y, Y_n, \sigma_n\}$ satisfy the hypotheses of Theorem 1 (except for possibly the symmetry condition) then

$$0<\limsup_{n\to\infty} \frac{\left|\sum_1^n \sigma_j Y_j - \operatorname{med} \sum_1^n \sigma_j Y_j\right|}{s_n q_n / n^{\frac{1}{2}}} < \infty, \quad \text{a.c.} \tag{100}$$

Proof. Only the case $EY^2=\infty$ need be considered in view of Theorem 3 of Teicher [13] and the elementary inequality (see, for example, [7, p. 244]) $|EX-\operatorname{med} X| \leqq (2\operatorname{Var} X)^{\frac{1}{2}}$. Let m be a median of Y and let $\{Y^*=Y-Y', Y_n^*=Y_n-Y_n'\}$ be a symmetrized version of $\{Y, Y_n\}$. Define for $y>0$

$$\bar{G}(y)=2\int_0^y tP\{|Y-m|>t\}\,dt, \qquad \bar{H}(y)=E\{(Y-m)^2 I_{[|Y-m|\leqq y]}\},$$

$$G^*(y)=2\int_0^y tP\{|Y^*|>t\}\,dt.$$

Since G is slowly varying, Lemma 3 and the observation just prior to it ensure that $E|Y|<\infty$ and since

$$H(\tfrac{1}{2}y) \leqq E\{Y^2 I_{[|Y-m|\leqq y]}\} \leqq H(y) \quad \text{for all large } y,$$

it follows from Lemma 3 that

$$\bar{H}(y)=E\{Y^2 I_{[|Y-m|\leqq y]}\}+O(1) \sim H(y)$$

and so $\bar{G}$ is slowly varying and

$$\bar{G}(y) \sim G(y). \tag{101}$$

Then, the Weak Symmetrization Inequality [7, p. 245]

$$P\{|Y-m|>y\} \leqq 2P\{|Y^*|>y\} \leqq 4P\{|Y-c|>\tfrac{1}{2}y\}, \quad \text{all } c \tag{102}$$

yields (take $c=m$)

$$\bar{G}(y) \leqq 2G^*(y) \leqq 16\bar{G}(y), \quad y \geqq 0 \tag{103}$$

and

$$\frac{y^2 P\{|Y^*|>y\}}{G^*(y)} \leqq \frac{2y^2 P\{|Y^*|>y\}}{\bar{G}(y)} \leqq \frac{16(\frac{1}{2}y)^2 P\{|Y-m|>\frac{1}{2}y\}}{\bar{G}(\frac{1}{2}y)} \to 0$$

by slow variation of $\bar{G}$ and Lemma 3. Again by Lemma 3, G^* is slowly varying.

Next, let $Q^*(y)=y^2/(G^*(y)\log_2 y)$ and let q_y^* denote the inverse function of Q^*. Now by (101) and (103) for some finite nonzero constants A and B and all large y, $Q(y)\leqq AQ^*(y)\leqq BQ(y)$ implying for all large n that $q_{n/B}\leqq q_{n/A}^*\leqq q_n$. Then, recalling (21),

$$0<\liminf_{n\to\infty} q_n^*/q_n \leqq \limsup_{n\to\infty} q_n^*/q_n<\infty. \tag{104}$$

It follows from (101), (103), (104), and slow variation of G^* that $G(q_n) =O(G^*(q_n^*))$, whence (34) and (102) (with $c=0$) ensure that for some $\lambda^*>0$

$$\sum_1^\infty P\{|Y^*|>\lambda^*(n \inf_{i\geqq n} \gamma_i^{-1} G^*(q_i^*)\log_2 i)^{\frac{1}{2}}\}<\infty.$$

Applying Theorem 1 and Corollary 1 to $\{Y_n^*\}$ yields

$$\limsup_{n\to\infty} \frac{\sum_1^n \sigma_j Y_j^*}{s_n q_n^*/n^{\frac{1}{2}}} = -\liminf_{n\to\infty} \frac{\sum_1^n \sigma_j Y_j^*}{s_n q_n^*/n^{\frac{1}{2}}} = 2^{\frac{1}{2}}, \quad \text{a.c.}$$

and then (100) follows from (104) and the argument of Kesten [5, Lemma 1]. (The only change needed in Kesten's proof is that the Kolmogorov $0-1$ Law is used rather than the Hewitt-Savage $0-1$ Law.) □

Remark 2. Some discussion about the significance of Theorem 2 vis-à-vis Theorem 1 is in order. If

$$(\gamma_n e_n)^{-1} G(q_n)\log_2 n=O(1) \tag{105}$$

(*a fortiori* if $\gamma_n^{-1} G(q_n)\log_2 n\uparrow$ or $\gamma_n=O(1)$) then, recalling (18) and (19), for some $M<\infty$ and all large n

$$\tfrac{1}{2}c_n^2 \leqq q_n^2/\gamma_n \leqq M c_n^2$$

implying via Lemma 10 that the series in (34) and (91) converge (for some and hence all $\lambda>0$) or diverge (for some and hence all $\lambda>0$) together. Therefore, under the assumptions of Theorem 1 *if* (105) *holds*, then (34) is both *necessary* and sufficient for (35). This leads to a question: Is (34) (or convergence of the series in (91)) *always* necessary and sufficient for (35)?

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