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On Convergence of Vector-Valued Asymptotic Martingales

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1. Introduction

Let (X_n) be a sequence of random variables taking values in a Banach space E; T the collection of bounded stopping times. Call (X_n) an *asymptotic martingale* if $(\int X_\tau)_{\tau\in T}$ converges. It was shown in [1] that an L_1-bounded asymptotic *real-valued* martingale converges almost everywhere. A vector version of this theorem is here proved, the convergence holding a.e. in the *weak* topology of E. A simple example shows that the weak topology cannot be replaced by the strong topology. Also a new proof of Chatterji's vector-valued martingale convergence theorem [6] is given.

2. Asymptotic Martingales

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, E a Banach space with norm $|\ |$. Our notation and terminology are close to that of [10]. A *random variable* is a strongly (or Bochner) measurable function with values in E. The *integral* of such a function is defined in the Bochner sense. We consider a sequence (X_n) of random variables and an increasing sequence $(\mathfrak{F}_n)$ of σ-fields such that each X_n is measurable with respect to $\mathfrak{F}_n$. We assume, without loss of generality, that $\mathfrak{F}$ is generated by $\bigcup \mathfrak{F}_n$. A *stopping time* τ is a random variable taking values in $\{1, 2, \ldots, \infty\}$ and such that for each n, $\{\tau = n\} \in \mathfrak{F}_n$. The set of all bounded stopping times is denoted by T, the set of bounded stopping times larger than a given stopping time σ is $T_{>\sigma}$. (X_n) is a *martingale* if $E^{\mathfrak{F}_n} X_{n+1} = X_n$ for each n. (X_n) is a martingale if and only if the expression $\int X_\tau$ does not depend upon the choice of $\tau \in T$; it seems therefore natural to call (X_n) an *asymptotic martingale* if

$$\lim_T \int X_\tau = z \text{ exists.} \tag{1}$$

(1) means that there is a vector $z\in E$ with the following property: For each $\varepsilon>0$ there exists a $\sigma\in T$ such that $|\int X_\tau - z| < \varepsilon$ if $\tau \in T_{>\sigma}$. A sequence (X_n) is called

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L_1-*bounded* if

$$\sup_n \int |X_n| < \infty. \tag{2}$$

The following theorem was proved in [1]; see also [5] for a related result.

Theorem 1 (Austin-Edgar-Tulcea). *If (X_n) is a real-valued L_1-bounded asymptotic martingale, then X_n converges a.e.*

A Banach space E is said to have the *Radon-Nikodým* property if every E-valued measure μ on $\mathfrak{F}$, of finite variation and vanishing on P-null sets, can be represented as an integral of a random variable X in the sense that

$$\mu(A) = \int_A X\,dP \qquad A \in \mathfrak{F}. \tag{3}$$

Here we prove the following.

Theorem 2. *Assume that E is a Banach space with a separable dual and the Radon-Nikodým property (for instance, a separable and reflexive Banach space). If (X_n) is an E-valued asymptotic martingale such that*

$$\sup_T \int |X_\tau| < \infty, \tag{4}$$

then X_n converges a.e. in the weak topology of E.

Proof. We at first prove a 'maximal' lemma.

Lemma 1. *Let (X_n) be a sequence of random variables satisfying* (4). *Then for each positive number a*

$$P\{\sup_n |X_n| \geqq a\} \leqq \frac{1}{a} \sup_T \int |X_\tau|. \tag{5}$$

Proof. Let N be a fixed positive integer and define $\sigma \in T$ as follows: If $n \leqq N$, $X_1, \ldots, X_{n-1} < a$, $X_n \geqq a$ $(X_0 = 0)$, let $\sigma = n$. If $\sup_{1 \leqq n \leqq N} |X_n| < a$, set $\sigma = N$. Let $A_N = \{\sup_{1 \leqq n \leqq N} |X_n| > a\}$. Now $\sup_T \int |X_\tau| \geqq \int |X_\sigma| \geqq \int_{A_N} |X_\sigma| \geqq a P(A_N)$. (5) follows on letting $N \uparrow \infty$.

Lemma 2. *Let k be a fixed positive integer, $A \in \mathfrak{F}_k$. If* (1) *holds then $(\int_A X_\tau)_{\tau \in T}$ converges (in fact, uniformly in $A \in \mathfrak{F}_k$).*

Proof. Given an $\varepsilon > 0$ find an integer $N \geqq k$ such that if $\sigma_1, \tau_1 \in T_{\geqq N}$ then $|\int X_{\sigma_1} - \int X_{\tau_1}| < \varepsilon$. Now given $\sigma, \tau \in T_{\geqq N}$, define σ_1, τ_1 as follows. Let N_1 be an integer $> \max(\sigma, \tau)$ and set $\sigma_1 = \sigma$ on A, $\tau_1 = \tau$ on A, $\sigma_1 = \tau_1 = N_1$ on A^c. One has $\{\sigma_1 < N\} = \{\sigma_1 > N_1\} = \emptyset$; $\{\sigma_1 = n\} = \{\sigma = n\} \cap A \in \mathfrak{F}_n$ for $n \in [N, N_1)$; $\{\sigma_1 = N_1\} = A^c \in \mathfrak{F}_k \subset \mathfrak{F}_{N_1}$. Thus σ_1 is a stopping time; similarly τ_1 is a stopping time. Now

$$|\int_A X_\sigma - \int_A X_\tau| = |\int X_{\sigma_1} - \int X_{\tau_1}| < \varepsilon.$$

This proves the lemma.

We now reduce the problem of convergence of an asymptotic martingale satisfying (4) to that of convergence of an asymptotic martingale such that

$\sup_n |X_n| \in L_1$. A similar device was used in the case of real martingales in [8] and [9]. Let a be a positive constant. Define a stopping time σ as follows: $\sigma = \infty$ if $|X_n| < a$ for all n; otherwise σ is the first n such that $|X_n| \geqq a$. Let $Y = \sup_n |X_{n \wedge \sigma}|$, we assert that $\int Y < \infty$. Indeed, $Y < a$ on $\{\sigma = \infty\}$. On $A \overset{\text{def}}{=} \{\sigma < \infty\}$, $|X_{n \wedge \sigma}| \to |X_\sigma|$, hence

$$\int_A X_\sigma \leqq \liminf_n \int_A |X_{n \wedge \sigma}| \leqq \liminf_n \int |X_{n \wedge \sigma}| \leqq \sup_T \int |X_\tau| = M < \infty \tag{6}$$

by Fatou's lemma and the observation that the infimum of two stopping times is a stopping time. Clearly $|X_{n \wedge \sigma}| \leqq |X_\sigma|$ on A, hence $\int Y \leqq a + M$.

Since

$$\int (X_{\tau \wedge \sigma} - X_{\tau' \wedge \sigma}) = \int (X_{(\tau \wedge \sigma) \vee (\tau \wedge \tau')} - X_{(\tau' \wedge \sigma) \vee (\tau \wedge \tau')}),$$

$(X_{n \wedge \sigma})$ is an asymptotic martingale.

Now, by the maximal Lemma 1, $(X_{n \wedge \sigma})$ coincides with (X_n) except on a set the measure of which is small if a is large. Thus we may and do assume without loss of generality that (X_n) *itself* is such that $\sup_n |X_n| = Y \in L_1$.

Define a generalized sequence of E-valued measures μ_τ, $\tau \in T$, by

$$\mu_\tau(A) = \int_A X_\tau \, dP \qquad A \in \mathfrak{F}. \tag{7}$$

By Lemma 2, $\lim_T \mu_\tau(A) = \mu(A)$ exists for each $A \in \bigcup \mathfrak{F}_n$. If $A \in \mathfrak{F}$ then for each $\varepsilon > 0$ there exists a set $A' \in \bigcup \mathfrak{F}_n$ such that $P(A \Delta A') < \varepsilon$. Since $|X_\tau| \leqq Y$ for all τ, $|\int_A X_\tau - \int_{A'} X_\tau| \leqq \varepsilon \int Y$. This implies that $\mu(A) = \lim_T \mu_\tau(A)$ exists for all $A \in \mathfrak{F}$. Clearly μ is a finitely additive measure of finite (bounded by $\int Y$) variation. We now state a well-known result:

Theorem of Vitali-Hahn-Saks. *Let μ_n be a sequence of E-valued finite measures on $\mathfrak{F}$ such that $\lim_n \mu_n(A) = \mu(A)$ exists for each $A \in \mathfrak{F}$. Then μ is a measure.*

A proof of this theorem is given in [7], p. 321. An elementary proof of a stronger result was published in [4].

Applying the Vitali-Hahn-Saks theorem and the Radon-Nikodým property of E, we obtain that there exists a random variable X_∞ such that

$$\lim_T \int_A X_\tau \, dP = \int_A X_\infty \, dP \qquad A \in \mathfrak{F}. \tag{8}$$

Let E' be the dual of the space E, and let $(x_i', i = 1, 2, \ldots)$ be a sequence dense in the unit ball of E'. Fix i; applying x_i' to (8) we obtain

$$\lim_T \int_A x_i'(X_\tau) = \int_A x_i'(X_\infty) \qquad A \in F. \tag{9}$$

Therefore by Theorem 1 $\lim_n x_i'(X_n)$ exists a.e., and, because of (9), this limit must be a.e. $x_i'(X_\infty)$, say $\lim_n x_i'(X_n) = x_i'(X_\infty)$ except on a null set Ω_i. The argument is valid for all i; hence X_n converges weakly to X_∞ outside of the null set $\bigcup \Omega_i$. Indeed, $\sup_n |X_n| \in L_1$ implies that $X_n(\omega)$ is bounded in n for each ω outside of a set of arbitrarily small measure. This proves the theorem.

We now give an example showing that strong convergence in Theorem 2 need not hold. Let E be an l_p space $(1<p<\infty)$, with the usual basis (e_n): each vector e_n is a sequence of real numbers with all terms equal 0 except for the n-th term which is 1. Let X_n be independent E-valued random variables defined as follows: $P(X_1=e_1)=1$; $P(X_2=e_i)=\frac{1}{2}$, $i=2,3$; $P(X_3=e_i)=\frac{1}{4}$, $i=4,5,6,7$; etc. Clearly $|X_n|=1$ for each n, and (X_n) is not Cauchy hence diverges at each point of Ω. But $|\int X_n|=2^{(n-1)(1-p)/p}\to 0$, and also $(\int X_\tau)_{\tau\in T}\to 0$, since $\tau\geqq n$ implies

$$|\int X_\tau|\leqq|\int X_n|. \tag{10}$$

To prove (10), proceed by induction on the number $N(\tau)$ of values taken on by τ. Passing from $N(\tau)=m$ to $N(\tau)=m+1$, we replace in the expression $\int X_\tau$ a vector u by two vectors, say v and w, such that $|u|=|v|=|w|=1$ and the mass of u is the sum of masses of v and w. This clearly diminishes $|\int X_\tau|$.

Another example, for which we are indebted to W.J. Davis, shows that in Theorem 2 the assumption (4) cannot be replaced by (2). Let $(\Omega,\mathfrak{F},P)$ be the interval $[0,1)$ with Borel sets and Lebesgue measure. E is the Hilbert space l_2. Let $A_1^1=[0,\frac{1}{2})$, $A_1^2=[\frac{1}{2},1)$, $A_2^1=[0,\frac{1}{4})$, ...; in general for each positive integer n, $A_n^k=\left[\frac{k-1}{2^n},\frac{k}{2^n}\right)$, $k=1,2,\ldots,2^n$. For each positive n let

$$Y_n^k=\sum_{i=1}^{2^n}\alpha_i 1_{A_n^i} e_n^i, \qquad k=1,2,\ldots,2^n$$

where $\alpha_i=1$ for $i\neq k$, $\alpha_k=n$, and the e_n^i are unit vectors in l_2 such that $e_n^i\perp e_{n'}^{i'}$ unless $i=i'$ and $n=n'$. Let (X_n) be a sequence of random variables considering of the Y_n^k's ordered so that Y_n^k is before $Y_{n'}^{k'}$ if $n<n'$, or if $n=n'$ and $k<k'$. Then $\sup_n\int|X_n|<\infty$, and $X_n(\omega)$ is at each point ω unbounded, hence it fails to converge weakly. But $\lim_T\int X_\tau=0$. To see this, write

$$X_\tau=\sum\alpha_i e_{n_i} 1_{A_{m_i}}$$

where the e_{n_i}'s are mutually orthogonal unit vectors in E, and if $P(A_{m_i})=2^{-k}$, then α_i is either 1 or k. Let $\pi_i=P(A_{m_i})$, then $\sum\pi_i=1$. Now

$$\begin{aligned}|\int X_\tau|^2&=|\sum\pi_i\alpha_i e_{n_i}|^2=\sum\alpha_i^2\pi_i^2\leqq\sum(\log_2\pi_i)^2\pi_i^2\\&\leqq\max_i\frac{\pi_i}{(\log_2\pi_i)^2}\sum_j\pi_j.\end{aligned}$$

The last expression converges to zero if $k\to\infty$ and $\tau>2^k$, since then $\max_i\pi_i\leqq 2^{-k}$.

3. Martingales

Chatterji [6] proved the following elegant result.

Theorem 3 (Chatterji). *If a Banach space E has the Radon-Nikodým property, then (and, as it is easy to see, only then) every E-valued L_1-bounded martingale converges a.e.*

The proof in [6] is rather difficult and, as observed in [10], "est plus une étude de decomposition de measures qu'une étude de martingales." We sketch here a proof of Theorem 3 along the lines of the proof in the previous section.

If (X_n) is a martingale, $(|X_n|)$ is a submartingale since $E^{\mathfrak{F}_n}|X_{n+1}| \geqq |E^{\mathfrak{F}_n} X_{n+1}| = |X_n|$. First, Lemma 1 is replaced by the maximal lemma for submartingales, due to Doob (see e.g. [10], p. 24); it suffices to assume (2) instead of (4). Lemma 2 is now a triviality, since if $A \in \mathfrak{F}_k$ then $\int_A X_n = \int_A X_k$ for $n \geqq k$. Since $|X_n|$ is a positive submartingale, $\int |X_{n \wedge \sigma}| \leqq \int |X_n|$ (see e.g. [10], p. 31); hence the last two inequalities in (6) may be replaced by: $\liminf_n \int |X_{n \wedge \sigma}| \leqq \sup_n \int |X_n| < \infty$. The proof that if (X_n) is a martingale then so is $(X_{n \wedge \sigma})$ is the same as in the real case (see [10], p. 73). Finally, for martingales the relation (8) (in which it suffices to take $\tau = n$) is more informative than for asymptotic martingales, since it implies that

$$X_n = E^{\mathfrak{F}_n} X_\infty \qquad n = 1, 2, \dots. \tag{11}$$

To see this, observe that both sides of (11) are measurable with respect to $\mathfrak{F}_n$, and both sides yield the same integral over a set A in $\mathfrak{F}_n$. The martingale convergence theorem is now reduced to the theorem about convergence of martingales of the form (11), for which simple proofs exist (see [6, 10], or [9]). □

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