Z. Wahrscheinlichkeitstheorie verw. Gebiete 33, 41 – 48 (1975)

Weak Convergence of a Sequence of Markov Chains

Alan F. Karr

Department of Mathematical Sciences, The Johns Hopkins University
Baltimore, Md. 21218, USA

1. Introduction

In this paper we study weak convergence of a sequence of Markov chains as a consequence of convergence of initial distributions and transition kernels. When the initial distributions converge weakly and transition kernels converge in a particular sense which is weaker than weak convergence in the dual of the set of continuous functions on the state space, then the associated sequence of Markov chains converges weakly. We establish this first for a state space which is a complete separable metric space and then for a separable metrizable Radon space. Under somewhat stronger conditions on the transition kernels, convergence of invariant measures also occurs. As an application we treat the weak convergence of sequences of Markov renewal processes and semi-Markov processes engendered by convergence of initial distributions and semi-Markov kernels and weak convergence of sequences of regular Markov processes resulting from convergence of initial distributions, jump functions, and transition kernels.

2. Continuity of Markov Chains

Our notation and terminology are those of [2]. Let E be a complete separable metric space with Borel σ-algebra $\mathscr{E}$. On the infinite product space $(\Omega, \mathscr{M})=(E, \mathscr{E})^{\mathbf{N}}$ we define the product topology (so that Ω is a complete separable metric space) and the coordinate stochastic process $(X_n)_{n\in\mathbf{N}}$. If we are given a probability measure α on $\mathscr{E}$ and a Markov kernel K on $(E, \mathscr{E})$ then there exists a probability measure P^α on $(\Omega, \mathscr{M})$ such that (X_n) is a Markov chain over the probability space $(\Omega, \mathscr{M}, P^\alpha)$ with state space E, initial distribution α, and transition kernel K; cf. [6, p. 162] for details. Given probability measures $\alpha_1, \alpha_2, \ldots$ on $\mathscr{E}$ and Markov kernels $K_1, K_2, \ldots$ we may form the corresponding measures $P_1^{\alpha_1}, P_2^{\alpha_2}, \ldots$ on $(\Omega, \mathscr{M})$; to simplify notation we write P_n for $P_n^{\alpha_n}$ and P for P^α.

We denote by $C(E)$ the Banach space of bounded continuous functions on E and by $UC(E)$ the set of functions in C which are uniformly continuous. If $g\in b\mathscr{E}$ (g is bounded and $\mathscr{E}$-measurable) and K is a Markov kernel then by Kg we mean the function on E defined by

$$Kg(x)=\int_E K(x,dy)\,g(y).$$

If μ is a measure on $\mathscr{E}$, μK is the measure defined by

$$\mu K(A)=\int_E \mu(dx)\,K(x,A).$$

Here is our main result.

(1) **Theorem.** *Let $K, K_1, K_2, \ldots$ be Markov kernels on $(E,\mathscr{E})$ and let $\alpha, \alpha_1, \alpha_2, \ldots$ be probability measures on $\mathscr{E}$. If*

a) $\alpha_n \Rightarrow \alpha$;

b) *$g\in UC$ implies $Kg\in C$;*

c) *for every $g\in UC$, $K_n g\to Kg$ uniformly on each compact subset of E,*

then $P_n \Rightarrow P$ on $(\Omega, \mathscr{M})$.

We state part of the proof as a Lemma for purposes of future reference. This part reduces weak convergence to convergence of finite-dimensional distributions, and is essentially contained in the proof of Prohorov's Theorem [1]. We omit the proof.

(2) **Lemma.** *If $\mu, \mu_1, \mu_2, \ldots$ are probability measures on $(\Omega,\mathscr{M})$ then $\mu_n \Rightarrow \mu$ if and only if $\mu_n \pi_k^{-1} \Rightarrow \mu\pi_k^{-1}$ on E^{k+1} for each k, where $\pi_k\colon \Omega\to E^{k+1}$ is the projection $\omega\to(X_0(\omega),\ldots,X_k(\omega))$.*

Proof of Theorem (1). It suffices by Lemma (2) to show that $P_n\pi_k^{-1} \Rightarrow P\pi_k^{-1}$ for each k, which we prove by induction.

By (1.a) we have $P_n\pi_0^{-1}=\alpha_n \Rightarrow \alpha = P\pi_0^{-1}$. Make the induction hypothesis that $P_n\pi_{k-1}^{-1} \Rightarrow P\pi_{k-1}^{-1}$, let F_1 be a closed subset of E^k, and let g be a function in UC with $\|g\|_\infty>0$. The indicator function of a set A is denoted by 1_A. By the Markov property

$$\begin{aligned}
&\left|\int P_n\pi_k^{-1}(dy)\,1_{F_1}(y_0,\ldots,y_{k-1})\,g(y_k)-\int P\pi_k^{-1}(dy)\,1_{F_1}(y_0,\ldots,y_{k-1})\,g(y_k)\right|\\
&=\left|\int_{F_1} P_n\pi_{k-1}^{-1}(dz)\int_E K_n(z_{k-1},dx)\,g(x)-\int_{F_1} P\pi_{k-1}^{-1}(dz)\int_E K(z_{k-1},dx)\,g(x)\right|\\
&\leqq \int_{E^k} |K_n g(z_{k-1})-Kg(z_{k-1})|\,P_n\pi_{k-1}^{-1}(dz)\\
&\quad+\left|\int_{F_1} Kg(z_{k-1})\,P_n\pi_{k-1}^{-1}(dz)-\int_{F_1} Kg(z_{k-1})\,P\pi_{k-1}^{-1}(dz)\right|.
\end{aligned}$$

Since by (1.b), $Kg\in C$, the induction hypothesis implies that the second term above goes to zero as n approaches infinity.

Moreover, the induction hypothesis also implies that $(P_n\pi_{k-1}^{-1})_{n\geqq 1}$ is tight. Hence for any prescribed $\varepsilon>0$ there is a compact set A_ε such that

$$P_n\pi_{k-1}^{-1}(A_\varepsilon)>1-\varepsilon/(4\,\|g\|_\infty)$$

for all n. The projection A'_ε of A_ε onto the last factor space of E^k, being the continuous image of a compact set, is compact and therefore

$$\int_{E^k} |K_n g(z_{k-1}) - K g(z_{k-1})| \, d(P_n \pi_{k-1})$$

$$= \Big(\int_{A_\varepsilon} + \int_{A_\varepsilon^c} \Big) |K_n g(z_{k-1}) - K g(z_{k-1})| \, d(P_n \pi_{k-1})$$

$$\leqq \sup_{x \in A'_\varepsilon} |K_n g(x) - K g(x)| + 2 \, \|g\|_\infty \sup_n P_n \pi_{k-1}^{-1}(A_\varepsilon^c)$$

$$\leqq \sup_{x \in A'_\varepsilon} |K_n g(x) - K g(x)| + \varepsilon/2.$$

By (1.c) this last quantity is less than ε for all sufficiently large n, so we have shown that

$$\text{(3)} \quad \lim_{n \to \infty} \int 1_{F_1}(y_0, \ldots, y_{k-1}) \, g(y_k) \, P_n \pi_k^{-1}(dy)$$
$$= \int 1_{F_1}(y_0, \ldots, y_{k-1}) \, g(y_k) \, P \pi_k^{-1}(dy).$$

If F_2 is a closed subset of E there exists a sequence (g_ℓ) in UC such that $g_\ell \downarrow 1_{F_2}$ pointwise as $\ell \to \infty$. For each n and ℓ

$$P_n \pi_k^{-1}(F_1 \times F_2) \leqq \int P_n \pi_k^{-1}(dy) \, 1_{F_1}(y_0, \ldots, y_{k-1}) \, g_\ell(y_k);$$

fixing ℓ and letting $n \to \infty$, we infer from (3) that

$$\limsup_{n \to \infty} P_n \pi_k^{-1}(F_1 \times F_2) \leqq \int P \pi_k^{-1}(dy) \, 1_{F_1}(y_0, \ldots, y_{k-1}) \, g_\ell(y_k).$$

Hence by the Bounded Convergence Theorem

$$\limsup_{n \to \infty} P_n \pi_k^{-1}(F_1 \times F_2) \leqq P \pi_k^{-1}(F_1 \times F_2),$$

from which $P_n \pi_k^{-1} \Rightarrow P \pi_k^{-1}$ follows; cf. [1, p. 11 and p. 20]. □

Next we give an equivalent set of sufficient conditions for the conclusion of Theorem (1).

(4) **Theorem.** *The conditions* (1.a, b, c) *are equivalent to*

a) $\alpha_n \Rightarrow \alpha$;

b) *If* $x_n \to x$ *in* E *and* $g \in UC$, *then* $K_n g(x_n) \to K g(x)$.

Proof. It suffices to show the equivalence of (1.b, c) and (4.b). Assume (1.b, c) hold. If g is bounded and uniformly continuous, then since $\{x, x_1, x_2, \ldots\}$ is a compact subset of E we have by (1.c) that

$$\lim_{n \to \infty} |K_n g(x_n) - K g(x_n)| = 0;$$

this together with (1.b) implies that $K_n g(x_n) \to K g(x)$. Conversely, if (4.b) holds then (1.b) clearly holds, while if (1.c) fails then there is a compact set A such that

$$\limsup_n \sup_{x \in A} |K_n g(x) - K g(x)| = \delta > 0;$$

we may then construct a convergent sequence $x_n \to x$ in A such that

$$\lim_n |K_n g(x_n) - K g(x)| = \delta,$$

which contradicts (4.b). □

Remark. Condition (4.b) is equivalent to the assertion that if $x_n \to x$ in E, then $K_n(x_n, \cdot) \Rightarrow K(x, \cdot)$. Condition (1.b) is a version of the Feller property: if $x_n \to x$, then $K(x_n, \cdot) \Rightarrow K(x, \cdot)$.

The conclusion of Theorem (1) remains valid for somewhat more general state spaces. By a *Radon space* we mean a Hausdorff space on which every Borel probability measure is tight.

(5) **Theorem.** *Theorem* (1) *is true if E is a separable metrizable Radon space.*

Proof. Two parts of the argument used to prove (1) need to be checked in the more general case; the restriction to finite-dimensional distributions proved in Lemma (2) and the fact that weak convergence implies tightness (i.e., Prohorov's Theorem). The latter holds because the Radon property implies that each individual probability measure on $\mathscr{E}$ is tight, because completeness can be suppressed [1, p. 241], and because weak convergence depends only on the topology of E so that E need only be metrizable. Since $(\Omega, \mathscr{M})$ is a countable product space it inherits the topological properties of E; the Radon property hence implies that a probability measure on $(\Omega, \mathscr{M})$ is determined by its finite-dimensional distributions [6, p. 84]. Finally, the fact that a sequence (μ_n) of probability measures on Ω is tight if and only if $(\mu_n \pi_k^{-1})_{n \geq 1}$ is tight for each k is easily seen to remain true in this case, so the argument of Lemma (2) is valid when E is a separable metrizable Radon space, and the Theorem follows. □

Remark. A complete separable metric space is a separable metrizable Radon space; whether the metrizability hypothesis can be suppressed is unknown.

By an invariant measure for a Markov kernel K we mean a probability measure μ on $\mathscr{E}$ such that $\mu K = \mu$. In the following result we consider the effect of the conditions of Theorem (1) on convergence of invariant measures. We assume here, and for the remainder of the paper, that E is a complete separable metric space; the notation $\mu(g) = \int g \, d\mu$ is used below.

(6) **Theorem.** *Assume that for each n the Markov kernel K_n has an invariant measure μ_n, that* (1.b, c) *hold, and that the family of probability measures $\{K_n(x, \cdot);\ n \geq 1, x \in E\}$ is tight. Then (μ_n) is tight and every limit point of (μ_n) is an invariant measure for K. In particular, there exists at least one invariant measure for K.*

Proof. Tightness of (μ_n) is easy: given $\varepsilon > 0$ choose a compact set A such that $K_n(x, A) > 1 - \varepsilon$ for all n and x. Then for each n

$$\mu_n(A) = \int \mu_n(dx) \, K_n(x, A) > (1 - \varepsilon) \int \mu_n(dx) = 1 - \varepsilon,$$

proving tightness.

If μ is a limit point of (μ_n) there is a subsequence $(\mu_{n'})$ such that $\mu_{n'} \Rightarrow \mu$. Suppose that $g \in UC$; then for each n'

$$|\mu(g) - \mu K(g)| \leq |\mu(g) - \mu_{n'}(g)| + |\mu_{n'}(g) - \mu_{n'}(K_n, g)| + |\mu_{n'}(K_{n'} g) - \mu_{n'}(K g)| + |\mu_{n'}(K g) - \mu(K g)|.$$

In this expression the first and fourth terms become zero in the limit as $n' \to \infty$

since $\mu_{n'} \Rightarrow \mu$ and g, $K g \in C$. The second term is identically zero since $\mu_{n'} K_{n'} = \mu_{n'}$ and the third goes to zero as $n' \to \infty$ by the argument used in the proof of (1). Thus $\mu(g) = \mu K(g)$ and hence $\mu = \mu K$.

Corollary. *If K can have at most one invariant measure then under the conditions of the Theorem, there exists an invariant measure μ for K and $\mu_n \Rightarrow \mu$.*

Proof. In this case every subsequence of (μ_n) will contain, by Prohorov's Theorem [1, p. 37], a further subsequence which is weakly convergent. The limit will be invariant for K by the Theorem and is then the unique invariant measure μ. That $\mu_n \Rightarrow \mu$ follows by a result in the theory of weak convergence [1, p. 16]. □

Remarks. 1. The assertion in (6) that every limit point of (μ_n) be invariant for K is true without the additional tightness hypothesis; the latter insures that the set of such limit points be nonempty.

2. If E is a compact metric space the tightness of (μ_n) (and of $\{K_n(x, \cdot)\}$) is automatic. In this case (1.b) is known, cf. [8, p. 101], to be sufficient for the existence of an invariant measure for K; here we have given an alternative and more intuitive way (than that of [8]) of constructing invariant measures for K. One possible choice for the approximating sequence (K_n) is the following. For each n let $\mathscr{A}_n = \{A_{n,1} \ldots, A_{n,k_n}\}$ be a partition of the state space E into sets of diameter at most n^{-1}, suppose that $x_{n,k} \in A_{n,k}$, and define

$$\alpha_{n,k,j} = K(x_{n,k}; A_{n,j}), \qquad j = 1, \ldots, k_n.$$

We may then construct an approximating sequence (K_n) by putting

$$K_n(x, \cdot) = \sum_{j=1}^{k_n} \alpha_{n,k,j}\, \varepsilon_{x_{n,j}} \qquad \text{if } x \in A_{n,k}$$

$K_n(x, \cdot)$ is then purely atomic and $K_n g$ is constant on each set $A_{n,k}$. One can show by straightforward computations that (K_n), K satisfy the hypothesis of (1), and the existence of an invariant measure for K_n (concentrated on $\{x_{n,1}, \ldots, x_{n,k_n}\}$) may be determined by elementary methods.

3. Conditions entailing unicity of an invariant measure may be found in [7]; these are of a recurrence nature. When they are satisfied, the assertion $\mu_n \Rightarrow \mu$ of the corollary to (6) justifies the interchange of two limits; namely, if $g \in C$

$$\lim_{n\to\infty} \lim_{k\to\infty} E_n[g(X_k)] = \lim_{k\to\infty} \lim_{n\to\infty} E_n[g(X_k)] = \lim_{k\to\infty} E[g(X_k)].$$

Another, and possibly more natural, set of sufficient conditions for the conclusion of (6) is the following.

(7) **Theorem.** *For each n let μ_n be an invariant measure for K_n, assume that* (1.b) *holds, that $K_n g \to K g$ uniformly whenever $g \in UC$, and that $\{K(x, \cdot); x \in E\}$ is tight. Then the conclusion of* (6) *and the corollary to* (6) *are valid.*

Proof. It suffices to verify the tightness of the sequence (μ_n). Given $\varepsilon > 0$ choose a compact set A such that $K(x, A^c) < \varepsilon$ for all $x \in E$. Let g be a continuous function which vanishes on A and such that $0 \leqq g \leqq 1$. Then for n large enough that

$$\|K_n g - K g\|_\infty < \varepsilon$$

we have

$$\mu_n(g)=\int \mu_n(dx)\, K_n\, g(x) \leqq \int \mu_n(dx)\, K\, g(x)+\varepsilon$$
$$\leqq \int \mu_n(dx)\, K(x, A^c)+\varepsilon \leqq 2\varepsilon,$$

from which tightness follows. □

More generally one defines an invariant measure for K to be any σ-finite measure μ such that $\mu K=\mu$. If we restrict attention to invariant Radon measures (a Radon measure is finite on each compact set) then one can easily extend (6) to the effect that if for each n μ_n is an invariant Radon measure for K_n, then any vague limit point of (μ_n) is invariant for K.

3. Markov Renewal and Semi-Markov Processes

As an application of the results of Section 2 we obtain here an analogous result for the continuity of a Markov renewal process with respect to its semi-Markov kernel, which leads to further results concerning continuity properties of certain classes of semi-Markov and regular Markov processes. We again denote by $(E, \mathscr{E})$ a complete separable metric space, only now we are considering Markov chains $((X_n, T_n))_{n\in\mathbf{N}}$ with state space $(L, \mathscr{L})=(E, \mathscr{E})\times(\mathbf{R}_+, \mathscr{R}_+)$, so that the canonical sample space $(\Omega, \mathscr{M})$ becomes $(L, \mathscr{L})^{\mathbf{N}}$, but remains a complete separable metric space.

Our notation and terminology are those of [3, 4]. Given a semi-Markov kernel Q on $(E, \mathscr{E})$ we define a Markov kernel K on $(L, \mathscr{L})$ by the relations

(8) $K((y, t); A\times B)=Q(y; A\times(B-t)).$

The Markov chain $(\Omega, \mathscr{M}, (X_n, T_n), P^{(y,t)})$ with transition kernel K is called the *Markov renewal process* induced by Q. One thinks of X_n as the n^{th} state entered by an evolving system which changes state only by jumps and of T_n as the time of the n^{th} transition. In particular, given the history generated by the Markov chain X, the process T has independent increments. For further details and interpretations we refer to [3, 4]. Applied to Markov renewal processes Theorem (1) yields the following result; the notation is that of Section 2.

(9) **Theorem.** *Let $\nu, \nu_1, \nu_2, \ldots$ be probability measures on $(L, \mathscr{L})$ and let $Q, Q_1, Q_2, \ldots$ be semi-Markov kernels on $(E, \mathscr{E})$. If*

a) $\nu_n \Rightarrow \nu$;

b) *whenever $f\in C(L)$ the function*

$$\int Q(\cdot, dz, du)\, f(z, u)$$

is in $C(E)$;

c) *For every uniformly bounded, equicontinuous subset $\mathscr{H}$ of $C(L)$ and compact subset A of E*

$$\lim_{n\to\infty} \sup_{y\in A,\, g\in\mathscr{H}} \left|\int Q_n(y; dz, du)\, g(z, u)-\int Q(y; dz, du)\, g(z, u)\right|=0$$

then $P_n \Rightarrow P$ on $(\Omega, \mathscr{M})$.

Proof. Define $K, K_1, K_2, \ldots$ from $Q, Q_1, Q_2, \ldots$, respectively, by (8); then we need only to show that the hypotheses of Theorem (1) are satisfied by K, K_1, $K_2, \ldots$. Since

$$K\,g(y,t)=\int Q(y;dz,du)\,g(z,t+u),$$

it is evident that (1.b) holds. Moreover, if $g\in UC(L)$ and $B\subset L$ is compact then

$$\begin{aligned}&\sup_{(y,t)\in B}|K_n\,g(y,t)-K\,g(y,t)|\\ &=\sup_{(y,t)\in B}\left|\int Q_n(y;dz,du)\,g(z,t+u)-\int Q(y;dz,du)\,g(z,t+u)\right|\\ &\leqq \sup_{y\in B_1,\,t\in B_2}\left|\int Q_n(y;dz,du)\,g_t(z,u)-\int Q(y;dz,du)\,g_t(z,u)\right|,\end{aligned}$$

where $g_t(z,u)=g(z,t+u)$ and B_1, B_2 are the (compact) projections of B onto E, $\mathbf{R}_+$, respectively. Because $g\in UC$, $\{g_t\colon t\in B_2\}$ is uniformly bounded and equicontinuous, so (9.c) implies that $K, K_1, K_2, \ldots$ satisfy (1.c). □

A semi-Markov kernel Q on $(E,\mathscr{E})$ is *regular* if $P^{(y,0)}\{T_0<T_1<\cdots, \sup T_n=+\infty\}=1$ for all $y\in E$; sufficient conditions for this may be found in [4]. The process $(Y_t; P^{(y,0)})$ defined by

$$Y_t=X_n \quad \text{if} \quad t\in[T_n, T_{n+1})$$

is the *semi-Markov process* induced by Q. The continuous time E-valued process (Y_t) is a deterministic function Ψ of $((X_n, T_n))$ (here $\Psi\colon \Omega\to D(E,\mathbf{R}_+)$, the space of right continuous functions from $\mathbf{R}_+$ to E with limits from the left) and (cf. [5]) it is known that Ψ is continuous almost surely with respect to each $P^{(y,0)}$ if Q is regular. Hence we may combine this observation, Theorem (9) and the Continuous Mapping theorem [1, p. 30] to obtain the following result for the continuity of a semi-Markov process with respect to its initial distribution α and semi-Markov kernel Q.

(10) **Theorem.** *Let $\alpha, \alpha_1, \alpha_2, \ldots$ be probability measures on $(E,\mathscr{E})$ and let Q, Q_1, $Q_2, \ldots$ be regular semi-Markov kernels on $(E,\mathscr{E})$. For each n let P_n be the probability law on Ω of the Markov renewal process with initial distribution $\alpha_n\times\varepsilon_0$ and semi-Markov kernel Q_n and let P be defined analogously from α, Q. If $\alpha_n\Rightarrow\alpha$ and if* (9.b, c) *hold, then $P_n\Psi^{-1}\Rightarrow P\Psi^{-1}$ on $D(E,\mathbf{R}_+)$.*

A regular Markov process (cf. [2] for definitions and details) is a semi-Markov process with regular semi-Markov kernel Q which is of the form

(11) $\quad Q(x; A\times(t,\infty))=e^{-\lambda(x)t}\,K(x,A),$

where $\lambda\colon E\to(0,\infty)$ is measurable and K is a Markov kernel on $(E,\mathscr{E})$ such that $K(x,\{x\})=0$ for all $x\in E$. Boundedness of the jump function λ is the simplest sufficient condition for regularity of Q, but is not necessary. Theorem (10) applies to yield the following continuity theorem for a regular Markov processes.

(12) **Theorem.** *Let $P, P_1, P_2, \ldots$ be the probability laws on $D(E,\mathbf{R}_+)$ of the regular Markov processes with initial distributions $\alpha, \alpha_1, \alpha_2, \ldots$, jump functions $\lambda, \lambda_1, \lambda_2$, and transition kernels $K, K_1, K_2, \ldots$, respectively. If*

a) $\alpha_n \Rightarrow \alpha$;

b) λ *is uniformly continuous;*

c) $\lambda_n \to \lambda$ *uniformly on compact subsets of* E;

d) $Kg \in C$ *whenever* $g \in UC$;

e) *For every compact subset* A *of* E *and uniformly bounded, equicontinuous subset* $\mathscr{H}$ *of* C,

$$\lim_{n\to\infty} \sup_{x\in A,\, g\in\mathscr{H}} |K_n g(x) - K g(x)| = 0,$$

then $P_n \Rightarrow P$ *on* $D(E, \mathbf{R}_+)$.

Proof. It is a matter only of defining, by (11), semi-Markov kernels $Q, Q_1, Q_2, \ldots$ from (λ, K), (λ_1, K_1), (λ_2, K_2), ..., respectively, and verifying that (9.b, c) hold, which is entirely straightforward.

Remarks. 1. A related result, with stronger conditions on $\lambda_n \to \lambda$ and weaker conditions on $K_n \to K$, may be obtained using infinitesimal generators as described in [9].

2. A regular Markov process with jump function λ and transition kernel K admitting an invariant measure μ has invariant measure η given by

$$\eta(A) = \frac{\int_A \frac{1}{\lambda(x)} \mu(dx)}{\int_E \frac{1}{\lambda(x)} \mu(dx)},$$

provided the denominator be finite. This observation, Theorem (6) or (7), and Theorem (12) can then be combined to yield a convergence theorem for invariant measures of regular Markov processes.

Acknowledgement. The author is indebted to the referee of an earlier version of this paper for numerous helpful suggestions.

References

1. Billingsley, P.: Convergence of Probability Measures. New York: Wiley 1968
2. Blumenthal, R. M., Getoor, R. K.: Markov Processes and Potential Theory. New York: Academic Press 1968
3. Çinlar, E.: Markov renewal theory. Advances in Appl. Probability **1**, 123–187 (1969)
4. Çinlar, E.: On semi-Markov processes on arbitrary spaces. Proc. Cambridge Philos. Soc. **66**, 381–392 (1969)
5. Kennedy, D. P.: The continuity of the single server quene. J. Appl. Probability **9**, 370–381 (1972)
6. Neveu, J.: Mathematical Foundations of the Calculus of Probability. San Francisco: Holden-Day 1965
7. Orey, S.: Limit Theorems for Markov Chain Transition Probabilities. London: Van Nostrand Reinhold 1971
8. Rosenblatt, M.: Markov Processes. Structure and Asymptotic Properties. Berlin, Heidelberg, New York: Springer-Verlag 1971
9. Skorohod, A. V.: Limit theorems for Markov processes. Theor. Probability Appl. **3**, 202–246 (1958)

Received February 1, 1975