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Stable Probability Measures on $\mathbb{R}^\nu$

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1. Introduction

The notion of stable probability measures on the real line $\mathbb{R}$ as introduced by Lévy [1] generalizes in a natural way to probability measures on $\mathbb{R}^\nu$, $\nu \geqq 1$, which are stable under a fixed subgroup of the general linear group $GL(\nu, \mathbb{R})$ on $\mathbb{R}^\nu$. The ultimate aim of a theory of stable measures on $\mathbb{R}^\nu$ is to find a method which assigns to every subgroup $A \subset GL(\nu, \mathbb{R})$ the set of all those probability measures on $\mathbb{R}^\nu$ which are stable under A. Once this has been achieved one can proceed to study the domains of attraction of these measures and related questions. For $\nu = 1$ the classification of stable measures is classical (see e.g. [2], p. 227). For higher dimensions the papers by Sharpe and Michalicek ([3, 4] and [7]) have investigated probability measures which are stable under a one parameter subgroup of $GL(\nu, \mathbb{R})$. A first result concerning stable measures under a bigger group is contained in [6]: There it is shown that the probability measures on $\mathbb{R}^\nu$ which are stable under the full group $GL(\nu, \mathbb{R})$ are precisely the nonsingular Gaussian and the degenerate probability measures. The main question left open in [6] in this context was how to find the stable measures for a subgroup A of $GL(\nu, \mathbb{R})$ which has only a small compact part. Here we try to solve this problem in some special cases. The contents of this paper are as follows: § 2 gives a short introduction to the terminology and some earlier results. § 3 is concerned with the supports of stable measures. § 4 shows that any measure which is stable under a subgroup A of $GL(\nu, \mathbb{R})$ is also stable under the closure of A in $GL(\nu, \mathbb{R})$. Furthermore the measure is stable under a one parameter subgroup of $GL(\nu, \mathbb{R})$ which may be assumed to lie in A. So every stable measure in our sense is also "operator stable" in the sense of Sharpe and Michalicek. § 5 contains the Lévy-Hinčin formula for all stable measures on $\mathbb{R}^\nu$. Some results of this section have been obtained earlier in [7], but it seemed natural to include the proofs. One consequence of this section is that every probability measure on $\mathbb{R}^\nu$ which is stable under a subgroup of $SL(\nu, \mathbb{R})$, may be translated onto a proper subspace of $\mathbb{R}^\nu$. Another result is an easily checked condition for a group A to have no stable nonsingular Gaussian measures. § 6 deals with an example: We determine all stable probability measures for the group of lower triangular matrices on $\mathbb{R}^\nu$. In § 7 we find all probability measures which are stable under the group of diagonal matrices on $\mathbb{R}^\nu$.

2. Some Basic Facts about Stable Probability Measures

Let $\mathbb{R}^\nu$ denote the ν-dimensional real Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We write $M_1(\mathbb{R}^\nu)$ for the set of probability measures on $\mathbb{R}^\nu$ and $\mu * \nu$ for the convolution of any $\mu, \nu \in M_1(\mathbb{R}^\nu)$. The Fourier transform (or characteristic function) $\hat{\mu}$ of $\mu \in M_1(\mathbb{R}^\nu)$ is defined as $\hat{\mu}(v) = \int \exp i\langle v, w\rangle \, d\mu(w)$ for any $v \in \mathbb{R}^\nu$. Let $P(\mathbb{R}^\nu) = \{\hat{\mu} : \mu \in M_1(\mathbb{R}^\nu)\}$ be the set of characteristic functions on $\mathbb{R}^\nu$. An element $\mu \in M_1(\mathbb{R}^\nu)$ is called *infinitely divisible* if, for any $n = 2, 3, \ldots$ there exists a $\mu_n \in M_1(\mathbb{R}^\nu)$ such that $\mu_n^{*n} = \mu$. μ is infinitely divisible if and only if $\hat{\mu} = \exp \psi$ where ψ is given by the Lévy-Hinčin formula

$$\psi(v) = i\langle v, a\rangle + \psi_1(v) + \psi_2(v) \tag{2.1}$$

with

$$\psi_1(v) = -\tfrac{1}{2}\langle Pv, v\rangle, \tag{2.2}$$

$$\psi_2(v) = \int_{\mathbb{R}^\nu - \{0\}} \left(\exp i\langle v, w\rangle - 1 - \frac{i\langle v, w\rangle}{1 + \|w\|^2} \right) dF(w). \tag{2.3}$$

Here a is an element of $\mathbb{R}^\nu$, P a real symmetric positive definite $\nu \times \nu$ matrix, and F is a σ-finite measure on $\mathbb{R}^\nu - \{0\}$ such that

$$\int (1 - \cos\langle v, w\rangle) \, dF(w) < \infty \tag{2.4}$$

for all $v \in \mathbb{R}^\nu$. If $P = 0$, μ is called a *Poisson* measure, and if $F = 0$, μ is called *Gaussian.* ψ_1 and ψ_2 are called the Gaussian and Poisson part of ψ respectively, and correspondingly we can decompose μ as $\mu = \mu_1 * \mu_2 * p_a$ where $\hat{\mu}_k = \exp \psi_k$, $k = 1, 2$, and where p_a denotes the probability measure concentrated in the point $a \in \mathbb{R}^\nu$. μ_1 and μ_2 are called the Gaussian and Poisson part of μ respectively. μ_1, μ_2 and a are uniquely determined by μ. For later use we define a kernel K associated with the infinitely divisible measure μ by

$$\begin{aligned} K(u, v) &= \psi(u - v) - \psi(u) - \psi(-v) \\ &= \langle Pu, v\rangle + \int (\exp i\langle u, w\rangle - 1)\overline{(\exp i\langle v, w\rangle - 1)} \, dF(w) \end{aligned} \tag{2.5}$$

for all $u, v \in \mathbb{R}^\nu$. Clearly K determines μ up to translation – that is, if two infinitely divisible probability measures μ and μ' give rise to the same K, then $\mu = \mu' * p_a$ for some $a \in \mathbb{R}^\nu$. Consider now a fixed $\mu \in M_1(\mathbb{R}^\nu)$ and an element $\alpha \in \mathrm{GL}(\nu, \mathbb{R})$, the set of nonsingular real $\nu \times \nu$ matrices. We define a measure $\mu\alpha$ on $\mathbb{R}^\nu$ by setting $\mu\alpha(E) = \mu(\alpha E)$ for any Borel set $E \subset \mathbb{R}^\nu$. Following [6] we have

Definition 2.1. Let A be a subgroup of $\mathrm{GL}(\nu, \mathbb{R})$. *A measure $\mu \in M_1(\mathbb{R}^\nu)$ is called stable under A if, for any $\alpha_1, \alpha_2 \in A$ there exists an $\alpha_3 \in A$ and an $x \in \mathbb{R}^\nu$ such that*

$$\mu\alpha_1 * \mu\alpha_2 = \mu\alpha_3 * p_x. \tag{2.6}$$

The following lemma contains some elementary facts about stable probability measures which are proved in [6].

Lemma 2.2. *Let A be a subgroup of* $\mathrm{GL}(\nu, \mathbb{R})$ *and let μ be a probability measure on $\mathbb{R}^\nu$ which is stable under A. Then the following conditions are satisfied:*

1. *μ is infinitely divisible. If K is given by* (2.1)–(2.5) *then there exists, for any α_1, $\alpha_2 \in A$, an $\alpha_3 \in A$ with*

$$K(\alpha_1^T u, \alpha_1^T v) + K(\alpha_2^T u, \alpha_2^T v) = K(\alpha_3^T u, \alpha_3^T v) \tag{2.7}$$

for all $u, v \in \mathbb{R}^\nu$, where α^T denotes the transpose of α for any $\alpha \in A$.

2. *Let $B \subset A$ be the subgroup of all elements $\beta \in A$ for which*

$$K(\beta^T u, \beta^T v) = c(\beta)\, K(u, v) \tag{2.8}$$

for all $u, v \in \mathbb{R}^\nu$, and for some constant $c(\beta) > 0$. Then the map $c(\cdot)$ is a homomorphism from B onto a subgroup of $\mathbb{R}^+ = (0, \infty)$ which contains all positive rational numbers.

3. *Let $\mu = \mu_1 * \mu_2 * p_a$ be the decomposition of μ into its Gaussian part μ_1 and its Poisson part μ_2. Then both μ_1 and μ_2 are stable under A.*

3. Full Stable Measures

Let A be a fixed subgroup of $\mathrm{GL}(\nu, \mathbb{R})$ and let $\mu \in M_1(\mathbb{R}^\nu)$ be stable under A.

Definition 3.1. μ is called full if $|\hat{\mu}(v)| \neq 1$ whenever $v \neq 0$.

In this section we shall show that any stable measure is the translate of a measure which is full on some subspace of $\mathbb{R}^\nu$. We define K by (2.5) and put $N = \{u \in \mathbb{R}^\nu;\, K(u, u) = 0\}$. Clearly N is a closed subgroup of $\mathbb{R}^\nu$, and K is constant on the cosets of $N \times N$ in $\mathbb{R}^\nu \times \mathbb{R}^\nu$. K induces therefore a continuous map $\tilde{K}$ of $X \times X$ (where $X = \mathbb{R}^\nu / N$) given by

$$\tilde{K}(x, y) = K(u, v)$$

for all $x = u + N$, $y = v + N \in X$.

Lemma 3.2. *N is a subspace of $\mathbb{R}^\nu$ which is invariant under every β^T, $\beta \in B$. (B is defined in Lemma* 2.2.)

Proof. That N is invariant under $B^T = \{\beta^T : \beta \in B\}$ is an immediate consequence of (2.8). To show that N is a subspace, we denote by $\tilde{\beta}$ the automorphism β^T induces on X. By structure theory, X is of the form $\mathbb{R}^\sigma \times C$ where $\sigma \leqq \nu$ and where C is compact. Clearly every $\tilde{\beta}$ must leave C invariant so that, for any $x \in C$,

$$\sup_{\beta \in B} \tilde{K}(\tilde{\beta} x, \tilde{\beta} x) \leqq \sup_{y \in C} \tilde{K}(y, y) < \infty,$$

which together with (2.8) implies that $\tilde{K}(x, x) = 0$ whenever $x \in C$. So $C = \{0\}$, and $X = \mathbb{R}^\sigma$. Consequently $N = \mathbb{R}^{\nu - \sigma}$, and the lemma is proved.

Lemma 3.3. *N is invariant under $A^T = \{\alpha^T : \alpha \in A\}$.*

Proof. Assume there exist $\alpha_1 \in A$ and $v_0 \in N$ such that $\alpha_1 v_0 \notin N$. Choose α_2 such that

$$K(\alpha_1^T u, \alpha_1^T u) + K(u, u) = K(\alpha_2^T u, \alpha_2^T u)$$

for all $u \in \mathbb{R}^\nu$. This equation implies $\alpha_2^{T-1} N = N \cap \alpha_1^{T-1} N$ where the right hand side has dimension $\leqq \nu - 1$. As α_2 is nonsingular, this leads to a contradiction and the lemma is proved.

Lemma 3.4. *There exists an element $a \in \mathbb{R}^\nu$ such that*

$$\psi'(v) = i\langle v, a\rangle + \psi(v)$$

is constant on the cosets of N in $\mathbb{R}^\nu$.

Proof. This is a standard argument.

We obtain

Theorem 3.5. *Let A be a subgroup of* $\mathrm{GL}(\nu, \mathbb{R})$ *and let μ be a probability measure on $\mathbb{R}^\nu$ which is stable under A. Then there exists a subspace $\mathbb{R}^\tau$ of $\mathbb{R}^\nu$ which is invariant under A, and an element $a \in \mathbb{R}^\nu$ such that*

(1) *$\mu' = \mu * p_a$ is concentrated on $\mathbb{R}^\tau$,*

(2) *μ' considered as a probability measure on $\mathbb{R}^\tau$, is full.*

Proof. Put $\mathbb{R}^\tau$ equal to the orthogonal complement of $N = \mathbb{R}^\sigma$ in $\mathbb{R}^\nu$. Defining ψ' by Lemma 3.4 and μ' by $\hat{\mu}' = \exp \psi'$ we have proved the theorem.

Remark 3.6. As a consequence of Theorem 3.5 it is no loss of generality to assume that a stable probability measure μ is full. Otherwise we can replace μ by a translate μ' concentrated on a subspace of lower dimension in $\mathbb{R}^\nu$ which is invariant under A. μ' will then be full on this subspace and stable under the restriction of A there.

4. Stability under Closed Subgroups and under One Parameter Subgroups of GL(ν, $\mathbb{R}$)

Theorem 4.1. *Let A be any subgroup of* $\mathrm{GL}(\nu, \mathbb{R})$ *and let μ be a probability measure on $\mathbb{R}^\nu$ which is stable under A. Then μ is stable under the closure $\bar{A}$ of A in* $\mathrm{GL}(\nu, \mathbb{R})$.

Proof. According to Remark 3.6 it will be sufficient to prove the theorem under the additional assumption that μ is full. Under this hypothesis, let $(\alpha_n^{(i)})$, $i = 1, 2$, be two sequences in A^T which converge to $\alpha_0^{(i)}$, $i = 1, 2$, in $\mathrm{GL}(\nu, \mathbb{R})$ respectively. Hence $\alpha_n^{(i)} v$ converges to $\alpha_0^{(i)} v$ for all $i = 1, 2$, $v \in \mathbb{R}^\nu$. By Lemma 2.2 there exists a sequence $(\alpha_n^{(3)}) \subset A$ such that

$$K(\alpha_n^{(1)T} u, \alpha_n^{(1)T} v) + K(\alpha_n^{(2)T} u, \alpha_n^{(2)T} v) = K(\alpha_n^{(3)T} u, \alpha_n^{(3)T} v)$$

for every $u, v \in \mathbb{R}^\nu$. The continuity of K implies the convergence of $K(\alpha_n^{(3)T} u, \alpha_n^{(3)T} v)$, and, since μ is full,

$$\lim_n \log K(\alpha_n^{(3)T} u, \alpha_n^{(3)T} u) = \lim_n \log(-2\psi(\alpha_n^{(3)T} u))$$

exists for all $u \neq 0$. Lemma 5.3 in [6] can now be applied to show that $(\alpha_n^{(3)})$ is relatively compact in $\mathrm{GL}(\nu, \mathbb{R})$. Choosing a convergent subsequence of $(\alpha_n^{(3)})$ with limit $\alpha_0^{(3)}$ say, we have $\alpha_0^{(3)} \in \bar{A}$ and

$$K(\alpha_0^{(1)T} u, \alpha_0^{(1)T} v) + K(\alpha_0^{(2)T} u, \alpha_0^{(2)T} v) = K(\alpha_0^{(3)T} u, \alpha_0^{(3)T} v)$$

for all u, $v \in \mathbb{R}^\nu$. Since $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ can be chosen arbitrarily in $\bar{A}$, the theorem is proved.

Lemma 4.2. *If μ is stable under a closed subgroup $A \subset \mathrm{GL}(\nu, \mathbb{R})$, then the subgroup B of A defined in Lemma 2.2 is closed.*

Proof. See Lemma 2.6 in [6].

Lemma 4.3. *Let μ be stable under a closed subgroup A of $\mathrm{GL}(\nu, \mathbb{R})$ and assume that μ is full. Then the map $c: B \to \mathbb{R}^+$ is surjective.*

Proof. The proof uses the same idea as the proof of Theorem 4.1 and will be omitted.

Lemma 4.4. *Under the assumptions of Lemma 4.3, $B_0 = \{\beta \in B: c(\beta) = 1\}$ is a maximal compact subgroup of A.*

Proof. Apply Lemma 6.4 and Lemma 6.5 in [6].

Lemma 4.5. *Under the assumptions of Lemma 4.3, we have*

1. $\lim\limits_{\eta \to 0} \sup\limits_{\{x \in \mathbb{R}^\nu: \|x\| = 1\}} \sup\limits_{\{\beta \in B: c(\beta) \leqq \eta\}} \|\beta x\| = 0.$
2. $\lim\limits_{\eta \to \infty} \inf\limits_{\{x \in \mathbb{R}^\nu: \|x\| = 1\}} \inf\limits_{\{\beta \in B: c(\beta) \geqq \eta\}} \|\beta x\| = \infty.$

Proof. Let C be any compact subset of $\mathbb{R}^\nu$. Since ψ is continuous, we have $\delta_1 = \sup\limits_{v \in C} |\mathrm{Re}\, \psi(v)| < \infty$ and $\inf\limits_{\|v\| = 1} |\mathrm{Re}\, \psi(v)| = \delta_2 > 0$. For any $\beta \in B$ with $c(\beta) \geqq 2\delta_1/\delta_2$, and for any v with $\|v\| = 1$, we get

$$|\mathrm{Re}\, \psi(\beta^T v)| = c(\beta)\, |\mathrm{Re}\, \psi(v)| \geqq 2\delta_1$$

which implies that $\beta^T v \notin C$. Hence

$$\lim_{\eta \to \infty} \inf_{\{x \in \mathbb{R}^\nu: \|x\| = 1\}} \inf_{\{\beta \in B: c(\beta) \geqq \eta\}} \|\beta^T x\| = \infty.$$

But this implies

$$\lim_{\eta \to \infty} \sup_{\{x \in \mathbb{R}^\nu, \|x\| = 1\}} \sup_{\{\beta \in B: c(\beta) \leqq \eta\}} \|\beta^T x\| = 0$$

which in turn implies (1). (2) follows from (1) in a straightforward way, and the proof is complete.

Lemma 4.6. *There exists a Borel subset $S \subset \mathbb{R}^\nu - \{0\}$ which intersects each orbit of B in $\mathbb{R}^\nu - \{0\}$ in exactly one point.*

Proof. Let $|||\cdot|||$ denote a norm on $\mathbb{R}^\nu$ which is invariant under the compact group $B_0 \subset B$. The maps $\beta \to |||\beta x|||$, $x \in \mathbb{R}^\nu$, are constant on the cosets of B_0 in B and can thus be considered as continuous maps on B/B_0, which is isomorphic to $\mathbb{R}^+$ by Lemma 5.3 in [6] and by Lemma 4.4 in this paper. Together with Lemma 4.5 this implies that any orbit of B intersects the set $F = \{x: |||x||| = 1\}$ in $\mathbb{R}^\nu$. Furthermore, for any $x \in F$, the set $B_x = \{\beta \in B: \beta x \in F\}$ is compact. We define, for any $x \in F$, $[x]$ to be the set $\{\beta x: \beta \in B x\}$ and denote by $\mathcal{O}$ the collection $\{[x], x \in F\}$. $\mathcal{O}$ is a Hausdorff space in the quotient topology defined by the map $x \to [x]$ from F onto $\mathcal{O}$. A theorem of Kuratowski (see [5], p. 23) shows that there exists a Borel subset S of F such that S intersects each $[x]$ in exactly one point. Clearly S is then a cross section for the orbits of B. The proof is complete.

Remark 4.7. We can in fact choose S in such a way that the closure $\bar{S}$ of S in $\mathbb{R}^\nu$ does not contain 0. The section S constructed in Lemma 4.6 certainly has this property.

We are thus led to

Theorem 4.8. *Let A be a subgroup of* $\mathrm{GL}(\nu, \mathbb{R})$ *and let μ be a probability measure on $\mathbb{R}^\nu$ which is stable under A. Then there exists a one parameter subgroup $G = \{\beta_t, t \in \mathbb{R}\}$ of* $\mathrm{GL}(\nu, \mathbb{R})$ *such that μ is stable under G. Furthermore, there exists a Borel subset of $\mathbb{R}^\nu - \{0\}$ which intersects each orbit of G in $\mathbb{R}^\nu - \{0\}$ in exactly one point. If A is closed and if μ is full, then G can be chosen as a one-parameter subgroup of A.*

Proof. Again we may assume A to be closed and μ to be full, by Remark 3.6 and Theorem 4.1. The subgroup B of A defined in Lemma 2.2 is then closed and the homomorphism $c(\cdot)$ from B to $\mathbb{R}^+$ is surjective, by the Lemmas 4.2 and 4.3. Lemma 2.2 also implies that μ is stable under B. Let B' be the connected component of the identity in B. It is easy to see that $c(B') = \mathbb{R}^+$ and that there exists a one parameter subgroup G in B' with $c(G) = \mathbb{R}^+$. So μ is stable under G, and we can apply Lemma 4.6 to the case $A = B = G$ to complete the proof.

5. The Lévy-Hinčin Formula for Stable Measures on $\mathbb{R}^\nu$

Let γ be a real $\nu \times \nu$ matrix which is a direct sum of $k_j \times k_j$ matrices $\zeta_j, j = 1, \ldots, n_1$, and of $2l_j \times 2l_j$ matrices $\xi_j, j = 1, \ldots, n_2$, of the form

$$\zeta_j = \begin{pmatrix} b_j & & & & 0 \\ 1 & b_j & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & b_j \end{pmatrix}, \qquad b_j \in \mathbb{R} \tag{5.1}$$

and

$$\xi_j = \left(\begin{array}{ccccc|ccccc} c_j & & & & 0 & d_j & & & & 0 \\ 1 & c_j & & & & & \ddots & & & \\ & 1 & \ddots & & & & & \ddots & & \\ & & \ddots & \ddots & & & & & \ddots & \\ 0 & & & 1 & c_j & 0 & & & & d_j \\ \hline -d_j & & & & 0 & c_j & & & & 0 \\ & \ddots & & & & 1 & c_j & & & \\ & & \ddots & & & & 1 & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ 0 & & & & -d_j & 0 & & & 1 & c_j \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} l_j \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} l_j \end{array} \qquad c_j, d_j \in \mathbb{R}, d_j \neq 0, \tag{5.2}$$

such that $\sum_{j=1}^{n_1} k_j + 2 \sum_{j=1}^{n_2} l_j = \nu$. If we assume that there exists a full probability measure μ stable under the group $\{\exp t\gamma, t \in \mathbb{R}\}$ then Lemma 4.5 implies (without loss of generality) that

$$\lim_{t\to\infty}(\exp - t\gamma)\, v = 0$$

for all $v \in \mathbb{R}^\nu$, and hence that

$$\inf_j \{b_j, c_j\} > 0. \tag{5.3}$$

If μ is in addition Gaussian, then $\hat{\mu}(v) = \exp(i\langle v, a\rangle - \frac{1}{2}\langle Pv, v\rangle)$ where $a \in \mathbb{R}^\nu$ and where P is a positive definite nonsingular real symmetric matrix. Lemma 2.2 then implies that

$$\exp t\gamma \cdot P \cdot \exp t\gamma^T = \exp \lambda t \cdot P$$

for all $t \in \mathbb{R}$, where $\lambda \neq 0$. Multiplying both sides by $P^{-1/2}$ from the left and from the right we see that $\exp t\gamma$ is similar to a scalar multiple of a rotation. The uniqueness of the Jordan form (5.1) and (5.2) implies that

$$\gamma = \begin{pmatrix} c & d_1 & & & & & & & & \\ -d_1 & c & & & & & & & & \\ & & c & d_2 & & & & & & \\ & & -d_2 & c & & & & 0 & & \\ & & & & \ddots & & & & & \\ & & & & & c & d_n & & & \\ & & 0 & & & -d_n & c & & & \\ & & & & & & & c & & 0 \\ & & & & & & & & \ddots & \\ & & & & & & & 0 & & c \end{pmatrix} \tag{5.4}$$

where $c > 0$ and where $d_1, \ldots, d_k \in \mathbb{R}$. Since both $\exp t\gamma$ and $P^{-1/2} \exp t\gamma P^{1/2}$ are scalar multiples of rotations, P must commute with γ. Note that in this case the map $c(\cdot)$ from $\{\exp t\gamma, t \in \mathbb{R}\}$ to $\mathbb{R}^+$ is given by

$$c(\exp t\gamma) = \exp 2ct. \tag{5.5}$$

Theorem 5.1. *Let $G = \{\exp t\delta, t \in \mathbb{R}\}$ be a one parameter group in* $\mathrm{GL}(\nu, \mathbb{R})$. *A necessary and sufficient condition for the existence of a full Gaussian probability measure μ which is stable under G is the following:*

1. *There exists an $\alpha \in \mathrm{GL}(\nu, \mathbb{R})$ such that $\gamma = \alpha\delta\alpha^{-1}$ is of the form* (5.4),
2. *If $\hat{\mu}(v) = \exp(i\langle v, a\rangle - \frac{1}{2}\langle Pv, v\rangle)$, then P commutes with δ.*

Proof. For any $\delta \in \mathrm{GL}(\nu, \mathbb{R})$ there exists an $\alpha \in \mathrm{GL}(\nu, \mathbb{R})$ such that $\gamma = \alpha\delta\alpha^{-1}$ is of the form

$$\gamma = \begin{pmatrix} \zeta_1 & & & & & \\ & \ddots & & & 0 & \\ & & \zeta_{n_1} & & & \\ & & & \xi_1 & & \\ & 0 & & & \ddots & \\ & & & & & \xi_{n_2} \end{pmatrix}$$

where the ζ_j and ξ_j are matrices given by (5.1) and (5.2). If μ is full, (5.3) must also be satisfied, and the discussion preceding the statement of the theorem shows that γ is of the form (5.4) and commutes with $\alpha P \alpha^{-1}$. The theorem is proved. Turning now to the case where μ is purely Poisson, full, and stable under $\{\exp t\gamma\}$ with γ given by (5.1)–(5.3), we can apply Theorem 3.4 in [6] and Theorem 4.8 in this paper to show that $\hat{\mu} = \exp \psi$ with

$$\psi(v) = \int_S \int_{\mathbb{R}} \left(\exp i\langle v, \exp t\gamma s\rangle - 1 - \frac{i\langle v, \exp t\gamma s\rangle}{1 + \|\exp t\gamma s\|^2} \right) \exp(-\lambda t)\, dt\, d\rho(s) \tag{5.6}$$

where S is a Borel cross section of the orbits of $\{\exp t\gamma\}$ in $\mathbb{R}^\nu - \{0\}$, ρ is a σ-finite measure on S, and λ is a real number such that, for all $v \in \mathbb{R}^\nu$,

$$\int_S \int_{\mathbb{R}} (1 - \cos\langle v, \exp t\gamma s\rangle) \exp(-\lambda t)\, dt\, d\rho(s) < \infty. \tag{5.7}$$

(5.7) is equivalent to

$$\int_S \int_{\mathbb{R}} \frac{\|\exp t\gamma s\|^2}{1 + \|\exp t\gamma s\|^2} \exp(-\lambda t)\, dt\, d\rho(s) < \infty \tag{5.8}$$

which in turn implies

$$\int_S \frac{\|s\|^2}{1 + \|s\|^2}\, d\rho(s) < \infty \tag{5.9}$$

and

$$0 < \lambda < 2 \min\{b_1, \ldots, b_{n_1}, c_1 \ldots c_{n_2}\} = 2 \min_{i=1,\ldots,k} \{\operatorname{Re} \lambda_i\} \tag{5.10}$$

where $\lambda_1, \ldots, \lambda_k$ denotes the eigenvalues of γ. If we choose S in such a way that $0 \notin \bar{S}$ (see Remark 4.7), then (5.9) implies that ρ must be totally finite on S. Conversely, if the closure of S does not contain 0, then (5.9) and (5.10) together imply (5.8). Note that the homomorphism $c(\cdot)$ is now given by

$$c(\exp t\gamma) = \exp \lambda t. \tag{5.11}$$

We now recall that any matrix γ is similar to one given by direct summands of the form (5.1) and (5.2) and obtain

Theorem 5.2. *Let* $\gamma \in \mathrm{GL}(\nu, \mathbb{R})$ *and let* $\lambda_1, \ldots, \lambda_k$ *denote the eigenvalues of* γ.

1. *There exists a full stable probability measure* μ *for the group* $G = \{\exp t\gamma, t \in \mathbb{R}\}$ *on* $\mathbb{R}^\nu$ *if and only if either* $\operatorname{Re} \lambda_i < 0$ *for all i or* $\operatorname{Re} \lambda_i > 0$ *for all* $i = 1, \ldots, k$.

2. *Assume that* $\operatorname{Re} \lambda_i > 0$ *for all* $i = 1, \ldots, k$. *Let S be a cross section of the orbits of G in* $\mathbb{R}^\nu - \{0\}$ *such that* $0 \notin \bar{S}$. *A Poisson measure* μ *on* $\mathbb{R}^\nu$ *is stable under G if and only if it is if the form*

$$\hat{\mu} = \exp \psi$$

where ψ *is given by*

$$\psi(v) = \int_S \int_{\mathbb{R}} \left(\exp i\langle v, \exp t\gamma s\rangle - 1 - \frac{i\langle v, \exp t\gamma s\rangle}{1 + \|\exp t\gamma s\|^2} \right) \exp(-\lambda t)\, dt\, d\rho(s). \tag{5.12}$$

Here ρ denotes any totally finite measure on S and λ is a real number with

$$0<\lambda<2 \min_{1\leq j\leq k} \operatorname{Re}\lambda_j. \tag{5.13}$$

Theorem 5.3. *Let A be a subgroup of $\mathrm{GL}(\nu, \mathbb{R})$. If μ is stable under A, and if $\mu=\mu_1 * \mu_2 * p_a$ is the decomposition of μ into its Gaussian part μ_1 and its Poisson part μ_2, then μ_1 and μ_2 are concentrated on subspaces N_1 and N_2 of $\mathbb{R}^\nu$ respectively, which are invariant under A and which have intersection $\{0\}$.*

Proof. This follows at once from (5.5), (5.10) and (5.11).

Corollary 5.4. *Let μ be a probability measure on $\mathbb{R}^\nu$. There exists a subgroup A of $\mathrm{GL}(\nu, \mathbb{R})$ such that μ is stable under A if and only if μ satisfies the following three conditions:*

1. *μ is infinitely divisible.*

2. *If $\mu=\mu_1 * \mu_2 * p_a$ is the decomposition of μ into its Gaussian part μ_1 and its Poisson part μ_2, then μ_1 and μ_2 are supported on subspaces N_1 and N_2 of $\mathbb{R}^\nu$ respectively with $N_1 \cap N_2=\{0\}$.*

3. *μ_2 on N_2 is given by* (5.12)–(5.13) *for some one parameter group $\{\exp t\gamma\}$ of linear transformations of N_2 such that all the eigenvalues $\lambda_1, \ldots, \lambda_k$ of γ have positive real parts.*

Corollary 5.5. *Let A be a subgroup of $\mathrm{SL}(\nu, \mathbb{R})$. Then there are no full stable probability measures for A.*

Proof. By Theorem 4.7, any full probability measure μ on $\mathbb{R}^\nu$ which is stable under A is also stable under a one parameter subgroup which lies in the closure of A and hence in $\mathrm{SL}(\nu, \mathbb{R})$. But no such one parameter group has a generator γ whose eigenvalues have all nonzero real parts of the same sign.

6. Probability Measures on $\mathbb{R}^\nu$ which Are Stable under the Group of Lower Triangular Matrices

Let $A^{(\nu)}$ be the group of matrices $\{(a_{ik}): a_{ik}\in\mathbb{R}, a_{ii}\neq 0, a_{ik}=0 \text{ for } i<k, i, k=1, \ldots, \nu\}$. The invariant subspaces of $\mathbb{R}^\nu$ are $N_k=\{(x_1, \ldots, x_\nu)\in\mathbb{R}^\nu: x_1=x_2=\cdots=x_k=0\}$, $k=1, \ldots, \nu$. Accordingly we conclude from Theorem 3.5 and Remark 3.6 that any probability measure μ on $\mathbb{R}^\nu$ which is stable under $A^{(\nu)}$ is either full, or it can be translated onto some N_k where it will be stable under $A^{(\nu-k)}$, $k\geq 1$. From Theorem 5.3 it also follows that any stable probability measure is either purely Gaussian or purely Poisson.

Lemma 6.1. *Any full Gaussian probability measure μ on $\mathbb{R}^\nu$ is stable under $A^{(\nu)}$.*

Proof. Let μ be a nonsingular Gaussian measure on $\mathbb{R}^\nu$ given by

$$\hat{\mu}(y)=\exp\{i\langle y, a\rangle-\tfrac{1}{2}\langle Py, y\rangle\}.$$

μ is stable under $A^{(\nu)}$ if and only if, for every $\alpha_1, \alpha_2\in A^{(\nu)}$, there exists an $\alpha_3\in A^{(\nu)}$ such that

$$\alpha_1 P\alpha_1^T+\alpha_2 P\alpha_2^T=\alpha_3 P\alpha_3^T$$

by Lemma 2.2. Let $\alpha_1, \alpha_2 \in A^{(\nu)}$ be fixed, and put $Q = \alpha_1 P \alpha_1^T + \alpha_2 P \alpha_2^T$ Q is a non-singular symmetric real positive definite matrix. We set $M_k = \{(x_1, \ldots, x) \in \mathbb{R}^\nu: x_{k+1} = \cdots = x_\nu = 0\}$. Let T be an orthogonal matrix on $\mathbb{R}^\nu$ which maps, for every $k = 1, \ldots, \nu$, the subspace $Q^{1/2} M_k$ onto the subspace $P^{1/2} M_k$ in $\mathbb{R}^\nu$. If e_i denotes the i-th unit vector in $\mathbb{R}^\nu$, we set

$$a_i = P^{-1/2} T Q^{1/2} e_i.$$

$a_1, \ldots, a_\nu$ forms a basis of $\mathbb{R}^\nu$ with $a_k \in M_k$ and with

$$\langle P a_j, a_k \rangle = \langle P^{1/2} a_j, P^{1/2} a_k \rangle = \langle T^{-1} P^{1/2} a_j, T^{-1} P^{1/2} a_k \rangle = \langle Q e_j, e_k \rangle,$$

for all j, $k = 1, \ldots, \nu$. Let α_3 be the element of $A^{(\nu)}$ whose k-th row is equal to a_k, $k = 1, \ldots, \nu$. Clearly we have

$$\alpha_3 P \alpha_3^T = Q,$$

and the lemma is proved.

Turning now to the Poisson measures, let us assume that μ is a full Poisson measure on $\mathbb{R}^\nu$ which is stable under $A^{(\nu)}$. Let U be the subgroup of orthogonal matrices in $A^{(\nu)}$. U is equal to the set of diagonal matrices with entries ± 1 along the diagonal. By Lemma 4.4 we can find an element α in $A^{(\nu)}$ such that $\mu\alpha$ is invariant under U. We shall assume for the sake of simplicity that μ itself is invariant under U. According to Theorem 4.8, μ is also stable under a one parameter subgroup $G = \{\exp t\gamma, t \in \mathbb{R}\}$ in $A^{(\nu)}$. By Theorem 5.2 γ does not have 0 as eigenvalue, so that the sets

$$S_1 = \{(x_1, \ldots, x_\nu) \in \mathbb{R}^\nu: x_1 = 1\} \quad \text{and} \quad S_{-1} = \{(x_1, \ldots, x_\nu): x_1 = -1\}$$

together intersect each orbit of G in $\mathbb{R}^\nu - \{0\}$ in at most one point. Let S be a Borel cross section of the orbits of G in $\mathbb{R}^\nu - \{0\}$ which contains S_1 and S_{-1}, and such that $0 \notin \bar{S}$ (see Remark 4.7). By Theorem 5.2 there exists a totally finite measure ρ on S such that μ is given by the formulae (5.12) and (5.13). Quite obviously, $\rho(S_1 \cup S_{-1}) > 0$, since μ is full. The invariance of μ under U implies further that the restrictions of ρ to S_1 and to S_{-1} define the same measure on $\mathbb{R}^{\nu-1}$. More precisely, let us denote by ρ_1 and ρ_{-1} the restriction of ρ to S_1 and to S_{-1} respectively. Define measures ν_1 and ν_{-1} on $\mathbb{R}^{\nu-1}$ by setting

$$d\rho_{\pm 1}((\pm 1, x_2, \ldots, x_\nu)) = d\nu_{\pm 1}(x_2, \ldots, x_\nu).$$

Then $\nu_1 = \nu_{-1}$. If we write $\tilde{\rho}$ for the restriction of ρ to $S_1 \cup S_{-1}$, then

$$\tilde{\psi}(y) = \int_{S_1 \cup S_{-1}} \int_{\mathbb{R}} \left(\exp i\langle y, \exp t\gamma s\rangle - 1 - \frac{i\langle y, \exp t\gamma s\rangle}{1 + \|\exp t\gamma s\|^2} \right) \exp(-\lambda t)\, dt\, d\tilde{\rho}(s) \quad (6.1)$$

defines a Poisson measure $\tilde{\mu}$ on $\mathbb{R}^\nu$ by setting $\hat{\tilde{\mu}} = \exp\tilde{\psi}$. It is easy to see that $\tilde{\mu}$ is again stable under $A^{(\nu)}$. We set $\tilde{K}(u, v) = \tilde{\psi}(u - v) - \tilde{\psi}(u) - \tilde{\psi}(-v)$, $u, v \in \mathbb{R}^\nu$, and we choose and fix an element $\alpha = (a_{ik})$ in $A^{(\nu)}$. For any $s \in S_1 \cup S_{-1}$, $t \in \mathbb{R}$, the element

$$\tilde{s} = \beta_{-t - \log|a_{11}|} \alpha \beta_t s$$

lies again in $S_1 \cup S_{-1}$. Consequently we can write

$$\tilde{K}(\alpha^T u, \alpha^T v) = \int_{S_1 \cup S_{-1}} \int_{\mathbb{R}} (\exp i\langle u, \alpha \exp t\gamma s\rangle - 1)$$

$$\cdot \overline{(\exp i\langle v, \alpha \exp t\gamma s\rangle - 1)} \exp(-\lambda t)\, dt\, d\tilde{\rho}(s)$$

$$= \int_{S_1 \cup S_{-1}} \int_{\mathbb{R}} (\exp i\langle u, \beta_{t+\log|a_{11}|} \tilde{s}\rangle - 1)$$

$$\cdot \overline{(\exp i\langle v, \beta_{t+\log|a_{11}|} \tilde{s}\rangle - 1)} \exp(-\lambda t)\, dt\, d\tilde{\rho}(s)$$

$$= \int_{\mathbb{R}} \int_{S_1 \cup S_{-1}} (\exp i\langle u, \beta_t s\rangle - 1)\overline{(\exp i\langle v, \beta_t s\rangle - 1)} \exp(-\lambda t)$$

$$\cdot |a_{11}|^\lambda\, d\rho_1(\beta_{-t} \alpha^{-1} \beta_{t+\log|a_{11}|} s)\, dt.$$

Using (2.7), the invariance of μ under U, and denoting by A^+ the subgroup of $A^{(\nu)}$ consisting of all elements $\alpha = (a_{ik})$ in $A^{(\nu)}$ with $a_{11} > 0$, we get the following relation: For any $\alpha_1, \alpha_2 \in A^+$ there exists an $\alpha_3 \in A^+$ such that

$$(a_{11}^{(1)})^\lambda\, d\tilde{\rho}(\beta_{-t} \alpha_1^{-1} \beta_{t+\log a_{11}^{(1)}} s) + (a_{11}^{(2)})^\lambda\, d\tilde{\rho}(\beta_{-t} \alpha_2^{-1} \beta_{t+\log a_{11}^{(2)}} s)$$
$$= (a_{11}^{(3)})^\lambda\, d\tilde{\rho}(\beta_{-t} \alpha_3^{-1} \beta_{t+\log a_{11}^{(3)}} s) \tag{6.2}$$

a.e.t, where $a_{11}^{(i)}$ denotes the top left entry of α_i, $i = 1, 2, 3$. But the terms of (6.2) are all continuous as functions of t in the weak*-topology for totally finite measures on S_1. Hence (6.2) holds for all t.

Lemma 6.2. *Any totally finite measure $\tilde{\rho}$ on $S_1 \cup S_{-1}$ satisfying* (6.2) *is zero.*

Proof. We can rewrite (6.2) in the following weaker form: for any $\alpha_1, \alpha_2 \in A^+$ with $a_{11}^{(i)} = 1$, $i = 1, 2$, and for any $c_1, c_2 > 0$ there exists an $\alpha_3 \in A^+$ with $a_{11}^{(3)} = 1$ and a $c_3 > 0$ such that

$$c_1 \rho_1 \alpha_1 + c_2 \rho_1 \alpha_2 = c_3 \rho_1 \alpha_3. \tag{6.3}$$

ρ_1 is totally finite and hence tight on S_1. Choose a compact set C in S_1 with $\rho_1(C) > \frac{9 \rho_1(S_1)}{10}$. Now choose elements $\alpha_1, \alpha_2 \in A^+$ as follows:

$$\alpha_i = \begin{pmatrix} 1 & & & & 0 \\ t_2^{(i)} & 1 & & & \\ t_3^{(i)} & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ t_\nu^{(i)} & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with $\alpha_i C \cap C = \emptyset$, $i = 1, 2$, and $\alpha_1 C \cap \alpha_2 C = \emptyset$. According to (6.3), there exist elements α_3 and α_4 in A^+ and constants $c_3, c_4 > 0$ with

$$\rho_1 + \rho_1 \alpha_1 = c_3 \rho_1 \alpha_3,$$

$$\rho_1 + \rho_1 \alpha_1 + \rho_1 \alpha_2 = c_4 \rho_1 \alpha_4$$

which immediately leads to a contradiction. The lemma is proved.

Theorem 6.3. *Let A be the group of lower triangular real $v \times v$ matrices with nonzero determinant. If $v>1$, then a full probability measure μ on $\mathbb{R}^v$ is stable under A if and only if it is Gaussian and nonsingular. If $v=1$, the stable probability measures for A on $\mathbb{R}$ are given by the classical formulae*

$$\hat{\mu}=\exp\psi$$

with

$$\psi(y)=iay-b\,|y|^{\lambda} \tag{6.4}$$

where $a\in\mathbb{R}$, $b\geqq 0$, and $\lambda\in(0,2]$.

Proof. That every nonsingular Gaussian probability measure on is stable under A has been shown in Lemma 6.1. The nonexistence of full stable Poisson measures for A follows from Lemma 6.2 and the arguments leading up to it. The case $v=1$ is classical (see [2], p. 327).

Remark 6.4. If A' denotes the group of lower diagonal real $v \times v$ matrices with positive determinant (or with positive entries on the diagonal) then the same argument as before shows that the full stable measures in the case $v>1$ are again the nonsingular Gaussian measures, while for $v=1$ the stable measures (in both cases) are given by $\hat{\mu}=\exp\psi$ with

$$\psi(y)=iay-b\,|y|^{\lambda}\left\{1+ic\frac{y}{|y|}\tan\frac{\pi}{2}\lambda\right\} \tag{6.5}$$

or

$$\psi(y)=iay-b\,|y|\left\{1+ic\frac{y}{|y|}\frac{2}{\pi}\log|y|\right\} \tag{6.6}$$

where $a\in\mathbb{R}$, $b\geqq 0$, $|c|\leqq 1$, $\lambda\in(0,1)\cup(1,2]$.

7. Probability Measures on $\mathbb{R}^v$ which Are Stable under the Group of Diagonal Matrices

Theorem 7.1. *Let D^+ denote the group of diagonal $v \times v$ matrices with positive entries on the diagonal. A probability measure is stable under A^+ if and only if it is of the form*

$$\hat{\mu}=\exp\psi$$

where

$$\psi(y)=\sum_{k=1}^{v}\psi_k(y_k) \quad \text{for any } y=(y_1,\dots,y_v) \tag{7.1}$$

and where $\psi_k\colon \mathbb{R}\to\mathbb{C}$ is given by

$$\psi_k(t)=ia_k t-b_k\,|t|^{\lambda_k}\left\{1+ic_k\frac{t}{|t|}\tan\frac{\pi}{2}\lambda_k\right\} \tag{7.2}$$

or

$$\psi_k(t)=ia_k t-b_k\,|t|\left\{1+ic_k\frac{t}{|t|}\frac{2}{\pi}\log|t|\right\} \tag{7.3}$$

with $a_k\in\mathbb{R}$, $b_k\geqq 0$, $|c_k|\leqq 1$, $\lambda_k\in(0,1)\cup(1,2]$.

In other words, every stable probability measure μ for A^+ is a cartesian product of classical stable probability measures on $\mathbb{R}$.

Theorem 7.2. *Let D denote the group of diagonal $\nu \times \nu$ matrices with real entries and nonzero determinant. A probability measure μ on $\mathbb{R}^\nu$ is stable under A if and only if $\hat{\mu} = \exp \psi$ with*

$$\psi(y) = \sum_{k=1}^{\nu} i a_k y_k - b_k |y_k|^{\lambda_k} \tag{7.4}$$

for any $y = (y_1, \ldots, y_\nu)$ and for $a_k \in \mathbb{R}$, $b_k \geqq 0$, $\lambda_k \in (0, 2]$.

Proof of 7.1 and 7.2. The idea of the proof is the same as the one used in the previous section, so we shall restrict ourselves to an outline. One first checks that any Gaussian measure μ which is stable under A^+ is given by

$$\hat{\mu}(y) = \exp\{i\langle y, a\rangle - \tfrac{1}{2}\langle P y, y\rangle\}$$

where P is a diagonal matrix. To find the full Poisson measures on $\mathbb{R}^\nu$ which are stable under A^+, let us assume that μ is full, stable, Poisson, and in addition stable under a one parameter subgroup $G = \{\exp t\gamma\}$ with $\gamma \in A^+$ (cf. Theorem 4.8). μ will then be given by a totally finite measure ρ on a suitable Borel cross section of the orbits of G in $\mathbb{R}^\nu - \{0\}$, and by a positive number λ (see (5.12) and (5.13)). We choose a Borel cross section $S^{(i)}$ containing the sets $S^{(i)}_{\pm 1} = \{(x_1, \ldots, x_\nu) \in \mathbb{R}^\nu : x_i = \pm 1\}$ and such that $0 \notin \overline{S^{(i)}}$. Deriving the equations analogous to (6.2) and (6.3) one sees easily, that the restriction of ρ to $S^{(i)}_{\pm 1}$ must in fact be concentrated in the points $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ with ± 1 in the i-th place. Varying now i between 1 and ν we see that the measure F in (2.1)–(2.3) is concentrated on the axes in $\mathbb{R}^\nu$. The rest of the proof is a well known computation (see p. 227 in [2]). To obtain the stable measures for A, we apply Lemma 4.4 to A and μ and see that the kernel K associated with μ must be invariant under the subgroup B_0 of A consisting of all matrices which have entries ± 1 along the diagonal. So μ must be invariant under B_0, which leads to (7.4). The proof is then complete.

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