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The Positive Rates Problem for Attractive Nearest Neighbor Spin Systems on $\mathbb{Z}$

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Summary. It is shown that a spin system on $\mathbb{Z}$ has only one invariant probability measure if it has attractive or repulsive nearest neighbor flip rates which are strictly positive and periodic under translation along $\mathbb{Z}$.

0. Introduction

A spin system on $\mathbb{Z}$ is a type of Markov process on the state space S = the subsets of $\mathbb{Z}$. An introduction to spin systems may be found in Liggett [12] or Durrett [3]. We will briefly describe them here to establish our notation and terminology. A rigorous construction will be given in Sect. 1.

We write ξ_t^A for the state of a spin system at time $t \geqq 0$ if A is the initial state. Thus, ξ_t^A is a random subset of $\mathbb{Z}$. We will find it convenient to view ξ_t^A as the set of *sites* (points in $\mathbb{Z}$) that are *occupied* at time *t*. We will also treat ξ_t^A as a function from $\mathbb{Z}$ to $\{0, 1\}$, with

$$\begin{aligned} \xi_t^A(x) &= 1 \quad \text{if } x \in \xi_t^A (x \text{ is occupied}) \\ &= 0 \quad \text{if } x \notin \xi_t^A (x \text{ is vacant}). \end{aligned}$$

We write (ξ_t^A) for the process with initial state A, and $(\xi_t^{\cdot})$ for the system of processes (ξ_t^A), $A \in S$.

Associated with each spin system is a set of *flip rates* $\gamma_x: S \to [0, \infty)$. These govern the transitions made by the system. If the system is in a state $\xi \in S$, then $\gamma_x(\xi)$ is the exponential rate at which the occupancy of the site x changes. Thus, if $x \notin \xi$, then $\gamma_x(\xi)$ is the rate at which the vacant site x becomes occupied, while if $x \in \xi$, then $\gamma_x(\xi)$ is the rate at which the occupied site x is vacated. For convenience, we will define *birth rates* β_x and *death rates* δ_x by

$$\beta_x(\xi) = \gamma_x(\xi \setminus \{x\}) \quad \text{and} \quad \delta_x(\xi) = \gamma_x(\xi \cup \{x\}).$$

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One of the most important problems in the study of spin systems is to determine whether or not a given system is *ergodic*, that is, whether it has a unique invariant probability measure. One of the oldest open questions concerning the ergodicity of spin systems on $\mathbb{Z}$ is to determine the validity of the so-called "positive rates conjecture", which is that the following three conditions ensure ergodicity:

Translation invariant: $\gamma_x(\xi)=\gamma_0(\xi-x)$ for all $x\in\mathbb{Z}$ and $\xi\in S$, where $\xi-x=\{y\in\mathbb{Z}: x+y\in\xi\}$.

Positivity: $\exists\,\varepsilon>0$ such that $\gamma_x(\xi)>\varepsilon$ for all $x\in\mathbb{Z}$ and $\xi\in S$.

Finite range: $\exists$ a positive integer r such that

$$\gamma_x(\xi)=\gamma_x(\xi\cap\{y: x-r\leqq y\leqq x+r\}) \quad \text{for all } x\in\mathbb{Z} \quad \text{and} \quad \xi\in S.$$

The conjecture is suggested by, among other things, certain facts from statistical mechanics. (For a survey of the connection between spin systems and statistical mechanics, see Kindermann and Snell [11] or Durrett [3].)

Examples are known of non-ergodic systems which satisfy any two of the above conditions. In fact, the case in which positivity fails has been studied extensively. See Durrett [2] and Gray and Griffeath [6] for recent results. If either of the other two conditions fails, almost anything can happen – non-ergodicity is only one of the milder possible types of behavior. However, no non-ergodic systems are known which satisfy positivity, finite range, and the condition that the rates be uniformly bounded. In this regard there is an example of a non-ergodic discrete time system due to Cirel'son [1] which has uniformly bounded, uniformly positive, finite range flip rates which depend on time as well as the state of the system.

Very little progress has been made on the general positive rates conjecture. In fact, Gach, Kurdymov and Levin [4] have suggested a possible counterexample in discrete time. Their evidence for non-ergodicity is based on computer simulations, which are of course not rigorous. See [11] for a discussion of this example.

Some progress has been made in the case that the rates satisfy the following monotonicity condition:

Attractive: If $\xi\supset\xi'$, then $\beta_x(\xi)\geqq\beta_x(\xi')$ and $\delta_x(\xi)\leqq\delta_x(\xi')$ for all $x\in\mathbb{Z}$.

In this case it is known that $(\xi_t^{\mathbb{Z}})$ and (ξ_t^{ϕ}) converge in distribution to invariant measures μ_1 and μ_0, and that the system is ergodic iff $\mu_0=\mu_1$ (this was first proved by Holley [9], although the result is almost taken for granted in light of current techniques). Under the further restriction that $r=1$, Liggett [13] showed that μ_1 and μ_0 are the only extreme invariant measures. (Liggett used a much weaker positivity condition). This condition that $r=1$ is called

Nearest neighbor: finite range with $r=1$.

In the very special "one-sided" nearest neighbor case (i.e., $\gamma_x(\xi)=\gamma_x(\xi\cap\{x, x+1\})$), Holley and Stroock [10] have proved the conjecture. Their result covers attractive as well as some non-attractive systems. See Sect. 3 below.

Our main result (Theorem 1) is to prove the positive rates conjecture in the attractive, nearest neighbor case. Actually we do not need the full strength of translation invariance – it is sufficient to assume

Periodicity: $\exists$ an integer d such that $\gamma_x(\xi)=\gamma_{x-d}(\xi-d)$ for all $x\in\mathbb{Z}$ and $\xi\in S$.

The paper is organized as follows. In Sect. 1 we give a particular construction of spin systems. This construction helps us to define and develop our main tool, called "edges". We use edges to prove the main result in Sect. 2. Variations and generalizations of Theorem 1 are discussed in Sect. 3.

Section 1. Edges

Edges are easiest to define if we construct spin systems using a graphical approach due to Harris [8]. The version given here is a more general form of the specific construction used by Gray and Griffeath [6].

The idea is to define a collection of independent random variables $S_n(x)$ and $C_n(x)$, $n\geqq 1$, $x\in\mathbb{Z}$, and then to define the system $(\xi_t^{\cdot})$ on the resulting probability space. The $S_n(x)$'s are exponentially distributed and are used to determine the times of possible flips at the site x. The $C_n(x)$'s are uniformly distributed and are used to introduce the interaction between x and other sites.

For each $x\in\mathbb{Z}$, let

$$\bar{\gamma}_x=\sup_{\xi}\delta_x(\xi)+\sup_{\xi}\beta_x(\xi).$$

We assume that $\bar{\gamma}_x<\infty$ for all x. Let $S_1(x)$, $S_2(x),\dots$ be i.i.d. exponentially distributed random variables with mean $1/\bar{\gamma}_x$ (let $S_n(x)\equiv\infty$ if $\bar{\gamma}_x=0$). Also, let $C_1(x)$, $C_2(x),\dots$ be i.i.d. random variables, uniformly distributed on $[0,\bar{\gamma}_x]$. Assume that all the random variables defined are mutually independent and let $(\Omega,\mathscr{F},P)$ be the resulting probability space. Let

$$T_n(x)=\sum_{j=1}^{n}S_j(x).$$

So far, we have made no restrictions on the rates except that $\bar{\gamma}_x<\infty$ for all x. In order to complete our construction, we need to impose a condition that will allow us to carry out a certain approximation procedure. We are not interested here in stating the most general conditions. Instead, we refer the reader to the conditions stated in Theorems 4.4 and 4.9 of Gray [5], which are sufficient. (The conditions of Theorem 4.9 in [5] are essentially the well-known uniqueness conditions due to Liggett. See also [12].) For our immediate purpose here, we impose the conditions of finite range and uniform boundedness. Then we claim that we can define a spin system $(\xi_t^{\cdot})$ on $(\Omega,\mathscr{F},P)$, with flip rates (γ_x), such that:

(1) $\xi_0^A=A$,

(2) a birth occurs at a site x at time s in the process (ξ_t^A) iff $\xi_{s-}^A(x)=0$ and there exists an $n\geqq 1$ such that $s=T_n(x)$ and $C_n(x)\geqq\bar{\gamma}_x-\beta_x(\xi_{s-}^A)$,

(3) a death occurs at a site x at time s in the process (ξ_t^A) iff $\xi_{s-}^A(x)=1$ and there exists an $n \geqq 1$ such that $s=T_n(x)$ and $C_n(x)<\delta_x(\xi_{s-}^A)$,

where $\xi_{s-}^A = \lim_{r \nearrow s} \xi_r^A$. The transitions described in (2) and (3) are made so as to make (ξ_t^A) right continuous in t. Intuitively, we could think of process (ξ_t^A) as evolving in the following manner. Start in state A. At each site x, wait until time $T_1(x)$. Then flip at x if $C_1(x)$ is the right size, depending on which of the other sites are occupied. Continue in this fashion, flipping at the successive times $T_2(x)$, $T_3(x)$, ... if the corresponding $C_n(x)$'s allow it. Of course, this is happening simultaneously at infinitely many sites, so there might be some difficulty with existence. In order to prove existence, we first define approximating systems $(\xi_{t,B}^{\cdot})$ for each finite set $B \in S$. Each process $(\xi_{t,B}^A)$ satisfies (1), but we only require it to satisfy (2) and (3) at sites $x \in B$. No flips occur at sites $x \notin B$. It is elementary to check that each system $(\xi_{t,B}^{\cdot})$ is uniquely defined on $(\Omega, \mathscr{F}, P)$ up to null sets. These "finite" systems have been used many times in existence and uniqueness proofs. In particular, they are special cases of the systems used in [5]. It is shown in the proof of Theorem 4.9 in [5] that under our assumptions,

$$\lim_{B \nearrow \mathbb{Z}} \inf_{\substack{B' \supset B \\ A \in S}} P(\xi_{t,B}^A(x) = \xi_{t,B'}^A(x),\ 0 \leqq t \leqq s) = 1$$

for all $x \in \mathbb{Z}$ and $s \geqq 0$. Thus, we can define $(\xi_t^{\cdot})$ to be the limit of the systems $(\xi_{t,B}^{\cdot})$ as $B \nearrow \mathbb{Z}$. It is easily checked that $(\xi_t^{\cdot})$ satisfies (1), (2) and (3), and standard results (such as the theorems in [5]) imply that $(\xi_t^{\cdot})$ has the desired flip rates.

The above construction has many advantages over more abstract approaches. One of these will not become apparent until the end of the proof of Theorem 1. A more obvious advantage is that it gives us a way to jointly define or *couple* all the processes that make up the system $(\xi_t^{\cdot})$. This coupling is known as the basic coupling (see Liggett [12]). It is most useful when the flip rates are attractive and nearest neighbor, in which case (2) and (3) imply the following well known result (see [12]), which is a sort of maximum principle:

Proposition 1. *Suppose that $(\xi_t^{\cdot})$ has attractive, nearest neighbor, uniformly bounded flip rates. Let I be a bounded or unbounded interval of integers, and let ∂I be the set of points in I^c with a neighbor in I (∂I is empty if $I=\mathbb{Z}$). Choose $A, B \in S$ and times s and u, with $0 \leqq s < u$. Suppose that*

$$\xi_s^A(x) \geqq \xi_s^B(x) \quad \textit{for all} \quad x \in I$$

and

$$\xi_t^A(x) \geqq \xi_t^B(x) \quad \textit{for all} \quad x \in \partial I \quad \textit{and} \quad t \in [s,u].$$

Then $\xi_t^A(x) \geqq \xi_t^B(x)$ for all $x \in I$ and $t \in [s,u]$. (To be precise, we should say something about null sets, but we will soon remove such sets from Ω. See (5) *below.)*

This proposition has an important special case, which we state as

Corollary 1.1. *If $\xi_s^A \supset \xi_s^B$, then $\xi_t^A \supset \xi_t^B$ for all $t \geqq s$.*

Proof. Let $I = \mathbb{Z}$ and $u \to \infty$ in Proposition 1.

We are now ready to define edges. As above and throughout the rest of this section, we assume that the flip rates are attractive, nearest neighbor and uniformly bounded above. We will define a *left edge process* (l_t^n) and a *right edge process* (r_t^n) for each $n \in \mathbb{Z}$. These processes are the same as those used by Durrett [2] if $\beta_x(\phi) = 0$ for all x, but in general they should be thought of as refined versions of the ideas used by Liggett in [13].

We start with left edges. Let $L_n = \{x: x \geqq n\}$. The process (l_t^n) will be defined in terms of $(\xi_t^{L_n})$. It will move about on the state space $\{x + \frac{1}{2}: x \in \mathbb{Z}\}$ and will be constructed so that

$$(4) \qquad \xi_t^{L_n}(l_t^n - \tfrac{1}{2}) = 0 \quad \text{and} \quad \xi_t^{L_n}(l_t^n + \tfrac{1}{2}) = 1 \qquad \text{for all } t \geqq 0.$$

Let $l_0^n = n - \frac{1}{2}$. Thus, (4) holds with $t = 0$. The transitions of (l_t^n) occur whenever the process $\xi_t^{L_n}$ has a birth at the site immediately to the left of l_t^n or a death at the site immediately to the right of l_t^n. As soon as such a flip occurs, (4) would be violated, and so l_t^n jumps to a new position that lies between a 0 on the left and a 1 on the right. In the case that there is a birth at the site to the left of l_t^n, the edge jumps to the nearest position on its left where (4) is satisfied. (The uniform boundedness of the flip rates ensures that there will always be such a site with probability 1.) If instead, there is a death at the site to the right of l_t^n, the edge jumps to the nearest position on its right where (4) holds. It is elementary to check that the uniform boundedness condition implies that the process (l_t^n) makes only finitely many transitions in a finite time with probability 1, so that it is defined for all $t \geqq 0$.

The right edge is defined similarly. Let $R_n = \mathbb{Z} \setminus L_n$. We wish (r_t^n) to satisfy

$$(4') \qquad \xi_t^{R_n}(r_t^n - \tfrac{1}{2}) = 1 \quad \text{and} \quad \xi_t^{R_n}(r_t^n + \tfrac{1}{2}) = 0 \qquad \text{for all } t \geqq 0.$$

Therefore, let $r_0^n = n - \frac{1}{2}$, and define the transitions of (r_t^n) so that the edge jumps to the right whenever the process $(\xi_t^{R_n})$ has a birth at the site to the right of the edge, and jumps to the left if there is a death at the site to the left of the edge. The edge always moves to the first position between a 0 and a 1.

In what follows, we can simplify things if we are allowed to ignore certain null sets. We therefore assume

(5) The following null sets have been removed from Ω:

$$\{\omega: \lim_{B \nearrow \mathbb{Z}} \xi_{t,B}^A(x) \text{ fails to exist for some } x \in \mathbb{Z},\ t \geqq 0, \text{ or } A \in S\},$$

$$\{\omega: \text{any edge process is undefined for some } t \geqq 0\},$$

$$\{\omega: T_n(x) = T_m(y) \text{ for some } n, m \in \mathbb{Z} \text{ and sites } x \neq y\}.$$

This last set is removed so that it is impossible for two flips to occur simultaneously at two different sites. It is a consequence of this that Proposition 1 is now correctly stated without any mention of null sets.

We now list some properties of edges. The first two are merely restatements of various features of our definitions. The third is the key to the applications in the rest of this paper.

E1 (4) and (4′) hold for all n.

E2 $(l_t^n)_{t \geqq s}$ is determined by l_s^n and $(\xi_t^{L_n})_{t \geqq s}$,
$(r_t^n)_{t \geqq s}$ is determined by r_s^n and $(\xi_t^{R_n})_{t \geqq s}$.

E3 $\xi_s^{\phi}(x) = \xi_s^{L_n}(x)$ for all $x < l_s^n$,
$\xi_s^{\phi}(x) = \xi_s^{R_n}(x)$ for all $x > r_s^n$,
$\xi_s^{\mathbb{Z}}(x) = \xi_s^{L_n}(x)$ for all $x > l_s^n$,
$\xi_s^{\mathbb{Z}}(x) = \xi_s^{R_n}(x)$ for all $x < r_s^n$.

E4 If $m \leqq n$, then $l_s^m \leqq l_s^n$ and $r_s^m \leqq r_s^n$.

E5 If $l_s^m = l_s^n$, then $l_t^m = l_t^n$ for all $t \geqq s$.
If $r_s^m = r_s^n$, then $r_t^m = r_t^n$ for all $t \geqq s$.

Property E3 (which we prove below) shows that the processes $(\xi_t^{L_n})$ and $(\xi_t^{R_n})$ are hybrids of (ξ_t^{ϕ}) and $(\xi_t^{\mathbb{Z}})$, with the edges marking the dividing lines. It will be used in the comparisons of (ξ_t^{ϕ}) and $(\xi_t^{\mathbb{Z}})$ that we need in the next section. Property E4 follows easily from Corollary 1.1, which implies that $\xi_t^{L_m} \supset \xi_t^{L_n}$ and $\xi_t^{R_m} \subset \xi_t^{R_n}$ for all $t \geqq 0$ if $m \leqq n$. Property E5 is used in the lemma in the next section. To prove E5, note that

$$\begin{aligned} l_s^m = l_s^n &\Rightarrow \xi_s^{L_m} = \xi_s^{L_n} \text{ by E3} \\ &\Rightarrow \xi_t^{L_m} = \xi_t^{L_n} \text{ for all } t \geqq s \text{ by Corollary 1.1} \\ &\Rightarrow l_t^m = l_t^n \text{ for all } t \geqq s \text{ by E2.} \end{aligned}$$

Proof of E3. We prove only the first and third equalities in E3. Note that these hold for $s=0$. We now argue that they are preserved by each transition of the system. If such a transition is a flip at any site which is not next to the edge, then the edge will not move, and the equalities will be preserved because of Proposition 1. Now suppose that there is a flip at the site to the left of the edge (the case of a flip to the right of the edge is treated similarly). Just before the flip, both processes (ξ_t^{ϕ}) and $(\xi_t^{L_n})$ have a vacancy at this site by E1 so we are only interested in the case that the flip is a birth in at least one of these two processes. But $\xi_t^{L_n} \supset \xi_t^{\phi}$ for all $t \geqq 0$ by Corollary 1.1, so a birth in the process (ξ_t^{ϕ}) implies a birth in the process $(\xi_t^{L_n})$. We can therefore assume that the flip is a birth in the process $(\xi_t^{L_n})$. By construction, the edge jumps to the left over a block of sites which are occupied by the process $(\xi_t^{L_n})$. Since the site where the flip occurred is now to the right of the edge, and since no other flips can occur at the same time by (5), the first equality in E3 is clearly preserved. The third equality is also clearly preserved at any site which is to the right of the previous position of the edge. But the additional sites that are to the right of the new position make up the block of sites over which the edge jumped. As stated above, these sites are occupied by $(\xi_t^{L_n})$, so they are also occupied by $(\xi_t^{\mathbb{Z}})$ because of Corollary 1.1. Thus, the third equality is preserved.

We now put E3 to work by proving the following proposition, which will help us in the next section to show that the processes (ξ_t^{ϕ}) and $(\xi_t^{\mathbb{Z}})$ tend to agree on larger and larger blocks of sites as $t \to \infty$:

Proposition 2. *Fix integers m and n and times s and u, with* $0 \leqq s < u$. *Then*

a) *If* $l_t^m + 2 \leqq r_t^n$ *for all* $t \in [s, u]$ *and if* $\xi_s^\phi(x) = \xi_s^{\mathbb{Z}}(x)$ *for* $l_s^m < x < r_s^n$, *then* $\xi_t^\phi(x) = \xi_t^{\mathbb{Z}}(x)$ *for all* $t \in [s, u]$ *and* $l_t^m < x < r_t^n$.

b) *If* $r_t^m + 2 \leqq l_t^n$ *for all* $t \in [s, u]$ *and if* $\xi_s^\phi(x) = \xi_s^{\mathbb{Z}}(x)$ *for* $r_s^m < x < l_s^n$, *then* $\xi_t^\phi(x) = \xi_t^{\mathbb{Z}}(x)$ *for all* $t \in [s, u]$ *and* $r_t^m < x < l_t^n$.

Remark. The proposition states that if $(\xi_t^{\mathbb{Z}})$ and (ξ_t^ϕ) agree at some time at all the sites that lie between any left edge and any right edge, then they will continue to agree at whatever sites lie between those two edges, as long as the edges remain at least two units apart. It is necessary that they remain two or more units apart because of the following situation which may occur if they are only one unit apart:

		l_t^m	r_t^n
$\xi_t^{\mathbb{Z}}$	1	1	1
$\xi_t^{L_m}$	0	1	1
$\xi_t^{R_n}$	1	1	0
ξ_t^ϕ	0	1	0

The picture shows the states of the four processes at three adjacent sites. The middle site lies between the two edges. A death could occur at that site in the process (ξ_t^ϕ) without one occurring in any of the other three processes. The edges would not move in this case, but (ξ_t^ϕ) and $(\xi_t^{\mathbb{Z}})$ would no longer agree at the site between the two edges.

Proof of Proposition 2. We prove a) only. As in the proof of E3, we will argue that the agreement between $(\xi_t^{\mathbb{Z}})$ and (ξ_t^ϕ) on the interval of sites between the edges (l_t^m) and (r_t^n) is preserved by any transition of the system, as long as the edges remain at least two units apart. We will use the following picture, which must be valid if $\xi_t^{\mathbb{Z}}(x) = \xi_t^\phi(x)$ for all x between l_t^m and r_t^n, and if $l_t^m + 2 \leqq r_t^n$:

		l_t^m		r_t^n	
$\xi_t^{\mathbb{Z}}$	†	†	1 * * 1	†	†
$\xi_t^{L_m}$	i)	0	1 * * 1	ii)	ii)
$\xi_t^{R_n}$	ii)	ii)	1 * * 1	0	i)
ξ_t^ϕ	†	0	1 * * 1	0	†

* indicates the (possibly empty) interval of sites between $l_t^m + 1$ and $r_t^n - 1$ where all four processes agree.

† indicates sites where $\xi_t^{\mathbb{Z}}$ or ξ_t^ϕ may be 0 or 1, subject to $\xi_t^{\mathbb{Z}} \supset \xi_t^\phi$.

i) indicates agreement with ξ_t^ϕ at this site.

ii) indicates agreement with $\xi_t^{\mathbb{Z}}$ at this site.

The picture shows the states of the four processes at sites between $l_t^m - 2$ and $r_t^n + 2$. It is correct because of E1 and E3. By Proposition 1, the picture remains unchanged by any flip that occurs at sites that are not next to one of the two edges. We now consider flips at the two sites $l_t^m \pm \frac{1}{2}$. (Flips at $r_t^n \pm \frac{1}{2}$ are treated similarly). First consider the possibility of a death in any of the four processes at $l_t^m+\frac{1}{2}$. Since $\xi_t^{L_m}$ is dominated by the other three processes at all sites to the left of r_t^n, such a death would necessarily occur in the process $(\xi_t^{L_m})$. By construction, the left edge would then jump to the right, and it is clear from the picture that either $(\xi_t^{\mathbb{Z}})$ would continue to agree with (ξ_t^{ϕ}) at all sites between the two edges, or the left edge would no longer be to the left of the right edge. Thus, our claim is true for flips at $l_t^m+\frac{1}{2}$. Now consider the possibility of a flip at $l_t^m-\frac{1}{2}$. We are only concerned here with a flip that affects the position of the left edge, that is, a birth at $l_t^m-\frac{1}{2}$ in the process $(\xi_t^{L_m})$. Since $\xi_t^{L_m}(x)=\xi_t^{\phi}(x)$ for $x=l_t^m-\frac{3}{2}$, $l_t^m-\frac{1}{2}$, and $l_t^m+\frac{1}{2}$, such a birth would also occur in the process (ξ_t^{ϕ}) by Proposition 1. Furthermore, after such a birth, the left edge would jump to the left over a block of sites which are occupied by $(\xi_t^{L_m})$. The first of these sites is $l_t^m-\frac{1}{2}$, which we have just argued is also now occupied by (ξ_t^{ϕ}). The rest of these sites are also occupied by (ξ_t^{ϕ}) because of E3. By Corollary 1.1, they are also occupied by $(\xi_t^{\mathbb{Z}})$, so the required agreement between (ξ_t^{ϕ}) and $(\xi_t^{\mathbb{Z}})$ is preserved.

Section 2. The Main Result

In order to set the stage, we first develop an ergodicity criterion. As mentioned in the introduction, it is well-known that the pair $(\xi_t^{\mathbb{Z}}, \xi_t^{\phi})$ converges in distribution to an invariant measure π on $S\times S$ as $t\to\infty$. Since $\xi_t^{\mathbb{Z}}\supset\xi_t^{A}\supset\xi_t^{\phi}$ for all t and $A\in S$, it follows that $\pi((B,C)\colon B\supset C)=1$ and $(\xi_t^{\cdot})$ is ergodic iff

$$\pi((B,C)\colon B=C)=1. \tag{6}$$

We will improve on (6) slightly in the following proposition, which may be found in the translation invariant case in Durrett [2]. The general result is actually a direct consequence of some facts from ergodic theory (see Chap. 10 of Phelps [14], for example), but we will give here a short proof based on our construction:

Proposition 3. *Let $(\xi_t^{\cdot})$ be a spin system with attractive nearest neighbor flip rates which are uniformly bounded. Then $(\xi_t^{\cdot})$ is ergodic iff*

$$\lim_{k\to\infty}\lim_{t\to\infty} P(\xi_t^{\mathbb{Z}}(x)=\xi_t^{\phi}(x),\ -k\leqq x\leqq k)>0. \tag{7}$$

Proof. Since $(\xi_t^{\mathbb{Z}}, \xi_t^{\phi})$ converges in distribution to π,

$$\pi((B,C)\colon B=C)=\lim_{k\to\infty}\lim_{t\to\infty} P(\xi_t^{\mathbb{Z}}(x)=\xi_t^{\phi}(x),\ -k\leqq x\leqq k). \tag{8}$$

Thus, (6) implies (7). To prove that (7) implies (6), let $(\mathscr{A},\mathscr{B})$ be a random initial state in $S\times S$ which is distributed according to π and which is independent of $(\xi_t^{\cdot})$. By Corollary 1.1, $\xi_t^{\mathbb{Z}}\supset\xi_t^{\mathscr{A}}\supset\xi_t^{\mathscr{B}}\supset\xi_t^{\phi}$ for all t, so

$$\begin{aligned} &P(\xi_t^{\mathscr{A}}(x)=\xi_t^{\mathscr{B}}(x),\ -k\leqq x\leqq k\mid \mathscr{A}\neq\mathscr{B})\\ &\quad\geqq P(\xi_t^{\mathbb{Z}}(x)=\xi_t^{\phi}(x),\ -k\leqq x\leqq k\mid \mathscr{A}\neq\mathscr{B})\\ &\quad= P(\xi_t^{\mathbb{Z}}(x)=\xi_t^{\phi}(x),\ -k\leqq x\leqq k)\end{aligned}$$

since $(\mathscr{A}, \mathscr{B})$ is independent of $(\xi_t^{\cdot})$. Also,

$$P(\xi_t^{\mathscr{A}}(x)=\xi_t^{\mathscr{B}}(x),\ -k\leqq x\leqq k \mid \mathscr{A}=\mathscr{B})=1,$$

so

$$\begin{aligned}P(\xi_t^{\mathscr{A}}(x)&=\xi_t^{\mathscr{B}}(x),\ -k\leqq x\leqq k)\\&\leqq \pi(\mathscr{A}=\mathscr{B})+(1-\pi(\mathscr{A}=\mathscr{B}))\,P(\xi_t^{\mathbb{Z}}(x)=\xi_t^{\phi}(x),\ -k\leqq x\leqq k).\end{aligned}$$

Now take limits to conclude $\pi(\mathscr{A}=\mathscr{B})=0$ or 1, by (8). It follows from (7) that $\pi(\mathscr{A}=\mathscr{B})=1$, so (6) holds.

We are now ready for

Theorem 1. *If* $(\xi_t^{\cdot})$ *is a spin system with periodic, attractive, nearest neighbor, strictly positive flip rates, then* $(\xi_t^{\cdot})$ *is ergodic.*

Note. Periodic nearest neighbor flip rates which are positive necessarily satisfy the positivity condition in the introduction. They are also uniformly bounded.

Proof. We will prove (7). The idea is that if k is fixed and s is large, we will be able to find a left and right edge pair which start together at time 0 but which lie on opposite sides of the interval $[-k, k]$ at time s with probability at least 1/2. By Proposition 2, $(\xi_t^{\mathbb{Z}})$ and (ξ_t^{ϕ}) agree on $[-k, k]$ at time s if they agree on the interval between the two edges at some time before s when these edges are close together. We then estimate the probability of this last event, thereby obtaining a lower bound for the expression in (7).

In order to find a suitable pair of edges, we will make use of the fact that the random variables l_t^m and r_t^n are negatively correlated for all $m, n\in\mathbb{Z}$ and $t\geqq 0$, in the strong sense that

$$P(l_t^m>j_1, \text{ and } r_t^n>j_2)\leqq P(l_t^m>j_1)\,P(r_t^n>j_2) \tag{9}$$

for all integers j_1 and j_2. This result is an easy consequence of Corollary 1.2 in Harris [7], which applied to our situation because of the attractive flip rates and the fact that edges are monotone functions of the processes used in their construction. The word "monotone" is used in the sense defined in [7].

We apply (9) as follows. For $s\geqq 0$, let

$$m(s)=\sup\{m\in\mathbb{Z}\colon P(r_s^m<0)>1/2\}.$$

The uniform boundedness of the flip rates implies that $m(s)<\infty$ for all s. Then $P(r_s^{m(s)}>0)<1/2$ and $P(r_s^{m(s)+1}>0)\geqq 1/2$, so (9) implies that

$$P(r_s^{m(s)}<0 \text{ and } l_s^{m(s)}>0)+P(r_s^{m(s)+1}>0 \text{ and } l_s^{m(s)}<0)>1/2. \tag{10}$$

We will show in the lemma below that for fixed k and large s, the probability is low that any edge lies in the interval $[-k, k]$ and the probability is high that $r_s^{m(s)}=r_s^{m(s)+1}$, so that (10) can be turned into a statement about the edges $r_s^{m(s)}$ and $l_s^{m(s)}$ lying on opposite sides of the interval $[-k, k]$ (see (13) below).

Lemma.

$$\lim_{s\to\infty} P(r_s^m=r_s^{m+1})=\lim_{s\to\infty} P(l_s^m=l_s^{m+1})=1$$

uniformly in $m \in \mathbb{Z}$, *and*

$$\lim_{s \to \infty} P(\textit{no left or right edge lies in } [-k, k] \textit{ at time } s) = 1$$

for all $k \geqq 0$.

Proof. This lemma is essentially the same as Lemma 2.2 in Liggett [13]. However, there are many superficial differences between our version and Liggett's which make it hard to translate one into the other. Also, Liggett's proof (as well as any completely rigorous proof that we could give) is somewhat tedious. We will therefore give an argument which is (we think) convincing and which avoids some of the mess.

First note that the second statement in the lemma follows from the first by periodicity. To prove the first statement, define

$$f(s) = \sup_m [P(r_s^m \neq r_s^{m+1}) + P(l_s^m \neq l_s^{m+1})].$$

By E5, $f(s)$ is non-increasing in s, so that $F = \lim_{s \to \infty} f(s)$ exists, with $F \geqq 0$. We wish to show that $F = 0$. Assume $F > 0$. Then the positions along the lattice where edges lie at time s maintain a minimum positive density for all s. More precisely, periodicity and the Ergodic Theorem imply that for all $s \geqq 0$,

$$\lim_{k \to \infty} \frac{1}{2k+1} \sum_{m=-k}^{k} (1_{\{r_s^m \neq r_s^{m+1}\}} + 1_{\{l_s^m \neq l_s^{m+1}\}}) \geqq F/d \quad \text{a.s.,} \tag{11}$$

where d is the period of the rates. By E4, right edges cannot cross over one another and neither can left edges, so (11) implies that there is a certain amount of "crowding" which does not decrease as $s \to \infty$. More precisely, there must exist M depending only on F and d such that

$$\lim_{k \to \infty} \frac{1}{2k+1} \sum_{m=-k}^{k} (1_{\{r_s^m < r_s^{m+1} < r_s^m + M\}} + 1_{\{l_s^m < l_s^{m+1} < l_s^m + M\}}) > F/2d \quad \text{a.s.} \tag{12}$$

It can be shown (this is the tedious part) that if $r_s^m < r_s^{m+1} < r_s^m + M$, then there is a positive probability (depending only on M) that the edges (r_t^m) and (r_t^{m+1}) will collide in the time interval $[s, s+1]$. The reasoning is that the positivity of the rates ensures the possibility that the edges move toward each other enough to collide. Once they collide, they coalesce forever by E5. The same is true of left edges. It is now easy to conclude from (12) and periodicity that $f(s)$ decreases at a minimum rate, independent of s, contradicting $F > 0$. Thus $F = 0$, and the first statement of the lemma follows.

Remark. The Lemma is the only place where we use periodicity in any essential way. Otherwise, we only need that the flip rates are uniformly bounded and uniformly positive.

Now we can apply the Lemma and immediately turn (10) into

$$\lim_{s \to \infty} [P(r_s^{m(s)} > k \text{ and } l_s^{m(s)} < -k) + P(r_s^{m(s)} < -k \text{ and } l_s^{m(s)} > k)] > 1/2. \tag{13}$$

Let $\mathscr{E}_k(s)$ be the union of the two events in (13); that is, $\mathscr{E}_k(s)$ is the event that $r_s^{m(s)}$ and $l_s^{m(s)}$ lie on opposite sides of the interval $[-k, k]$. For $\omega \in \mathscr{E}_k(s)$, define

$$\begin{aligned}\sigma(\omega) &= \sup\{t \leqq s\colon r_t^{m(s)} < l_t^{m(s)} + 2\} \quad \text{if } r_s^{m(s)}(\omega) > k\\ &= \sup\{t \leqq s\colon l_t^{m(s)} < r_t^{m(s)} + 2\} \quad \text{if } r_s^{m(s)}(\omega) < -k.\end{aligned}$$

Then by Proposition 2

$$\begin{aligned}&P(\xi_s^{\mathbb{Z}}(x) = \xi_s^{\phi}(x),\ -k \leqq x \leqq k)\\ &\quad \geqq P(\xi_\sigma^{\mathbb{Z}}(x) = \xi_\sigma^{\phi}(x) \text{ for all } x \text{ between } r_\sigma^{m(s)} \text{ and } l_\sigma^{m(s)} \mid \mathscr{E}_k(s))\, P(\mathscr{E}_k(s)).\end{aligned}$$

By (13), $P(\mathscr{E}_k(s)) > 1/2$ for sufficiently large s (depending on k) so we have proved the ergodicity criterion (7) if we can show that there exists $\delta > 0$ which does not depend on s and k, such that

$$P(\xi_\sigma^{\mathbb{Z}}(x) = \xi_\sigma^{\phi}(x) \text{ for all } x \text{ between } r_\sigma^{m(s)} \text{ and } l_\sigma^{m(s)} \mid \mathscr{E}_k(s)) > \delta. \tag{14}$$

The proof of (14) is somewhat long, but it is not too difficult until we get down to the inequalities in (18) below.

To simplify our notation, fix k and s, and let $\mathscr{E} = \mathscr{E}_k(s)$ and $m = m(s)$. Also write

$$\begin{aligned}\mathscr{E}^+ &= \mathscr{E} \cap \{r_s^m > k\},\\ \mathscr{E}^- &= \mathscr{E} \cap \{r_s^m < -k\}.\end{aligned}$$

Consider first $\omega \in \mathscr{E}^+$. Then by definition of σ, $r_\sigma^m \geqq l_\sigma^m + 2$ and $r_{\sigma-}^m \leqq l_{\sigma-}^m + 1$. Thus, at time σ, the right edge jumps to the right and/or the left edge jumps to the left (both can jump only if $r_{\sigma-}^m = l_{\sigma-}^m - 1$). We consider three cases:

(i) $r_{\sigma-}^m < l_{\sigma-}^m$,
(ii) $r_{\sigma-}^m = l_{\sigma-}^m$,
(iii) $r_{\sigma-}^m = l_{\sigma-}^m + 1$.

Assume that the right edge jumps at time σ. Then we claim that ξ_σ^{ϕ} and $\xi_\sigma^{\mathbb{Z}}$ necessarily agree at all sites between l_σ^m and r_σ^m, except possibly at the site $l_\sigma^m + \frac{1}{2}$ in case (ii) and possibly at either of the sites $l_\sigma^m + \frac{1}{2}$ or $l_\sigma^m + \frac{3}{2}$ in case (iii). This claim follows easily from E3 and the construction of edges, since it is always the case that if the right edge jumps to the right at time σ, $\xi_\sigma^{\phi}(x) = \xi_\sigma^{R_m}(x) = \xi_\sigma^{\mathbb{Z}}(x)$ at all sites x between $r_{\sigma-}^m$ and r_σ^m except possibly $x = r_{\sigma-}^m + \frac{1}{2}$. Since $l_{\sigma-}^m = l_\sigma^m$ in cases (ii) and (iii) and $l_{\sigma-}^m \geqq l_\sigma^m$ in case (i), the claim follows. A similar argument, using the sites $r_\sigma^m - \frac{1}{2}$ and $r_\sigma^m - \frac{3}{2}$ instead of $l_\sigma^m + \frac{1}{2}$ and $l_\sigma^m + \frac{3}{2}$, works for the case that the left edge jumps at time σ. Thus, the event in (14) always occurs in case (i). In case (ii), it occurs if ξ_σ^{ϕ} and $\xi_\sigma^{\mathbb{Z}}$ agree at one special site, while agreement at two special sites is needed in case (iii). In order to talk about these special sites, we define

$$\begin{aligned}\bar{x} &= l_\sigma^m + \tfrac{1}{2} \quad \text{if the right edge jumps at time } \sigma\\ &= r_\sigma^m - \tfrac{1}{2} \quad \text{if the left edge jumps at time } \sigma\end{aligned}$$

with $\bar{x}$ undefined if both edges jump at time σ (both edges can jump only in case (i)). The site $\bar{x}$ is the special site mentioned above for case (ii), and it is one of the two special sites in case (iii). The other is $\bar{x}-1$ if the right edge jumps at time σ or $\bar{x}+1$ if the left edge jumps. We now claim that even in case (iii), $\bar{x}$ is the crucial site. More precisely,

(15) In cases (ii) and (iii) ξ_σ^ϕ and $\xi_\sigma^{\mathbb{Z}}$ agree at all sites between l_σ^m and r_σ^m if they agree at $\bar{x}$.

To prove (15), assume that the right edge jumps at time σ (the case of the left edge is similar). We have already taken care of case (ii), so assume that we are in case (iii), that is, that $r_{\sigma-}^m=l_{\sigma-}^m+1$. Then $\bar{x}=l_{\sigma-}^m+\frac{1}{2}=r_{\sigma-}^m-\frac{1}{2}$. Since the right edge jumps toward the right at time σ, there must be a birth at $\bar{x}+1$ at time σ in the process $(\xi_t^{R_m})$. Furthermore, properties E1 and E3 imply that the following picture is valid if $\xi_\sigma^\phi(\bar{x})=\xi_\sigma^{\mathbb{Z}}(\bar{x})$:

	l_σ^m	$\bar{x}$	$r_{\sigma-}^m$	$\bar{x}+1$	
$\xi_{\sigma-}^{R_m}$		1		0	*
$\xi_{\sigma-}^{\phi}$		1		0	*

* the two processes agree here.

By Proposition 1 and the above picture, the birth that occurs in the process $(\xi_t^{R_m})$ at $\bar{x}+1$ at time σ also occurs in the process (ξ_t^ϕ), so $\xi_\sigma^\phi(\bar{x}+1)=\xi_\sigma^{\mathbb{Z}}(\bar{x}+1)=1$, and (15) is proved. Thus we have

(16) $$P(\xi_\sigma^{\mathbb{Z}}(x)=\xi_\sigma^\phi(x) \text{ for all } x \text{ between } l_\sigma^m \text{ and } r_\sigma^m \mid \mathscr{E}^+)$$
$$=P(r_{\sigma-}^m<l_{\sigma-}^m \mid \mathscr{E}^+)+P(\xi_\sigma^{\mathbb{Z}}(\bar{x})=\xi_\sigma^\phi(\bar{x}) \text{ and } r_{\sigma-}^m \geqq l_{\sigma-}^m \mid \mathscr{E}^+).$$

The case when $\omega\in\mathscr{E}^-$ can be treated similarly. Thus, if we define

$$\begin{aligned}\bar{\bar{x}}&=r_\sigma^m+\tfrac{1}{2} && \text{if the left edge jumps at time } \sigma,\\ &=l_\sigma^m-\tfrac{1}{2} && \text{if the right edge jumps at time } \sigma,\end{aligned}$$

we have

(17) $$P(\xi_\sigma^{\mathbb{Z}}(x)=\xi_\sigma^\phi(x) \text{ for all } x \text{ between } l_\sigma^m \text{ and } r_\sigma^m \mid \mathscr{E}^-)$$
$$=P(r_{\sigma-}^m>l_{\sigma-}^m \mid \mathscr{E}^-)+P(\xi_\sigma^{\mathbb{Z}}(\bar{\bar{x}})=\xi_\sigma^\phi(\bar{\bar{x}}) \text{ and } r_{\sigma-}^m \leqq l_{\sigma-}^m \mid \mathscr{E}^-).$$

Because of (16) and (17), the occurrence of the event in (14) really hinges on what happens at one (random) site. In fact, we claim that

(18) $$P(\xi_\sigma^{\mathbb{Z}}(\bar{x})=\xi_\sigma^\phi(\bar{x}) \text{ and } r_{\sigma-}^m \geqq l_{\sigma-}^m \mid \mathscr{E}^+)>\delta P(r_{\sigma-}^m \geqq l_{\sigma-}^m \mid \mathscr{E}^+)$$

and

$$P(\xi_\sigma^{\mathbb{Z}}(\bar{\bar{x}})=\xi_\sigma^\phi(\bar{\bar{x}}) \text{ and } r_{\sigma-}^m \leqq l_{\sigma-}^m \mid \mathscr{E}^-)>\delta P(r_{\sigma-}^m \geqq l_{\sigma-}^m \mid \mathscr{E}^-)$$

where $\delta>0$ does not depend on s or k. Note that (16), (17) and (18) imply (14), so we are done once we prove (18).

The proof of (18) is somewhat technical and is based on our construction of $(\xi_t^{\cdot})$ in Sect. 1. However, the intuitive ideas behind the proof are not too hard, and we will attempt to point them out as we proceed.

We will prove the first inequality in (18) only. We will have to consider the cases (ii) and (iii) above separately, so instead of (18), we will prove the stronger

(18′) There exists $\delta>0$ which does not depend on s or k such that

$$P(\xi_\sigma^{\mathbb{Z}}(\bar{x})=\xi_\sigma^{\phi}(\bar{x})\,|\,r_{\sigma-}^m=l_{\sigma-}^m \text{ and } \mathscr{E}^+)>\delta$$

and

$$P(\xi_\sigma^{\mathbb{Z}}(\bar{x})=\xi_\sigma^{\phi}(\bar{x})\,|\,r_{\sigma-}^m=l_{\sigma-}^m+1 \text{ and } \mathscr{E}^+)>\delta.$$

The first inequality in (18′) is the easiest to deal with. Furthermore, its proof will help the reader understand the proof of the second inequality.

Let $\mathscr{E}_0^+=\mathscr{E}^+\cap\{l_{\sigma-}^m=r_{\sigma-}^m\}$. It is enough to prove

(19) There exists $\delta>0$ which does not depend on s or k such that

$$P(\xi_\sigma^{\phi}(\bar{x})=1\,|\,\mathscr{E}_0^+)>\delta.$$

Choose $\omega\in\mathscr{E}_0^+$. Then by definition of $\bar{x}$ and σ, one of the two edges jumps over the site $\bar{x}$ at time σ, and $l_\sigma^m<\bar{x}<r_\sigma^m$. It follows that at least one of the two processes $(\xi_t^{L_m})$ and $(\xi_t^{R_m})$ has a birth at $\bar{x}$ at time σ. Thus $\xi_\sigma^{\mathbb{Z}}(\bar{x})=1$, since $\xi_\sigma^{\mathbb{Z}}\supset\xi_\sigma^{L_m}\cup\xi_\sigma^{R_m}$. By E3, $\xi_\sigma^{R_m}(\bar{x})=\xi_\sigma^{L_m}(\bar{x})=1$. In particular, the processes $(\xi_t^{L_m})$ and $(\xi_t^{R_m})$, and hence the event $\mathscr{E}_0^+$, do not seem to be directly affected by the value of $\xi_\sigma^{\phi}(\bar{x})$, since the values of $\xi_\sigma^{L_m}(\bar{x})$ and $\xi_\sigma^{R_m}(\bar{x})$ are determined. This is the intuitive reason that underlies (19). To be more rigorous, we need to look at the construction of $(\xi_t^{\cdot})$ given in Sect. 1.

Since there is a birth at $\bar{x}$ at time σ, (2) implies that there must exist an integer $\nu\geqq 1$ such that $T_\nu(\bar{x})=\sigma$ and

$$C_\nu(\bar{x})\geqq\bar{\gamma}_{\bar{x}}-\beta^*, \tag{20}$$

where the value of β^* is determined by the state of the process in which the birth occurs. If $\bar{x}=l_\sigma^m+\frac{1}{2}$, the birth occurs in $(\xi_t^{R_m})$, whereas if $\bar{x}=r_\sigma^m-\frac{1}{2}$ the birth occurs in $(\xi_t^{L_m})$. Thus

$$\begin{aligned}\beta^*&=\beta_{\bar{x}}(\xi_{\sigma-}^{R_m}) &&\text{if } \bar{x}=l_\sigma^m+\tfrac{1}{2}=r_{\sigma-}^m+\tfrac{1}{2}\\ &=\beta_{\bar{x}}(\xi_{\sigma-}^{L_m}) &&\text{if } \bar{x}=r_\sigma^m-\tfrac{1}{2}=l_{\sigma-}^m-\tfrac{1}{2}\end{aligned}$$

As long as (20) holds, the states $\xi_\sigma^{L_m}$ and $\xi_\sigma^{R_m}$ are unaffected by changes in the value of $C_\nu(\bar{x})$. To make the term "unaffected" precise, we must say something about measureability. Such a statement is slightly complicated by the fact that ν, $\bar{x}$, and β^* are random variables. To overcome this difficulty we partition according to the values of these variables. For $n'\geqq 1$ and $x'\in\mathbb{Z}$, let $\mathscr{F}(n',x',b)$ be the σ-algebra generated by the event $\{C_{n'}(x')\geqq\bar{\gamma}_{x'}-b\}$ and by all the $T_n(x)$'s and $C_n(x)$'s except for $C_{n'}(x')$, where b is any of the (finitely many) possible values of β^*. Let $\mathscr{E}_0^+(n',x',b)=\mathscr{E}_0^+\cap\{\nu=n'\}\cap\{\bar{x}=x'\}\cap\{\beta^*=b\}$. Then from our previous discussion, $\mathscr{E}_0^+(n',x',b)\in\mathscr{F}(n',x',b)$, so it is enough to prove

(21) $P(\xi^{\phi}_{\sigma}(x')=1 \mid \mathscr{F}(n', x', b))>\delta$ for all n', x' and b and almost all $\omega\in\mathscr{E}_0^+(n', x', b)$,

where $\delta>0$ does not depend on n', x', b, s or k. Let $\eta>0$ be a lower bound for the flip rates. By (2), $\xi^{\phi}_{\sigma}(x')=1$ if $\omega\in\mathscr{E}_0^+(n', x', b)$ and $C_{n'}(x')\geqq\bar{\gamma}_{x'}-\eta$. Thus

$$P(\xi^{\phi}_{\sigma}(x')=1 \mid \mathscr{F}(n', x', b))\geqq P(C_{n'}(x')\geqq\bar{\gamma}_{x'}-\eta \mid \mathscr{F}(n', x', b))$$
$$\text{for almost all } \omega\in\mathscr{E}_0^+(n', x', b)$$

But $C_{n'}(x')$ is independent of all the $T_n(x)$'s and of all the $C_n(x)$'s except for $C_{n'}(x')$, so

$$\begin{aligned}
P(C_{n'}(x')\geqq\bar{\gamma}_{x'}-\eta \mid \mathscr{F}(n', x', b))&\\
&=P(C_{n'}(x')\geqq\bar{\gamma}_{x'}-\eta \mid C_{n'}(x')\geqq\bar{\gamma}_{x'}-b)\\
&=\eta/b \quad \text{for almost all } \omega\in\mathscr{E}_0^+(n', x', b).
\end{aligned}$$

Since the rates are uniformly bounded, $\eta/b>\delta$ for some $\delta>0$ which does not depend on n', x', b, s or k. This proves (21) and completes the proof of the first inequality in (19).

The proof of the second inequality in (19) is similar but more involved. The reason for the complication is that when $r^m_{\sigma-}=l^m_{\sigma-}+1$, there is no birth (or death) at $\bar{x}$ at time σ. In order to get $\xi^{\phi}_{\sigma}(\bar{x})=1$, we will need to find a time before σ when a birth occurs at $\bar{x}$.

Let $\mathscr{E}_1^+=\mathscr{E}^+\cap\{r^m_{\sigma-}=l^m_{\sigma-}+1\}$. For $\omega\in\mathscr{E}_1^+$, define

$$\begin{aligned}
\tau_1&=\sup\{t<\sigma\colon l^m_t>\bar{x} \text{ or } r^m_t<\bar{x}\},\\
\tau_2&=\sup\{t<\sigma\colon t=T_n(\bar{x}) \text{ for some } n\geqq 1\},\\
\tau&=\tau_1\vee\tau_2.
\end{aligned}$$

Since $l^m_{\sigma-}<\bar{x}<r^m_{\sigma-}$ when $\omega\in\mathscr{E}_1^+$, τ_1 is the supremum of a non-empty set. By E1, $\xi^{Lm}_{\sigma-}(\bar{x})=\xi^{Rm}_{\sigma-}(\bar{x})=1$, so τ_2 is also the supremum of a non-empty set. The time τ will play a role in this part of the proof that is similar to the role played by σ in the proof of (19). In fact, (2) implies that no flips can occur at $\bar{x}$ in the time interval $(\tau_2, \sigma]$. Thus, $\xi^{Lm}_{\tau}(\bar{x})=\xi^{Rm}_{\tau}(\bar{x})=\xi^{\mathbb{Z}}_{\tau}(\bar{x})=1$ and $\xi^{\phi}_{\tau}(\bar{x})=\xi^{\phi}_{\sigma}(\bar{x})$. It is therefore enough to prove

$$P(\xi^{\phi}_{\tau}(\bar{x})=1 \mid \mathscr{E}_1^+)>\delta \tag{22}$$

where δ is as chosen above in the proof of (19). We distinguish two cases: $\tau>\tau_2$ and $\tau=\tau_2$. If $\tau>\tau_2$, then τ is a time when at least one of the two edges jumps over the site $\bar{x}$ in such a way that $l^m_{\tau}<\bar{x}<r^m_{\tau}$. Since $\tau>\tau_2$, no flip can occur at $\bar{x}$ at time τ, so the way in which edges are constructed implies that $\xi^{\phi}_{\tau}(\bar{x})=1$ (this is similar to the "case (i)" discussed earlier). Therefore, to prove (22), it is enough to prove

$$P(\xi^{\phi}_{\tau}(\bar{x})=1 \mid \tau=\tau_2 \text{ and } \mathscr{E}_1^+)>\delta. \tag{23}$$

The proof of (23) is very similar to the proof of (19). If $\omega\in\{\tau=\tau_2\}\cap\mathscr{E}_1^+$, we know that $\tau=T_{\nu}(\bar{x})$ for some $\nu\geqq 1$. We also know that $\xi^{Lm}_{\tau}(\bar{x})=\xi^{Rm}_{\tau}(\bar{x})=1$.

Furthermore, if $C_v(\bar{x}) \geqq \bar{\gamma}_{\bar{x}} - \eta$, then $\xi_\tau^\phi(\bar{x}) = 1$ by (2), where η is as above. We can argue almost exactly as before that

$$P(C_v(\bar{x}) \geqq \bar{\gamma}_{\bar{x}} - \eta \mid \tau = \tau_2 \text{ and } \mathscr{E}_1^+) > \delta,$$

which implies (23). The only difference is that β^* should now be defined according to the behavior of the processes $(\xi_\tau^{L_m})$ and $(\xi_\tau^{R_m})$ at $\bar{x}$ at time τ (we do not know quite so much about these processes at $\bar{x}$ at time τ as we did at time σ in the proof of (19)). In particular, if $\xi_{\tau-}^{L_m}(\bar{x}) = 0$, then we know that $C_v(\bar{x}) \geqq \bar{\gamma}_{\bar{x}} - \beta_{\bar{x}}(\xi_{\tau-}^{L_m})$, and similarly for the other process $(\xi_t^{R_m})$. On the other hand, if $\xi_{\tau-}^{L_m}(\bar{x}) = \xi_{\tau-}^{R_m}(\bar{x}) = 1$, then we only know that $C_v(\bar{x}) \geqq \delta_{\bar{x}}(\xi_{\tau-}^{L_m}) \vee \delta_{\bar{x}}(\xi_{\tau-}^{R_m})$. The random variable β^* should be defined accordingly. Otherwise, there is no essential difference. This complete the proof.

Section 3. Variations

Our Theorem 1 has an easy corollary which is of some interest because of work done by Holley and Stroock [10]. They used so-called "duality" techniques to treat the translation invariant "one-sided" nearest neighbor case. The one-sidedness condition is that $\gamma_x(\xi) = \gamma_x(\xi \cap [x, \infty))$ for all x. They divided one-sided nearest neighbor flip rates into eight classes, and proved the positive rates conjecture for six of these eight. These six cases include all attractive, one-sided, nearest neighbor, translation invariant flip rates. The remaining two cases include the *repulsive* case:

$$\text{Repulsive: If } \xi \supset \xi', \text{ then } \beta_x(\xi) \leqq \beta_x(\xi') \text{ and } \delta_x(\xi) \geqq \delta_x(\xi')$$

as well as some flip rates which are not repulsive. It is easy to see that repulsive nearest neighbor rates can be turned into attractive nearest neighbor rates simply by switching the identities of 0 and 1 at every other site. Under such a transformation, translation invariant rates become periodic, and so our Theorem 1 applies to repulsive as well as attractive rates (in the general nearest neighbor case, as well as in the one-sided case) and we have

Theorem 2. *The attractiveness condition in Theorem* 1 *may be replaced by the repulsiveness condition given above.*

Another variation is to consider the analogous discrete time setting. Unfortunately, just as in Liggett [13], our results do not carry over to all discrete time models which satisfy conditions analogous to those given in Theorems 1 and 2. See [13] for the reasons for the breakdown. As in [13], our techniques do apply to discrete time models which are one-sided or which have constant birth rates or constant death rates.

Finally, we conjecture that periodicity is not needed: it can be replaced by the condition that the flip rates be uniformly bounded away from 0 and ∞. We only used periodicity in the Lemma. Without periodicity, we can still prove that $\lim_{s \to \infty} P$ (at most two edges lie in $[-k, k]) = 1$ for all $k \geqq 0$. This is almost enough to carry out the rest of the proof, but our attempts fall just short.

Bibliography

1. Cirel'son, B.S.: Locally interacting systems and their application in biology. Lecture Notes in Mathematics No. **653**, 15–30. Berlin-Heidelberg-New York: Springer 1978
2. Durrett, R.: On the growth of one dimensional contact processes. Ann. Probab. **8**, 890–907 (1980)
3. Durrett, R.: An introduction to infinite particle systems. Stoch. Proc. Appl. **11**, 109–150 (1981)
4. Gach, P., Kurdymov, G.L., Levin, L.A.: One dimensional uniform arrays that wash out finite islands. Problemy Peredachi Informatsii **14**, 3, 92–96 (1978)
5. Gray, L.: Controlled spin systems. Ann. Probab. **6**, 953–974 (1978)
6. Gray, L., Griffeath, D.: A stability criterion for attractive nearest neighbor spin systems on $\mathbb{Z}$. Ann. Probab. **10**, 67–85 (1982)
7. Harris, T.E.: A correlation inequality for Markov processes in partially ordered state spaces. Ann. Probab. **5**, 451–454 (1977)
8. Harris, T.E.: Additive set-valued Markov processes and graphical methods. Ann. Probab. **6**, 355–378 (1978)
9. Holley, R.: An ergodic theorem for interacting systems with attractive interactions. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24**, 325–334 (1972)
10. Holley, R., Stroock, D.: Dual processes and their application to infinite interacting systems. Advances in Math. **32**, 149–174 (1979)
11. Kindermann, R., Snell, J.L.: Markov random fields and their applications. Contemporary Mathematics, vol. 1. Providence, R.I.: Amer. Math. Soc. Publications 1980
12. Liggett, T.M.: The stochastic evolution of infinite systems of interacting particles. Lecture Notes in Mathematics No. **598**, 187–248. Berlin-Heidelberg-New York: Springer 1977
13. Liggett, T.M.: Attractive nearest neighbor spin systems on the integers. Ann. Probability **6**, 629–636 (1978)
14. Phelps, R.: Lectures on Choquet's theorem. Van Nostrand Mathematical Studies No. 7. Princeton, New Jersey: van Nostrand 1966

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