Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 347-353 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Occurrence of Large Values in Stationary Sequences

G.L. O'Brien*

Dept. of Mathematics, York University, Downsview, Ontario, M3J 1P3, Canada

Summary. A direct proof is given of the Tanny (1974) result that for certain non-decreasing sequences $\{a_n\}$, it is true that $\limsup a_n^{-1} X_n = 0$ or $+\infty$ with probability one for all ergodic stationary sequences $\{X_n\}$. The condition on $\{a_n\}$ is shown to be necessary. For all non-decreasing $\{a_n\}$ and stationary $\{X_n\}$, $\limsup a_n^{-1} X_n = \limsup a_n^{-1} X_{-n}$ a.s. Similar continuous-time theorems are also given.

1. Introduction

Let $\{a_n\}$ be an eventually non-decreasing sequence of positive real numbers. Let $\{X_n\}$ be an ergodic strictly stationary sequence on a probability space $(\Omega, \mathscr{F}, P)$. Consider the following two equations:

$$\limsup \frac{X_n}{a_n} = \infty \text{ a.s.}; \tag{1}$$

$$\limsup \frac{X_n}{a_n} = 0 \text{ a.s.} \tag{2}$$

Here and throughout, all limits are taken to be "as $n \to \infty$". We will prove the following theorem:

Theorem 1. *One of* (1) *and* (2) *holds for every ergodic stationary sequence* $\{X_n\}$ *iff*

$$\liminf \frac{a_{rn}}{a_n} > 1 \quad \textit{for some } r > 0. \tag{3}$$

Theorem 1 is proved in Sect. 2. The main step in proving the sufficiency of (3) is Lemma 4, which deals with the case $a_n \equiv n$. The sufficiency was first

* This research supported in part by the Natural Sciences and Engineering Research Council of Canada

0044-3719/82/0061/0347/$01.40

obtained by Tanny (1974), whose proof involves the theory of branching processes in a random environment. One of our purposes is to give a direct proof. H. Furstenberg and S. Goldstein have independently obtained another proof which is similar but not identical to the one offered here. Aaronson and Tanny have found yet another variation based on Aaronson's (1981) result on large values for partial sums of stationary sequences. These proofs were communicated to me privately.

The part of Theorem 1 stating the necessity of (3) is new and we in fact show a stronger result, namely:

Corollary 1. *One of* (1) *and* (2) *holds for every independent identically distributed sequence* $\{X_n\}$ *only if* (3) *holds.*

Our proof of Theorem 1 is facilitated by first obtaining a zero-one law (Lemma 2), which is of some independent interest. We note that Tanny (1974) obtained this result as a *consequence* of Theorem 1.

In Sect. 3, we give several corrollaries of Theorem 1. We show that $\limsup a_n^{-1} X_n = \limsup a_n^{-1} X_{-n}$ a.s. for any positive increasing sequence $\{a_n\}$. Here, we have extended $\{X_n\}$ to a doubly infinite sequence $\{X_n, n = \ldots, -1, 0, 1, 2, \ldots\}$, as can (and frequently will) be done without loss of generality. Also, we obtain a continuous parameter analogue of Theorem 1.

The question of occurrence of large values in a stationary sequence was drawn to our attention by L. Arnold and V. Wihstutz in connection with their work (1981) on solutions of linear systems of equations with stationary noise.

We conclude this section with some notational conventions and preliminary remarks. First, none of (1), (2) or (3) is affected by changing finitely many a_n's, so we may assume without loss of generality that $\{a_n\}$ is nondecreasing and positive for all n. Likewise, multiplying all a_n's by a factor b_n which converges to 1 and for which $\{a_n b_n\}$ is increasing does not affect (1), (2) or (3), so we may assume $\{a_n\}$ is actually strictly increasing.

We use the phrase "infinitely often (i.o.)" to mean "for infinitely many positive n". If $P(X_n \leqq 0) = 1$, there is some $c < 0$ for which $P(X_n \geqq c \text{ i.o.}) = 1$ so that (2) must hold. In the complementary case, $P(X_n > 0 \text{ i.o.}) = 1$ and $\limsup a_n^{-1} X_n = \limsup a_n^{-1} \max(X_n, 0)$, so we may as well assume $P(X_n \geqq 0) = 1$.

The ergodicity assumption is essential only in a minor way. Without it, we only obtain the result that $P\left(\limsup \frac{X_n}{a_n} \in \{0, \infty\}\right) = 1$. All the results in this paper can be modified so as to apply to the non-ergodic case. In particular, Corollary 4 holds without modification in the non-ergodic case. The reader is referred to Breiman (1968) or Doob (1953) for background material on this question.

Condition (3) holds for a wide variety of sequences. These include all regularly varying sequences with positive exponent and all increasing sequences which yield convex sequences when raised to some positive power. Slowly varying sequences do not satisfy (3).

2. Proof of Theorem 1

We begin with an easy consequence of the Borel-Cantelli Lemma.

Lemma 1. *If* $EX_1 < \infty$, *then* $\limsup n^{-1} X_n = 0$ a.s. *If* $\{X_n\}$ *is an* i.i.d. *sequence and* $EX_1 = \infty$, *then* $\limsup n^{-1} X_n = \infty$ a.s.

The second statement can be generalized to the case of φ-mixing sequences, since the Borel-Cantelli Lemma holds in that case. See Iosefescu and Theodorescu (1969) or the author (1977). It cannot be generalized to the case of strongly-mixing sequences as can be seen from the Markov chain example in Sect. 3.

The proof of Theorem 1 is simplified by establishing the following 0–1 law, which Tanny (1974) obtained as a corollary of his main theorem (our Lemma 4). We give a more direct proof.

Lemma 2. *Let* $\{A_n\}$ *be a non-increasing sequence of measurable subsets of* $\mathbb{R}_1$. *Let* $\{X_n\}$ *be ergodic and stationary. Then* $P(X_n \in A_n \text{ i.o.}) = 0$ *or* 1.

Proof. The events $\{X_{n+k} \in A_n \text{ i.o.}\}$ form a non-decreasing sequence in the σ-algebra $\mathscr{F}$ as k increases. By stationarity, these sets have the same probability and hence, except for a null set in $\mathscr{F}$, the event $\{X_n \in A_n \text{ i.o.}\}$ is invariant with respect to the shift operator. By ergodicity, it has probability 0 or 1. □

If $b > 0$ and $\{a_n\}$ is non-decreasing, it follows from Lemma 2 that $\{X_n \geqq b a_n$ i.o.$\}$ has probability 0 or 1. Consequently we have the following result.

Lemma 3. *If* $\{a_n\}$ *is a non-decreasing positive sequence and* $\{X_n\}$ *is ergodic and stationary then* $\limsup a_n^{-1} X_n$ *is* a.s. *constant.*

Lemma 4. *If* $\{X_n\}$ *is ergodic and stationary, then*

$$\limsup n^{-1} X_n = 0 \text{ a.s. } or \limsup n^{-1} X_n = \infty \text{ a.s.} \tag{4}$$

Proof. By Lemma 3, $\limsup n^{-1} X_n = c$ a.s. for some $c \in [0, \infty]$. Assume $0 < c < \infty$. Define

$$U_n = \max\{(2c)^{-1} X_{n+k} - k,\ k = 0, 1, \ldots\}. \tag{5}$$

Clearly U_n is defined a.s. and the sequence $\{U_n\}$ is ergodic and stationary. Also, $U_{n-1} = \max\{(2c)^{-1} X_{n-1}, U_n - 1\}$ so that

$$(2c)^{-1} X_n \leqq U_n \leqq U_{n-1} + 1 \text{ a.s. for all } n. \tag{6}$$

For some $t \in \mathbb{R}$, $P(U_n \leqq t) > 0$ so that $P(U_n \leqq t \text{ i.o.}) = 1$. Let $n_1, n_2, \ldots$ be the random positive epochs for which $U_n \leqq t$, with $n_i < n_{i+1}$ for all i. Let $I_n = 1$ if $U_n \leqq t$ and $I_n = 0$ otherwise. Applying the ergodic theorem to the ergodic stationary sequence $\{I_n\}$, we see that

$$n_i^{-1} n_{i-1} \to 1 \text{ a.s. as } i \to \infty. \tag{7}$$

For $n_{i-1} \leqq n < n_i$, we deduce from (6) and (7) that

$$(2cn)^{-1}X_n \leqq n^{-1}U_n \leqq (U_{n_{i-1}} + n - n_{i-1})n^{-1} \leqq (t + n_i - n_{i-1})n_{i-1}^{-1} \to 0$$

as $i \to \infty$. This implies the first equation in (4) and thereby proves the lemma. □

We next prove Theorem 1. The sufficiency half of the proof, which involves extending Lemma 4 to the general case, is similar to that of Tanny (1974) and is included only for the sake of completeness.

Proof of the Sufficiency of (3). Assume (3) holds. Let f: $[0, \infty) \to [0, \infty)$ be a continuous increasing function such that $f(n) = a_n$ and such that there exists a $c > 1$ with

$$f(rx) \geqq cf(x) \tag{8}$$

for all x sufficiently large. Suppose $p = P(X_n \geqq c^{-1}df(n) \text{ i.o.}) > 0$ where $d > 0$. Then $p = 1$ by Lemma 2. By (8),

$$P(f^{-1}(d^{-1}X_n) \geqq f^{-1}(c^{-1}f(n)) \geqq r^{-1}n \text{ i.o.}) = 1.$$

Since $\{f^{-1}(d^{-1}X_n)\}$ is ergodic and stationary, it follows from Lemma 4 that

$$P(X_n \geqq df(n) \text{ i.o.}) = P(f^{-1}(d^{-1}X_n) \geqq n \text{ i.o.}) = 1.$$

Then (1) or (2) must hold since d is arbitrary. □

Proof of the Necessity of (3). Assume (3) fails. If $\{a_n\}$ converges, take $P(X_k = \lim a_n) = 1$ for all k. Then $\limsup a_n^{-1}X_n = 1$ a.s. We may therefore assume $\{a_n\}$ diverges. There exist numbers $n_0 = 0, n_1, n_2, \ldots$ such that $kn_{k-1} < n_k$ for all k and $a_{kn_{k-1}} < 2a_{n_{k-1}}$ for all k. Let H be a non-increasing continuous function such that $H(a) = (k^2 n_k)^{-1}$ for $a_{kn_{k-1}} \leqq a \leqq a_{(k+1)n_k - 1}$. Then

$$\sum_{n=0}^{\infty} H(a_n) = \sum_{k=1}^{\infty} (k^2 n_k)^{-1}[(k+1)n_k - kn_{k-1}]$$
$$\geqq \sum_{k=1}^{\infty} (k^2 n_k)^{-1} kn_k = \infty.$$

On the other hand,

$$\sum_{n=1}^{\infty} H(2a_n) = \sum_{k=1}^{\infty} \sum_{n=n_{k-1}+1}^{n_k} H(2a_n)$$
$$\leqq \sum_{k=1}^{\infty} (n_k - n_{k-1}) H(2a_{n_{k-1}})$$
$$\leqq \sum_{k=1}^{\infty} n_k H(a_{kn_{k-1}})$$
$$= \sum_{k=1}^{\infty} n_k (k^2 n_k)^{-1} < \infty.$$

Now let $\{X_n\}$ be an i.i.d. sequence with $P(X_n \geqq x) = H(x)$. By the Borel-Cantelli Lemma, $P(X_n \geqq a_n \text{ i.o.}) = 1$ but $P(X_n \geqq 2a_n \text{ i.o.}) = 0$, so that $1 \leqq \limsup a_n^{-1}X_n \leqq 2$ a.s. □

3. Further Results

In this section, we obtain several consequences of Theorem 1 by means of some simple transformations. The simplest of all is the following result obtained by applying the theorem to $Y_n = -X_n$.

Corollary 2. *Let $\{a_n\}$ be an eventually non-increasing sequence of positive numbers. Then*

$$P\left(\liminf \frac{X_n}{a_n} = -\infty\right) = 1 \quad \text{or} \quad P\left(\liminf \frac{X_n}{a_n} = 0\right) = 1 \tag{9}$$

for every ergodic stationary sequence $\{X_n\}$ iff (3) *holds.*

Let us now consider the sequences $\{a_n\}$ which are eventually nondecreasing but have a finite limit L. If $L \neq 0$, the choice $X_n \equiv L$ a.s. shows (1) or (2) need not hold. If $L = 0$, we obtain the following result by applying Theorem 1 to the stationary sequence $\{(X_n)^{-1}\}$.

Corollary 3. *Let $\{a_n\}$ be an eventually non-increasing sequence of positive real numbers. Then*

$$P\left(\liminf \frac{X_n}{a_n} = 0\right) = 1 \quad \text{or} \quad P\left(\liminf \frac{X_n}{a_n} = \infty\right) = 1$$

for every positive ergodic stationary sequence $\{X_n\}$ iff

$$\liminf \frac{a_n}{a_{rn}} > 1 \quad \text{for some } r > 0.$$

It is easy to show that the second statement of Lemma 1 does not extend to general stationary sequences. Let $\{X_n\}$ be the stationary Markov chain with statespace $\{1, 2, \ldots\}$ given by $P(X_n \geqq k) = k^{-1}$ for all k, $P(X_n = k+1 | X_{n-1} = k) = \frac{k}{k+2}$ and $P(X_n = 1 | X_{n-1} = k) = \frac{2}{k+2}$. Then $X_n \leqq X_0 + n$ a.s. for all n so $P(\limsup n^{-1} X_n \leqq 1) = 1$. Theorem 1 then shows that (2) holds, even though $EX_n = \infty$. Tanny (1974) gave a more elaborate argument that shows in effect that (2) is true for the stationary sequence $\{X_{-n}\}$. The fact that (2) holds in both cases is not coincidental, as the next corollary shows. We remark that this esthetically appealing result seems to be difficult to obtain without resorting to Theorem 1.

Corollary 4. *Let $\{X_n\}$ be an ergodic stationary sequence and let $\{a_n\}$ be an eventually non-decreasing sequence of positive real numbers. Then*

$$\limsup \frac{X_n}{a_n} = \limsup \frac{X_{-n}}{a_n} \text{ a.s.} \tag{10}$$

Proof. It is sufficient to consider the case when $\{a_n\}$ is nondecreasing throughout. Suppose (10) fails. By Lemma 3, each side of (10) is almost surely constant. Thus we may assume without loss of generality that $P(X_n \geqq a_n \text{ i.o.}) = 1$ while

$P(X_{-n} \geqq a_n \text{ i.o.})=0$. Now define a stationary sequence $\{Y_n\}$ by $Y_n=0$ if $X_{n-k}<a_k$ for all $k>0$ and

$$Y_n=\max\{k: X_{n-k} \geqq a_k\} \qquad \text{otherwise.} \tag{11}$$

If $Y_n=k$, then $X_{n-i}<a_i$ for $i>k$. Thus $X_{(n+1)-(i+1)}<a_{i+1}$ for $i>k$, so that $Y_{n+1} \leqq Y_n+1$. This implies that $\limsup n^{-1} Y_n=0$, by Theorem 1. On the other hand, $P(X_{n-\frac{1}{2}n} \geqq a_{\frac{1}{2}n} \text{ i.o.})=1$ so that $P(Y_n \geqq \frac{1}{2}n \text{ i.o.})=1$. This contradiction gives us the required result. □

Our final application of Theorem 1 is a continuous time version of the same theorem.

Corollary 5. *Let $f\colon [0,\infty)\to\mathbb{R}$ be an eventually non-decreasing positive function. Then*

$$\limsup \frac{X_t}{f(t)}=\infty \quad \text{a.s.,} \quad \textit{or} \tag{12}$$

$$\limsup \frac{X_t}{f(t)}=0 \quad \text{a.s.} \tag{13}$$

for all ergodic stationary processes $\{X_t, t\in\mathbb{R}\}$ iff

$$\liminf_{t\to\infty} \frac{f(rt)}{f(t)}>1 \qquad \textit{for some } r>0. \tag{14}$$

Proof. Assume (14) holds. For $\{X_t\}$ as in the statement of the corollary, define

$$Y_n=\sup\{X_t: n \leqq t<n+1\}.$$

Then $\{Y_n\}$ is stationary and ergodic. (This fact is of course weaker than the ergodicity of $\{X_t\}$.) Let $a_n=f(n)$. The sequence $\{a_n\}$ is eventually non-decreasing and positive and (3) holds. Let $[\cdot]$ denote the greatest integer function. Then

$$\begin{aligned}\limsup \frac{Y_n}{a_{n+1}} &= \limsup \frac{X_t}{f([t]+1)} \\ &\leqq \limsup \frac{X_t}{f(t)} \\ &\leqq \limsup \frac{X_t}{f([t])} \\ &= \limsup \frac{Y_n}{a_n}.\end{aligned} \tag{15}$$

The sufficiency result now follows from Theorem 1 and the fact that the first and last terms in (15) have the same distributions by stationarity. The necessity can be concluded from a minor modification of the proof of the discrete time theorem. For example, let $\{X_t\}$ be the pure jump Markov process with mean time in every state being one and with the conditional distribution of the location after a jump being given by the distribution of X_n as specified in the necessity part of the proof of Theorem 1, no matter what state occurred prior to the jump. □

We note that Corollaries 2, 3 and 4 have similar continuous-time versions.

Acknowledgement. I am indebted to H. Kesten, D. Tanny, and W. Vervaat for several useful comments about my manuscript.

References

1. Aaronson, J.: An ergodic theorem with large normalizing constants. Israel J. Math. **38**, 182–188 (1981)
2. Arnold, L., Wihstutz, V.: Stationary solutions of linear systems with additive and multiplicative noise. [To appear (1981)]
3. Breiman, L.: Probability. Reading, Massachusetts: Addison-Wesley 1968
4. Doob, J.L.: Stochastic processes. New York: Wiley 1953
5. Iosifescu, M., Theodorescu, R.: Random processes and learning. Berlin-Heidelberg-New York: Springer 1961
6. O'Brien, G.L.: Path properties of successive sample minima from stationary processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **38**, 313–327 (1977)
7. Tanny, D.: A zero-one law for stationary sequences. Z. Wahrscheinlichkeitstheorie verw. Gebiete **30**, 139–148 (1974)

Received November 28, 1981; in revised form May 25, 1982