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On a Condition Satisfied by Certain Random Walks

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Introduction

Let $X_1, X_2, \ldots$ etc. be a sequence of independent, identically distributed random variables. Define successive partial sums of these random variables

$$S_k = \sum_{j=1}^{k} X_j \quad \text{for } k \geqq 1.$$

The purpose of this paper is to elucidate the circumstances under which the following condition is satisfied

$$n^{-1} \cdot \sum_{k=1}^{n} \Pr(S_k \leqq 0) \to \gamma \quad \text{as } n \to \infty,$$

where $0 < \gamma < 1$. This condition has been used by a number of authors, see for instance Spitzer [9], Heyde [6], Bingham [1] and Emery [2].

Preliminary Discussion

When X_1 etc. have a symmetric common distribution, i.e. when $\Pr(X_1 \geqq x) = \Pr(X_1 < -x)$ at all continuity points x, then it is easy to show that the condition is satisfied with $\gamma = \frac{1}{2}$. If X_1 etc. all have zero mean and finite second moment, then it follows from the Central Limit Theorem that

$$\Pr(S_n \leqq 0) \to \tfrac{1}{2} \quad \text{as } n \to \infty,$$

and thus again the condition is satisfied with $\gamma = \frac{1}{2}$.

The condition is also satisfied when X_1 etc. belong to the domain of attraction of a stable law of exponent $\alpha \neq 1$, with the additional restriction that the mean of X_1 etc. is zero when $\alpha > 1$, and no additional restriction when $\alpha < 1$. Under these circumstances there exists a sequence $A_1, A_2, \ldots$ etc. of constants such S_n/A_n converges in distribution to a stable random variable (see [4], p. 75, Theorem 2 and footnote). Thus it follows that

$$\Pr(S_n \leqq 0) = \Pr(S_n/A_n \leqq 0) \to \gamma \quad \text{as } n \to \infty.$$

If the characteristic function of the stable limit law is

$$\exp\left\{-c\,|t|^{\alpha}\left(1 + i\beta \cdot \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right\},$$

where t is rea and α, β and c are real constants, satisfying $0<\alpha<1$ or $1<\alpha\leqq 2$, $c>0$ and $-1\leqq\beta\leqq 1$, then

$$\gamma=\tfrac{1}{2}-(\pi\alpha)^{-1}\cdot\arctan\left(-\beta\tan\left(\frac{\pi\alpha}{2}\right)\right).$$

Another class of random variables satisfying the condition can be constructed from the symmetric Cauchy distribution. Let $Y_1, Y_2, \ldots$ etc. be a sequence of independent, identically distributed random variables such that

$$\Pr(Y_1<x)=\frac{1}{\pi}\int_{-\infty}^{x}\frac{du}{1+u^2}=\frac{1}{2}+\frac{1}{\pi}\arctan x.$$

If $R_k=\sum_{j=1}^{k} Y_j$ for $k\geqq 1$ denotes the partial sums of this sequence, then it is an elementary property of the symmetric Cauchy distribution that R_k/k has the same distribution as Y_1 etc. Thus if $X_k=Y_k-a$ for $k\geqq 1$, where a is a constant, then

$$\begin{aligned}\Pr(S_n\leqq 0)&=\Pr(R_n-na\leqq 0)=\Pr(R_n\leqq na)\\&=\Pr\left(\frac{R_n}{n}<a\right)=\Pr(Y_1<a)=\frac{1}{2}+\frac{1}{\pi}\arctan a.\end{aligned}$$

By varying a the condition can be satisfied with any value of γ between 0 and 1.

Spitzer has given an example which shows that the limit $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\Pr(S_k\leqq 0)$ need not exist (see [10] p. 230, $E2$). It is easy to see that the ordinary limit $\lim_{n\to\infty}\Pr(S_n\leqq 0)$ need not exist by considering, as Spitzer [9] suggests, the Universal Laws of Doeblin. It is not known, to the author, whether $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\Pr(S_k\leqq 0)$ can exist whilst $\lim_{n\to\infty}\Pr(S_n\leqq 0)$ fails to exist.

An examination of the examples given above might lead one to think that random variables in the domain of attraction of a stable law are the only ones which satisfy the condition with $\gamma\neq\frac{1}{2}$. This is not so, for, as will be shown, certain random variables whose distributions have slowly varying tails satisfy the condition with $\gamma\neq\frac{1}{2}$. If, however, it is known that, in addition to satisfying the condition with $\gamma\neq\frac{1}{2}$, one tail of the distribution is exponentially small, then it does indeed follow that the random variables belong to the domain of attraction of a (completely asymmetric) stable law.

Demonstration of Results

Denote the characteristic function of X_1 etc. by $\phi(t)$, thus

$$\phi(t)=\int_{-\infty}^{+\infty} e^{itx}\Pr(X_1\in dx).$$

Then $\phi^k(t)$ is the characteristic function of S_k, and let $u_k(t)$ and $v_k(t)$ denote respectively the real and imaginary parts of $\phi^k(t)$. Where convienient we shall write $u(t)$ and $v(t)$ instead of $u_1(t)$ and $v_1(t)$.

Lemma 1. *If X_1 etc. are non-degenerate, then for every $\varepsilon>0$*

$$\Pr(S_k>0)-\Pr(S_k<0)=\frac{2}{\pi}\lim_{\delta\downarrow 0}\int_\delta^\varepsilon \frac{v_k(t)}{t}\,dt+O(k^{-\frac{1}{4}}).$$

(*Note that* $\int_0^\varepsilon \frac{v_k(t)}{t}\,dt$ *does not necessarily exist in an absolute sense, but* $\lim_{\delta\downarrow 0}\int_\delta^\varepsilon \frac{v_k(t)}{t}\,dt$ *always exists.*)

Proof of Lemma 1. We have that

$$\int_\delta^\varepsilon \frac{v_k(t)}{t}\,dt=\int_\delta^\varepsilon\left\{\int_{-\infty}^{+\infty}\sin xt\cdot\Pr(S_k\in dx)\right\}\frac{dt}{t}$$
$$=\int_{-\infty}^{+\infty}\left\{\int_\delta^\varepsilon \frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx).$$

The interchange of orders of integration is justified because the integrand is bounded over the region of integration. Thus

$$\int_\delta^\varepsilon \frac{v_k(t)}{t}\,dt=\int_{-\infty}^{+\infty}\left\{\int_{\delta\cdot x}^{\varepsilon\cdot x}\frac{\sin\theta}{\theta}\,d\theta\right\}\Pr(S_k\in dx).$$

Now since $\left|\int_a^b \frac{\sin\theta}{\theta}\,d\theta\right|<A$, where A is a constant independent of a and b, it follows by the Dominated Convergence Theorem that

$$\int_\delta^\varepsilon \frac{v_k(t)}{t}\,dt\to\int_{-\infty}^{+\infty}\left\{\int_0^{\varepsilon\cdot x}\frac{\sin\theta}{\theta}\,d\theta\right\}\Pr(S_k\in dx)$$

as $\delta\downarrow 0$. Hence

$$\lim_{\delta\downarrow 0}\int_\delta^\varepsilon \frac{v_k(t)}{t}\,dt=\int_{-\infty}^{+\infty}\left\{\int_0^\varepsilon \frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx)$$
$$=\int_{-\infty}^{+\infty}\left\{\int_0^\infty \frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx)-\int_{-\infty}^{+\infty}\left\{\int_\varepsilon^\infty \frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx).$$

Since

$$\int_0^\infty \frac{\sin xt}{t}\,dt=\begin{cases}+\pi/2 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -\pi/2 & \text{if } x<0,\end{cases}$$

it follows that

$$\Pr(S_k>0)-\Pr(S_k<0)=\frac{2}{\pi}\lim_{\delta\downarrow 0}\int_0^\varepsilon \frac{v_k(t)}{t}\,dt+\frac{2}{\pi}\int_{-\infty}^{+\infty}\left\{\int_\varepsilon^\infty \frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx).$$

Choose $n_k > 0$ and estimate as follows

$$\left|\int_{-\infty}^{+\infty}\left\{\int_{\varepsilon}^{\infty}\frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx)\right|$$
$$\leqq \int_{|x|\geqq n_k}\left|\int_{\varepsilon\cdot x}^{\infty}\frac{\sin\theta}{\theta}\,d\theta\right|\Pr(S_k\in dx)+\int_{|x|<n_k}\left|\int_{\varepsilon\cdot x}^{\infty}\frac{\sin\theta}{\theta}\,d\theta\right|\Pr(S_k\in dx)$$
$$\leqq \int_{|x|\geqq n_k}\frac{B}{|x|}\Pr(S_k\in dx)+\int_{|x|<n_k}A\cdot\Pr(S_k\in dx).$$

Here we are making use of the fact that $\left|\int_{\varepsilon\cdot x}^{\infty}\frac{\sin\theta}{\theta}\,d\theta\right|<A$ for some constant A, and $\left|\int_{\varepsilon\cdot x}^{\infty}\frac{\sin\theta}{\theta}\,d\theta\right|<\frac{B}{|x|}$ for some constant B. Hence

$$\left|\int_{-\infty}^{+\infty}\left\{\int_{\varepsilon}^{\infty}\frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx)\right|\leqq\frac{B}{n_k}+A\cdot\Pr(|S_k|<n_k).$$

Setting $n_k=\frac{1}{2}k^{\frac{1}{4}}$, it now follows from Theorem 1 (a) of Rosén [8] that

$$\left|\int_{-\infty}^{+\infty}\left\{\int_{\varepsilon}^{\infty}\frac{\sin xt}{t}\,dt\right\}\Pr(S_k\in dx)\right|=O(k^{-\frac{1}{4}}),$$

and the lemma follows.

Lemma 2. *If* $\frac{v(t)}{t}\in L_1(0,\varepsilon)$, *then*

$$n^{-1}\cdot\sum_{k=1}^{n}\{\Pr(S_k>0)-\Pr(S_k<0)\}$$

tends to a limit as $n\to\infty$ *if and only if*

$$\frac{2}{\pi}\int_0^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}$$

tends to a limit as $s\uparrow 1$, *and the two are equal.*

Proof of Lemma 2. First we observe that

$$v_k(t)=\operatorname{Im}\phi^k(t)=\operatorname{Im}(u(t)+i\,v(t))\,(u_{k-1}(t)+i\,v_{k-1}(t))$$
$$=u(t)\,v_{k-1}(t)+v(t)\,u_{k-1}(t),$$

and therefore

$$|v_k(t)|\leqq|v_{k-1}(t)|+|v(t)|,$$

and so it follows, by induction, that $|v_k(t)|\leqq k\,|v(t)|$. Thus $\frac{v_k(t)}{t}\in L_1(0,\varepsilon)$ for $k\geqq 1$, and hence by Lemma 1

$$\Pr(S_k>0)-\Pr(S_k<0)=\frac{2}{\pi}\int_0^{\varepsilon}\frac{v_k(t)}{t}\,dt+O(k^{-\frac{1}{4}}).$$

Multiplying through by s, where $0<s<1$, and summing we obtain

$$\sum_{k=1}^{\infty} s^k \{\Pr(S_k>0)-\Pr(S_k<0)\} = \frac{2}{\pi} \sum_{k=1}^{\infty} s^k \int_0^{\varepsilon} \frac{v_k(t)}{t}\, dt + \sum_{k=1}^{\infty} O(k^{-\frac{1}{4}})\, s^k$$

$$= \frac{2}{\pi} \int_0^{\varepsilon} \left\{ \sum_{k=1}^{\infty} s^k\, v_k(t) \right\} \frac{dt}{t} + \sum_{k=1}^{\infty} O(k^{-\frac{1}{4}})\, s^k,$$

where the interchange of orders of integration and summation may be justified by using $|v_k(t)| \leq k\, |v(t)|$ and appealing to the Dominated Convergence Theorem. Since

$$\operatorname{Im} \sum_{k=1}^{\infty} s^k\, \phi^k(t) = \operatorname{Im} \sum_{k=0}^{\infty} s^k\, \phi^k(t) = \operatorname{Im} \frac{1}{1-s\,\phi(t)},$$

we conclude that

$$(1-s) \sum_{k=1}^{\infty} s^k \{\Pr(S_k>0)-\Pr(S_k<0)\}$$

$$= \frac{2}{\pi} \int_0^{\varepsilon} \operatorname{Im} \left(\frac{1-s}{1-s\,\phi(t)} \right) \frac{dt}{t} + (1-s) \sum_{k=1}^{\infty} O(k^{-\frac{1}{4}})\, s^k$$

for $0<s<1$.

The result now follows by well-known Abelian and Tauberian theorems (see [5] Theorems 55 and 92).

Lemma 3. *Suppose that* $\phi(t)$ *is such that*

(i) $v(t) \geqq 0$ *for* $0 \leqq t \leqq \varepsilon$ *where* $\varepsilon>0$,

and

(ii) $\dfrac{v(t)}{1-u(t)} \to 0$ *as* $t \downarrow 0$,

then

$$n^{-1} \sum_{k=1}^{n} \{\Pr(S_k>0)-\Pr(S_k<0)\}$$

tends to a limit as $n \to \infty$ *if and only if*

$$\frac{2}{\pi} \int_0^{\varepsilon} \frac{1-s}{(1-s\,u(t))^2} \cdot \frac{v(t)}{t}\, dt$$

tends to a limit as $s \uparrow 1$, *and the two are equal.* (*It is not obvious that there exists any characteristic function which satisfies* (i) *and* (ii). *We postpone discussion of this point until after the proof of the lemma.*)

Proof of Lemma 3. Since $\lim_{\delta \downarrow 0} \int_{\delta}^{\varepsilon} \frac{v(t)}{t}\, dt$ always exists, (i) implies that $\frac{v(t)}{(t)} \in L_1(0, \varepsilon)$. Thus Lemma 2 is in force, and we need only show that

$$\lim_{s \uparrow 1} \frac{2}{\pi} \int_0^{\varepsilon} \operatorname{Im} \left(\frac{1-s}{1-s\,\phi(t)} \right) \frac{dt}{t}$$

exists if and only if

$$\lim_{s\uparrow 1}\frac{2}{\pi}\int_0^{\varepsilon}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\,dt$$

exists, and the two are equal. Note that

$$\operatorname{Im}\frac{1}{1-s\,\phi(t)}=\frac{s\,v(t)}{(1-s\,u(t))^2+s^2\,v^2(t)},$$

and define

$$\rho(\varepsilon)=\sup_{0<t\leqq\varepsilon}\frac{v(t)}{1-u(t)}.$$

Assume that X_1 etc. are non-degenerate, so that there exists $\varepsilon>0$ such that $u(t)\neq 1$ for $0<t\leqq\varepsilon$. Since $u(t)$ is continuous, $1-u(t)$ is bounded away from 0 for $0<\varepsilon'\leqq t\leqq\varepsilon$ (by a suitable choice of ε we may assume the same value of ε throughout). Thus $\operatorname{Im}\frac{1}{1-s\,\phi(t)}$ is bounded above for $0<\varepsilon'\leqq t\leqq\varepsilon$ and $0<s\leqq 1$, and therefore

$$\frac{2}{\pi}\int_{\varepsilon'}^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\cdot\frac{dt}{t}\to 0 \quad\text{as}\quad s\uparrow 1,$$

and

$$\frac{2}{\pi}\int_{\varepsilon'}^{\varepsilon}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\,dt\to 0 \quad\text{as}\quad s\uparrow 1.$$

Thus it follows that

$$\overline{\lim_{s\uparrow 1}}\int_0^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}=\overline{\lim_{s\uparrow 1}}\int_0^{\varepsilon'}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t},$$

and

$$\overline{\lim_{s\uparrow 1}}\int_0^{\varepsilon}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\,dt=\overline{\lim_{s\uparrow 1}}\int_0^{\varepsilon'}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\,dt.$$

In other words these limits have values that are independent of ε so long as ε is sufficiently small.

Now, for $0<t<\varepsilon$ and $0<s<1$, we have

$$\frac{s(1-s)\,v(t)}{(1-s\,u(t))^2+s^2\,v^2(t)}\leqq\frac{s(1-s)\,v(t)}{(1-s\,u(t))^2},$$

and

$$\frac{s(1-s)\,v(t)}{(1-s\,u(t))^2}\leqq\frac{(1+\rho^2(\varepsilon))\,s(1-s)\,v(t)}{(1-s\,u(t))^2+s^2\,v^2(t)}.$$

Hence

$$\overline{\lim_{s\uparrow 1}}\frac{2}{\pi}\int_0^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}\leqq\overline{\lim_{s\uparrow 1}}\frac{2}{\pi}\int_0^{\varepsilon}\frac{1-s}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\cdot dt,$$

and

$$\overline{\lim_{s\uparrow 1}}\frac{2}{\pi}\int_0^{\varepsilon}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\cdot dt\leqq(1+\rho^2(\varepsilon))\overline{\lim_{s\uparrow 1}}\frac{2}{\pi}\int_0^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}.$$

From (ii) $\rho(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$, and therefore since the values of these limits are independent of ε, it follows that

$$\overline{\lim_{s\uparrow 1}}\,\frac{2}{\pi}\int_0^{\varepsilon}\mathrm{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}=\overline{\lim_{s\uparrow 1}}\,\frac{2}{\pi}\int_0^{\varepsilon}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\cdot dt,$$

and similarly

$$\underline{\lim_{s\uparrow 1}}\,\frac{2}{\pi}\int_0^{\varepsilon}\mathrm{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}=\underline{\lim_{s\uparrow 1}}\,\frac{2}{\pi}\int_0^{\varepsilon}\frac{(1-s)}{(1-s\,u(t))^2}\cdot\frac{v(t)}{t}\,dt,$$

so that the lemma is established.

Let $L(x)$ be a bounded non-negative function defined for positive x, which is slowly varying at infinity, and such that $L(x)/x$ is non-increasing and $\int_0^{\infty}\frac{L(x)}{x}\,dx<+\infty$. One could, for instance, take $L(x)=\min\left(x,(\log_e x)^{-2}\right)$. If we set $K(x)=\int_x^{\infty}\frac{L(u)}{u}\,du$, then $K(x)$ is slowly varying at infinity and

$$\frac{L(x)}{K(x)}\to 0 \quad\text{as}\quad x\to+\infty.$$

For if we put

$$\frac{K(x)}{L(x)}=\frac{1}{L(x)}\int_x^{\infty}\frac{L(u)}{u}\,du=\int_1^{\infty}\frac{L(x\,t)}{L(x)}\cdot\frac{dt}{t},$$

then it follows by Fatou's Lemma that

$$\underline{\lim_{x\to+\infty}}\,\frac{K(x)}{L(x)}\geqq\int_1^{\infty}\underline{\lim_{x\to\infty}}\,\frac{L(x\,t)}{L(x)}\cdot\frac{dt}{t}=\int_1^{\infty}\frac{dt}{t}=+\infty.$$

To show that $K(x)$ is slowly varying, choose any positive λ which is less than 1, and note that

$$\frac{K(\lambda\,x)}{K(x)}-1=\left(\frac{K(\lambda\,x)-K(x)}{L(x)}\right)\cdot\frac{L(x)}{K(x)}.$$

Since $\frac{L(x)}{K(x)}\to 0$ as $x\to\infty$ the result will follow if we can show that $\frac{K(\lambda x)-K(x)}{L(x)}$ remains bounded as $x\to\infty$. This is easily demonstrated using that fact that $L(x)/x$ is non-increasing, for

$$\begin{aligned}\frac{K(\lambda x)-K(x)}{L(x)}&=\frac{1}{L(x)}\int_{\lambda x}^{x}\frac{L(u)}{u}\,du\\&\leqq\frac{1}{L(x)}\cdot\frac{L(\lambda x)}{\lambda\,x}\cdot\int_{\lambda x}^{x}du=\frac{L(\lambda\,x)}{L(x)}\left(\frac{1-\lambda}{\lambda}\right),\end{aligned}$$

and the latter is bounded because $\frac{L(\lambda\,x)}{L(x)}\to 1$ as $x\to\infty$. The case of $\lambda>1$ is treated in a like manner.

Let us assume that $L(x)$ contains a suitable scaling factor so that $\int_0^\infty \frac{L(x)}{x} dx = 1$. Then we may consider $\frac{L(x)}{x}$ to be the probability density function of the set of random variables X_1 etc. (in this case, of course, X_1 etc. will be positive). In this case

$$\phi(t) = \int_0^\infty e^{itx} \cdot \frac{L(x)}{x} \cdot dx,$$

$$u(t) = \int_0^\infty \cos xt \cdot \frac{L(x)}{x} dx, \quad \text{and} \quad v(t) = \int_0^\infty \sin xt \cdot \frac{L(x)}{x} \cdot dx.$$

The conditions of Theorem 2 of Pitman's paper [7] are satisfied and therefore $1-u(t) \sim K\left(\frac{1}{t}\right)$ as $t \downarrow 0$. We shall show later that $v(t) \sim \frac{\pi}{2} L(1/t)$ as $t \downarrow 0$. Thus (i) and (ii) of lemma 3 are satisfied and so, because X_1 etc. are positive, it follows that

$$\frac{2}{\pi} \int_0^\varepsilon \frac{(1-s)}{(1-su(t))^2} \cdot \frac{v(t)}{t} dt \to 1 \quad \text{as } s \uparrow 1.$$

If $X_1^*, X_2^*, \ldots$ etc. is a sequence of independent, identically distributed random variables with a probability density function equal to $\mu \frac{L(x)}{x}$ for $x > 0$, and $(1-\mu) \frac{L(-x)}{-x}$ for $x < 0$, where μ is a constant satisfying $\frac{1}{2} < \mu < 1$, then $u^*(t)$ and $v^*(t)$, the real and imaginary parts of the characteristic function of X_1^* etc. satisfy $u^*(t) = u(t)$ and $v^*(t) = (2\mu - 1) v(t)$. Hence

$$\frac{2}{\pi} \int_0^\varepsilon \frac{(1-s)}{(1-su^*(t))^2} \frac{v^*(t)}{t} dt = (2\mu-1) \frac{2}{\pi} \int_0^\varepsilon \frac{(1-s)}{(1-su(t))^2} \frac{v(t)}{t} dt \to 2\mu - 1 \quad \text{as } s \uparrow 1.$$

Now $u^*(t)$ *and* $v^*(t)$ also satisfy (i) and (ii), and so applying Lemma 3, we have that

$$n^{-1} \sum_{k=1}^{n} \{\Pr(S_k^* > 0) - \Pr(S_k^* < 0)\} \to 2\mu - 1 \quad \text{as } n \to +\infty,$$

where S_k^* denotes the kth partial sum of the sequence $X_1^*, X_2^*, \ldots$ etc. Using Theorem 1(d) of Rosén [8] it follows easily that the condition is satisfied with $\gamma = 1 - \mu$. Since the distribution of X_1^* etc. has a slowly varying tail it cannot belong to the domain of attraction of a stable law.

It remains to show that $v(t) \sim \frac{\pi}{2} L(1/t)$ as $t \downarrow 0$. Since $\frac{L(x)}{x}$ is non-increasing we have that

$$\int_0^\infty \sin tx \cdot \frac{L(x)}{x} \cdot dx \leqq \int_0^{(2n+1)\pi/t} \sin tx \cdot \frac{L(x)}{x} dx,$$

and

$$\int_0^\infty \sin tx \cdot \frac{L(x)}{x} dx \geqq \int_0^{2n\pi/t} \sin tx \cdot \frac{L(x)}{x} dx,$$

where n is any positive integer. Hence

$$\frac{v(t)}{L(1/t)} \leqq \int_0^{(2n+1)\pi} \frac{\sin u}{u} \cdot \frac{L(u/t)}{L(1/t)} \cdot du,$$

and

$$\frac{v(t)}{L(1/t)} \geqq \int_0^{2\pi n} \frac{\sin u}{u} \cdot \frac{L(u/t)}{L(1/t)} \cdot du.$$

Now by Lemma 2 of Pitman [7], we have that for ε positive and t sufficiently small, there exist a positive constant C such that

$$\frac{L(u/t)}{L(1/t)} < C u^{-\frac{1}{2}} \quad \text{when } u \leqq \varepsilon,$$

and

$$\frac{L(u/t)}{L(1/t)} < C u^{\frac{1}{2}} \quad \text{when } u > \varepsilon.$$

Thus, for t sufficiently small, $\frac{\sin u}{u} \cdot \frac{L(u/t)}{L(1/t)}$ is bounded by a function which is integrable over any finite interval, and consequently

$$\overline{\lim_{t \downarrow 0}} \frac{v(t)}{L(1/t)} \leqq \int_0^{(2n+1)\pi} \frac{\sin u}{u} \cdot du,$$

and

$$\underline{\lim_{t \downarrow 0}} \frac{v(t)}{L(1/t)} \geqq \int_0^{2n} \frac{\sin u}{u} \cdot du.$$

Since n is arbitrary it follows that

$$\frac{v(t)}{L(1/t)} \to \int_0^{\infty} \frac{\sin u}{u} du = \pi/2$$

as $t \downarrow 0$. This concludes the proof of Theorem 1.

Theorem 1. *There exist sequences of independent, identically distributed random variables which satisfy the condition with $\gamma \neq \frac{1}{2}$, and which do not belong to the domain of attraction of a stable law.*

We now state the other main result of this paper.

Theorem 2. *Suppose that X_1 etc. are such that $\int_{-\infty}^{+\infty} e^{-\eta x} \Pr(X_1 \in dx) < +\infty$ for all real η satisfying $0 \leqq \eta < \eta_0$, where η_0 is a positive constant (possibly $+\infty$). Then if the condition is satisfied with $\gamma \neq \frac{1}{2}$ it follows that X_1 etc. belong to the domain of attraction of a (completely asymmetric, of exponent $1/\gamma$) stable law.*

Notice that the condition $\int_{-\infty}^{+\infty} e^{-\eta x} \Pr(X_1 \in dx) < +\infty$ for $\eta > 0$ is equivalent to requiring that the left hand tail of the distribution of X_1 etc. is exponentially small. This theorem is clearly a generalization of Theorem 2 of [2].

Since $\phi(t)=\int_{-\infty}^{+\infty} e^{itx}\Pr(X_1\in dx)$, the condition of Theorem 2 implies that $\phi(t)$ is analytic for t satisfying $0<\operatorname{Im} t<\eta_0$, and is bounded and continuous for $0\leqq \operatorname{Im} t\leqq \eta_1$, where η_1 is any positive real number satisfying $\eta_1<\eta_0$. We shall henceforth regard t as a complex variable.

Lemma 4. *There exist $\varepsilon>0$ and $\delta>0$ such that for all s satisfying $1-\delta<s<1$, the equation $1-s\phi(t)=0$ has at most a single, simple root in the closed half-disc determined by $|t|\leqq\varepsilon$ and $\operatorname{Im} t\geqq 0$.*

Proof of Lemma 4. We make use of notation and results established in Feller Vol. II Chapters XII and XVIII [3]. Let $N=\min\{n|S_n\leqq 0\}$ be the epoch of first entry into $(-\infty,0]$, and let S_N be the place of first entry. We put $H_n(I)=\Pr\{N=n;\ S_N\in I\}$ where I is any interval contained in $(-\infty,0]$, and $H_n(I)=0$ when I is contained in $(0,+\infty)$. We also define

$$G_n(I)=\Pr\{S_1>0,\dots,S_{n-1}>0;\ S_n\in I\}$$

when I is contained in $(0,+\infty)$, and $G_n(I)=0$ when I is contained in $(-\infty,0]$. For convienience put $F(I)=\Pr(X_1\in I)$ for any interval I.

By Eq. (1.5a) of Chapter XVIII

$$H_{n+1}(I)=\int_0^\infty G_n(dy)\,F(I-y).$$

Hence if $0\leqq\eta<\eta_0$, then for every positive c

$$\begin{aligned}\int_{-c}^{0} e^{-\eta x}H_{n+1}(dx)&=\int_0^\infty G_n(dy)\cdot\int_{-c}^{0} e^{-\eta x}F(dx-y)\\ &=\int_0^\infty e^{-\eta y}G_n(dy)\cdot\int_{-y-c}^{-y} e^{-\eta u}F(du)\leqq\int_0^\infty e^{-\eta y}G_n(dy)\cdot\int_{-\infty}^{0} e^{-\eta u}F(du).\end{aligned}$$

Therefore, by the Monotone Convergence Theorem, $\int_{-\infty}^{0} e^{-\eta x}H_{n+1}(dx)$ exists for $0\leqq\eta<\eta_0$, and

$$\int_{-\infty}^{0} e^{-\eta x}H_{n+1}(dx)\leqq\int_0^\infty e^{-\eta y}G_n(dy)\cdot\int_{-\infty}^{0} e^{-\eta u}F(du).$$

The duality lemma of Chapter XII, Section 2 shows that $G_n(I)$ can be interpreted as the probability that the nth ascending ladder point belongs to the interval I, and consequently $G(I)=\sum_{n=0}^{\infty} G_n(I)$ is a renewal measure. Hence

$$\sum_{n=0}^{\infty}\int_0^\infty e^{-\eta y}G_n(dy)=\int_0^\infty e^{-\eta y}G(dy)<+\infty$$

for every $\eta>0$. Thus it follows that, for $0<\eta<\eta_0$, the sum

$$\sum_{n=1}^{\infty}\int_{-\infty}^{0} e^{-\eta x}H_n(dx)$$

is finite.

If we put

$$\chi(s,t)=\sum_{n=1}^{\infty} s^n \int_{-\infty}^{0} e^{itx} H_n(dx),$$

and

$$\gamma(s,t)=\sum_{n=0}^{\infty} s^n \int_{0}^{\infty} e^{itx} G_n(dx),$$

then it follows that $\chi(s,t)$ is analytic for $-\infty<\operatorname{Im} t<\eta_0$ when $0\leqq s\leqq 1$, and $\gamma(s,t)$ is analytic for $\operatorname{Im} t>0$ when $0\leqq s\leqq 1$. Further, if $\operatorname{Im} t\leqq\eta_1<\eta_0$ and $0\leqq s\leqq 1$, then the series defining $\chi(s,t)$ is dominated term-wise by the convergent series

$$\sum_{n=1}^{\infty} \int_{-\infty}^{0} e^{-\eta_1 x} H_n(dx),$$

and therefore $\chi(s,t)$ tends uniformly to $\chi(1,t)$ as $s\uparrow 1$ for $\operatorname{Im} t\leqq\eta_1$.

The equation $1-\chi(1,t)$ has a root at $t=0$. Since

$$\chi'(1,0)=i\sum_{n=1}^{\infty} \int_{-\infty}^{0} x H_n(dx)\neq 0$$

this root is simple. Because $\chi(1,t)$ is analytic in the neighbourhood of $t=0$, we can find $\varepsilon>0$ such that the equation $1-\chi(1,t)=0$ has no root but $t=0$ which satisfies $|t|\leqq\varepsilon$.

Let $m=\inf_{|t|=\varepsilon}|1-\chi(1,t)|$, so that $m>0$. By choosing s sufficiently close to 1 we can ensure that $|\chi(s,t)-\chi(1,t)|<\frac{1}{2}m$ for $|t|=\varepsilon$. Applying Rouche's Theorem we find that $1-\chi(s,t)$ and $1-\chi(1,t)$ have the same number of zeros within the circle $|t|=\varepsilon$. Hence for s sufficiently close to 1, the equation $1-\chi(s,t)=0$ has exactly one simple root within the circle $|t|=\varepsilon$.

By Chapter XVIII, Eq. (1.9)

$$\gamma(s,t)\{1-s\phi(t)\}=1-\chi(s,t)$$

for real t, and $0<s<1$. Now $\chi(s,t)$, $\gamma(s,t)$ and $\phi(t)$ are all defined for $0\leqq\operatorname{Im} t<\eta_0$, and therefore the above equation holds throughout this strip. It follows that the equation $1-s\phi(t)=0$ has at most a single, simple root satisfying $|t|\leqq\varepsilon$ and $\operatorname{Im} t\geqq 0$, when s is sufficiently close to 1. Thus the lemma is proved.

Proof of Theorem 2. We put $X_j=X_j^+ + X_j^-$, where $X_j^+=\max\{0,X_j\}$ and $X_j^-=\min\{0,X_j\}$. Then

$$\mathbb{E}(X_j^+)=\int_0^{\infty} xF(dx)\leqq+\infty, \quad \text{and} \quad \mathbb{E}(X_j^-)=\int_{-\infty}^{0} xF(dx)>-\infty.$$

If we set

$$S_n^+=\sum_{j=1}^{n} X_j^+ \quad \text{and} \quad S_n^-=\sum_{i=1}^{n} X_j^-,$$

then $S_n = S_n^+ + S_n^-$. The Weak Law of Large Numbers implies that $\frac{S_n^+}{n} \to \mathbb{E}(X_1^+)$ in probability, and $\frac{S_n^-}{n} \to \mathbb{E}(X_1^-)$ in probability, as $n \to \infty$.

Now if $\mathbb{E}(X_1^+) = +\infty$ then $\frac{S_n}{n} \to +\infty$ in probability, and hence $\Pr(S_n \leqq 0) \to 0$, contrary to $\gamma > 0$. Hence $\mathbb{E}(X_1^+) < +\infty$, and so $\mathbb{E}(X_1)$ is finite. If $\mathbb{E}(X_1) \neq 0$ then $\frac{S_n}{n} \to \mathbb{E}(X_1)$ in probability, and therefore $\Pr(S_n \leqq 0) \to 0$ or 1, contrary to $0 < \gamma < 1$. Thus $\mathbb{E}(X_1) = 0$, and so $\frac{v(t)}{t} \to 0$ as $t \downarrow 0$. This implies that $\frac{v(t)}{t} \in L_1(0, \varepsilon)$, and so Lemma 2 holds.

By Theorem 1(d) of Rosén [8], $\Pr(S_n = 0) \to 0$ as $n \to \infty$, and therefore

$$n^{-1} \sum_{k=1}^{n} \{\Pr(S_k > 0) - \Pr(S_k < 0)\} \to 1 - 2\gamma$$

as $n \to \infty$. Applying Lemma 2

$$\lim_{s \uparrow 1} \frac{2}{\pi} \int_0^{\varepsilon} \operatorname{Im}\left(\frac{1-s}{1-s\phi(t)}\right) \frac{dt}{t} = 1 - 2\gamma.$$

It is easily seen that the equation $1 - s\phi(t) = 0$ has a root on the positive imaginary axis. Indeed, if we put

$$f(\eta) = \phi(i\eta) = \int_{-\infty}^{+\infty} e^{-\eta x} F(dx),$$

then $f(0) = 1$, and $f(\eta) \to +\infty$ as $\eta \uparrow \eta_0$. Therefore $f(\eta(s)) = 1/s > 1$ for some $\eta(s) > 0$. Now $f'(0) = \mathbb{E}(X_1) = 0$ and since

$$f''(\eta) = \int_{-\infty}^{+\infty} x^2 e^{-\eta x} F(dx) > 0,$$

the graph of $f(\eta)$ is concave. It follows that $\eta(s) \downarrow 0$ as $s \uparrow 1$. By Lemma 4 therefore $1 - s\phi(t) = 0$ has a single simple root satisfying $|t| \leqq \varepsilon$ and $\operatorname{Im} t \geqq 0$, when s is sufficiently close to 1.

Thus, for s sufficiently close to 1, the function $(1 - s\phi(t))^{-1}$ is analytic in the interior of the closed half-disc $|t| \leqq \varepsilon$, $\operatorname{Im} t \geqq 0$, except for a simple pole at $t = i\eta(s)$. This function is also continuous and bounded on the boundary of this half-disc. Take Γ to be the composite contour $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ where

$$\Gamma_1 = \{t \mid t \text{ real}, -\varepsilon \leqq t \leqq -\varepsilon'\},$$

$$\Gamma_2 = \{t \mid |t| = \varepsilon', \operatorname{Im} t \geqq 0\},$$

$$\Gamma_3 = \{t \mid t \text{ real}, \varepsilon' \leqq t \leqq \varepsilon\},$$

$$\Gamma_4 = \{t \mid |t| = \varepsilon, \operatorname{Im} t \geqq 0\}.$$

Then, using the Residue Theorem, we obtain

$$\int_{\Gamma} \left(\frac{1-s}{1-s\phi(t)}\right) \frac{dt}{t} = 2\pi i \cdot \frac{1-s}{(-s\phi'(i\eta(s)))\, i\eta(s)}.$$

Since $\frac{1}{t}\operatorname{Im}\left(\frac{1-s}{1-s\phi(t)}\right)$ is an even function of t for real t

$$\int_{\Gamma_1+\Gamma_3}\left(\frac{1-s}{1-s\phi(t)}\right)\frac{dt}{t}=2i\int_{\varepsilon'}^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\phi(t)}\right)\frac{dt}{t},$$

and the latter tends to

$$2i\int_0^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\phi(t)}\right)\frac{dt}{t}\quad\text{as } \varepsilon'\downarrow 0.$$

From the fact that $\phi(t)\to 1$ as $t\to 0$, it follows that

$$\int_{\Gamma_2}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}\to -\pi i \quad\text{as}\quad \varepsilon'\downarrow 0.$$

Hence letting $\varepsilon'\downarrow 0$, we obtain

$$2i\int_0^{\varepsilon}\operatorname{Im}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}-\pi i+\int_{\Gamma_4}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}=\frac{2\pi i(1-s)}{i\eta(s)\left(-s\,\phi'(i\eta(s))\right)}.$$

Letting $s\uparrow 1$ we find that

$$\int_{\Gamma_4}\left(\frac{1-s}{1-s\,\phi(t)}\right)\frac{dt}{t}\to 0,$$

and consequently

$$\frac{2\pi i(1-s)}{i\eta(s)\left(-s\,\phi'(i\eta(s))\right)}\to\pi i(1-2\gamma)-\pi i+0=-2\pi i\gamma$$

as $s\uparrow 1$. This is the same as $\lim_{s\uparrow 1}\frac{1-s}{s\eta(s)f'(\eta(s))}=\gamma$. Since $\eta(s)\downarrow 0$ as $s\uparrow 1$, this can be re-expressed as

$$\lim_{\eta\downarrow 0}\frac{\eta f'(\eta)}{f(\eta)-1}=\frac{1}{\gamma}<+\infty.$$

It follows that $f(\eta)-1=\Lambda(1/\eta)\eta^{1/\gamma}$, where Λ is a function which is slowly varying at infinity. For, given any $\varepsilon>0$, we can find $\delta(\varepsilon)>0$ such that, for every x satisfying $0<x<\delta$, it is the case that

$$\left(\frac{1}{\gamma}-\varepsilon\right)\frac{1}{x}<\frac{f'(x)}{f(x)-1}<\left(\frac{1}{\gamma}+\varepsilon\right)\frac{1}{x}.$$

Now let k be any constant greater than 1. We choose η so that $k\eta<\delta$, and integrate from η to $k\eta$. Thus

$$\left(\frac{1}{\gamma}-\varepsilon\right)\log_e k\leqq\log_e\left(\frac{f(k\eta)-1}{f(\eta)-1}\right)\leqq\left(\frac{1}{\gamma}+\varepsilon\right)\log_e k$$

for $\eta<\delta/k$, and therefore

$$\frac{f(k\eta)-1}{f(\eta)-1}\to k^{1/\gamma}\quad\text{as}\quad \eta\downarrow 0.$$

If we now put $\lambda(\eta)=\eta^{-1/\gamma}(f(\eta)-1)$, then $\frac{\lambda(k\eta)}{\lambda(\eta)}\to 1$ as $\eta\downarrow 0$, and so the assertion is proved, with $\Lambda(1/\eta)=\lambda(\eta)$.

Using the fact that $\mathbb{E}(X_1)=0$, we can re-state this result in the form

$$\int_{-\infty}^{+\infty}(e^{-\eta x}+\eta x-1)\,F(dx)=\Lambda\left(\frac{1}{\eta}\right)\eta^{1/\gamma}.$$

By the Dominated Convergence Theorem

$$\frac{1}{\eta^2}\int_{-\infty}^{0}(e^{-\eta x}+\eta x-1)\,F(dx)\to\int_{-\infty}^{0}\tfrac{1}{2}x^2\,F(dx)$$

as $\eta\downarrow 0$. Since $(e^{-\eta x}+\eta x-1)\geqq 0$ for $x\geqq 0$, Fatou's Lemma gives

$$\varliminf_{\eta\downarrow 0}\frac{1}{\eta^2}\int_0^{\infty}(e^{-\eta x}+\eta x-1)\,F(dx)\geqq\int_0^{\infty}\tfrac{1}{2}x^2\,F(dx).$$

If $1/\gamma>2$ then

$$\int_{-\infty}^{+\infty}\tfrac{1}{2}x^2\,F(dx)\leqq\varliminf_{\eta\downarrow 0}\Lambda\left(\frac{1}{\eta}\right)\eta^{1/\gamma-2}=0,$$

and it would follow that $F(x)$ is degenerate. Hence $1/\gamma\leqq 2$, and since by assumption $\gamma\neq\frac{1}{2}$, we deduce that $1/\gamma<2$. So now we have that

$$\int_0^{\infty}(e^{-\eta x}+\eta x-1)\,F(dx)\sim\Lambda\left(\frac{1}{\eta}\right)\eta^{1/\gamma}\quad\text{as}\quad\eta\downarrow 0.$$

If we put $M(x)=\int_x^{\infty}\{1-F(u)\}\,du$ then, integrating by parts twice, we have

$$\int_0^{\infty}(e^{-\eta x}+\eta x-1)\,F(dx)=\eta^2\int_0^{\infty}e^{-\eta x}M(x)\,dx.$$

Hence

$$\int_0^{\infty}e^{-\eta x}M(x)\,dx\sim\Lambda\left(\frac{1}{\eta}\right)\eta^{-(2-1/\gamma)}\quad\text{as}\quad\eta\downarrow 0.$$

Since $M(x)$ is non-increasing, it follows from Feller XIII.5, p. 421, Theorem 2 [3], that

$$M(x)\sim\frac{\Lambda(x)\cdot x^{1-1/\gamma}}{\Gamma(2-1/\gamma)}\quad\text{as}\quad x\to+\infty,$$

Finally, because $1-F(x)$ is also non-increasing, we deduce that

$$1-F(x)\sim\frac{\Lambda(x)\,(1/\gamma-1)}{\Gamma(2-1/\gamma)}\,x^{-1/\gamma}\quad\text{as}\quad x\to+\infty.$$

The result now follows from a classical limit theorem (see [4]).

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