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Almost Everywhere Convergence of a Class of Integrable Functions

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Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and let $\{f_k\}$ be a sequence of integrable functions on Ω. In Loève [4, p. 406] a necessary and sufficient condition is given for the a.e. convergence of $\{f_k\}$ provided that $|f_k| \leqq g$ for all k where g is integrable. In practice this last condition is sometimes difficult to verify and therefore it is important to find other conditions for a.e. convergence which may depend on the sequences being studied. In this paper we consider sequences of integrable functions satisfying conditions on successive terms in the sequence and prove a necessary and sufficient condition for a.e. convergence. The condition is then applied to prove martingale type theorems for these sequences and to give a necessary and sufficient condition for the $L^p(\mu)$ ergodic theorem $(1 < p < \infty)$ on a finite measure space.

If $\{f_k\}_{k \geqq 0}$ is a sequence of integrable functions on Ω and α is a real number, define $B_{n,n} = \{f_n > \alpha\}$ and $B_{n,k}(\alpha) = \{f_n \leqq \alpha, f_{n+1} \leqq \alpha, \cdots, f_k > \alpha\}$ for $k > n \geqq 0$. Note that for n fixed the sets $\{B_{n,k}\}_{k \geqq n}$ are disjoint and

$$B_n = B_n(\alpha) \equiv \sum_{k \geqq n} B_{n,k} = \{\sup_{k \geqq n} f_k > \alpha\}.$$

Moreover, $B_n(\alpha) \downarrow B(\alpha)$ as $n \to \infty$ where $B(\alpha)$ satisfies $\{\overline{\lim}\, f_k > \alpha\} \subseteq B(\alpha) \subseteq \{\overline{\lim}\, f_k \geqq \alpha\}$. With these definitions we have

Lemma. *Let $\{f_k\}_{k \geqq 0}$ be a sequence of integrable functions such that $|f_k - f_{k-1}| \leqq g$ for all $k \geqq 1$ where g is integrable, and $\sup_k \int |f_k|\, d\mu = M < \infty$. Then for each real α and $A \in \mathscr{A}$, $\sum_{k \geqq n} \int_{AB_{n,k}} f_k\, d\mu$ converges absolutely and is uniformly bounded in n.*

Proof. Consider first the case where $\alpha \geqq 0$. Since $f_k > \alpha$ on $B_{n,k}(\alpha)$ the terms in the sum are positive. Now if $k \geqq n+1$ then $f_{k-1} \leqq \alpha$ on $B_{n,k}(\alpha)$ and therefore for all integers $N > n$

$$\begin{aligned}
\sum_{k=n}^{N} \int_{AB_{n,k}} f_k &= \sum_{k=n}^{N} \int_{AB_{n,k}} (f_k - f_{k-1}) + \sum_{k=n}^{N} \int_{AB_{n,k}} f_{k-1} \\
&\leqq \sum_{k=n}^{N} \int_{AB_{n,k}} |f_k - f_{k-1}| + \int_{AB_{n,n}} |f_{n-1}| + \sum_{k=n+1}^{N} \alpha \mu(AB_{n,k}) \\
&\leqq \sum_{k=n}^{N} \int_{AB_{n,k}} g + M + \alpha \mu\left(\sum_{k=n+1}^{N} AB_{n,k}\right) \leqq \int g\, d\mu + M + \alpha \mu(\Omega) < \infty.
\end{aligned}$$

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Hence $\sum_{k\geqq n}\int_{AB_{n,k}} f_k\, d\mu \leqq \int g + M + \alpha\mu(\Omega)$ uniformly in n and the assertion is proved for $\alpha \geqq 0$. The case where $\alpha < 0$ is handled easily by writing

$$AB_{n,k} = AB_{n,k}\{f_k \geqq 0\} + AB_{n,k}\{\alpha < f_k < 0\}$$

and using the above argument along with the fact that $|f_k| \leqq \alpha$ on $AB_{n,k}\{\alpha < f_k < 0\}$.

We now prove (cf. [4, p. 492]).

Theorem. *Let $\{f_k\}_{k\geqq 0}$ be a sequence of integrable functions satisfying*

1) $|f_k - f_{k-1}| \leqq g$ for all $k \geqq 1$ where g is integrable,

2) $f_k - f_{k-1} \to 0$ a.e. as $k \to \infty$, and

3) $f_k \to 0$ in $L^1(\mu)$ as $k \to \infty$.

Then for all $\alpha \geqq 0$, $A \in \mathscr{A}$, and integers $m \geqq 1$

$$\lim_{n\to\infty} \sum_{k\geqq n} \int_{AB_{n,k}} f_k\, d\mu = \lim_{n\to\infty} \sum_{k\geqq n} \int_{AB_{n,k}} f_{k+m}\, d\mu = \alpha\mu(AB(\alpha)).$$

Hence $\mu\{\overline{\lim}\, f_k > 0\} = 0$ if and only if the above limit is zero for all $\alpha > 0$ and $A = \Omega$.

Proof. Note first that the sequence $\{\sum_{k\geqq n}\int_{AB_{n,k}} f_k\}_{n\geqq 0}$ is uniformly bounded in n for each $\alpha \geqq 0$ and $A \in \mathscr{A}$ from the Lemma. Let β be a limit point of this sequence so there is a subsequence $\{n'\}$ such that $\beta = \lim_{n'} \sum_{k\geqq n'}\int_{AB_{n',k}} f_k$. Since $f_k > \alpha$ on $B_{n',k} = B_{n',k}(\alpha)$ we have $\beta \geqq \lim_{n'} \sum_{k\geqq n'} \alpha\mu(AB_{n',k}) = \alpha \lim_{n'} \mu(AB_{n'}(\alpha)) = \alpha\mu(AB(\alpha))$. On the other hand,

$$\Big|\sum_{k\geqq n'} \int_{AB_{n',k}} f_{k-1} - \sum_{k\geqq n'} \int_{AB_{n',k}} f_k\Big| \leqq \sum_{k\geqq n'} \int_{AB_{n',k}} |f_{k-1} - f_k| \xrightarrow{n'} 0$$

from conditions 1) and 2) and [4, p. 406]. Hence $\beta = \lim_{n'} \sum_{k\geqq n'}\int_{B_{n',k}} f_{k-1}$, and using the fact that $f_{k-1} \leqq \alpha$ on $B_{n',k}$ for $k \geqq n'+1$ we obtain (since $\int |f_{n'}| \to 0$)

$$\begin{aligned}\beta &\leqq \lim_{n'} \int_{B_{n',n'}} |f_{n'-1}| + \alpha \sum_{k\geqq n'+1} \mu(AB_{n',k}) \\ &= \alpha \lim_{n'} \mu\big(A(B_{n'}(\alpha) \setminus B_{n',n'}(\alpha))\big) \\ &= \alpha\mu(AB(\alpha)) - \alpha \lim_{n'} \mu(AB_{n',n'}).\end{aligned}$$

If $\alpha = 0$, $\beta \leqq 0$, hence $\beta = 0$. If $\alpha > 0$, then $\lim_{n'} \mu(AB_{n',n'}(\alpha)) = 0$ as

$$\mu(B_{n',n'}(\alpha)) = \mu(\{f_{n'} > \alpha\}) \to 0$$

as $n' \to \infty$. Hence $\beta \leqq \alpha\mu(AB(\alpha))$ and therefore $\beta = \alpha\mu(AB(\alpha))$ is the unique limit point of the sequence $\{\sum_{k\geqq n}\int_{AB_{n,k}} f_k\, d\mu\}_{n\geqq 0}$ so that $\lim_{n\to\infty} \sum_{k\geqq n}\int_{AB_{n,k}} f_k = \alpha\mu(AB(\alpha))$. If $m \geqq 1$ is an integer then $|f_{k+m} - f_k| = |(f_{k+m} - f_{k+m-1}) + \cdots + (f_{k+1} - f_k)| \leqq mg$ and $f_{k+m} - f_k \to 0$ a.e. It follows as above that

$$\Big|\sum_{k=n} \int_{AB_{n,k}} f_{k+m} - \sum_{k\geqq n} \int_{AB_{n,k}} f_k\Big| \leqq \sum_{k\geqq n} \int_{AB_{n,k}} |f_{k+m} - f_k| \to 0 \quad \text{as } n \to \infty$$

so that $\lim_{n\to\infty} \sum_{k\geqq n}\int_{AB_{n,k}} f_{k+m} = \alpha\mu(AB(\alpha))$ and the first assertion in the theorem is proved. The second assertion follows from the fact that $\mu\{\overline{\lim}\, f_k > 0\} = 0$ iff for all $\alpha>0$, $\mu\{\overline{\lim} f_k \geqq \alpha\} = \mu\{\overline{\lim} f_k > \alpha\} = 0$ iff $\alpha\mu(B(\alpha))=0$ for all $\alpha>0$. The proof of the theorem is complete.

Remark 1. The sequence $\{-f_k\}$ satisfies the same hypotheses of the theorem and therefore the theorem holds with $\{f_k\}$ replaced by $\{-f_k\}$. If the associated limits are also zero for all $\alpha>0$, then

$\mu\{\overline{\lim} - f_k > 0\} = \mu\{\underline{\lim}\, f_k < 0\} = 0$ and hence $f_k \to 0$ a.e.

Remark 2. Suppose $\{g_k\}_{k\geqq 0}$ is another sequence of integrable functions satisfying $|f_k - g_k| \leqq h$ for all $k \geqq 0$ where h is integrable, and $f_k - g_k \to 0$ a.e. Then the above proof shows that

$$\lim_{n\to\infty} \sum_{k\geqq n} \int_{AB_{n,k}} g_k = \alpha\mu(AB(\alpha)) \quad \text{for all } \alpha \geqq 0 \quad \text{and} \quad A \in \mathscr{A}.$$

To give an example of the type of result one can prove using the theorem, we consider a sequence $\{f_k\}$ of integrable functions on Ω satisfying the hypotheses of the theorem. If $k \geqq 0$ let $\mathscr{A}_k = \sigma(f_i: i \leqq k)$ be the σ-algebra generated by the f_i, $i \leqq k$. If h is an integrable function we denote by $E^{\mathscr{A}_k}(h)$ the conditional expectation of h with respect to $\mathscr{A}_k$, i.e. the $\mathscr{A}_k$-measurable function satisfying

$$\int_A E^{\mathscr{A}_k}(h)\, d\mu = \int_A h\, d\mu$$

for all $A \in \mathscr{A}_k$. Define for $m \geqq 1$, $F_m = \sup_k |E^{\mathscr{A}_k}(f_{k+m})|$. If, for example, the $\{f_k\}$ were independent with $\int f_k = 0$ for all $k \geqq 0$, then $F_m = 0$ a.e. for all $m \geqq 1$. The following result is an immediate consequence of our theorem.

Corollary 1. *If* $\int F_m\, d\mu \xrightarrow{m} 0$, *then* $f_k \to 0$ *a.e. as* $k \to \infty$.

Proof. Let $\alpha \geqq 0$ and note that $B_{n,k}(\alpha) \in \mathscr{A}_k$ for $k \geqq n \geqq 0$. Then the theorem implies that

$$\alpha\mu(B(\alpha)) = \lim_{n\to\infty} \sum_{k\geqq n} \int_{B_{n,k}} f_{k+m} = \lim_{n\to\infty} \sum_{k\geqq n} \int_{B_{n,k}} E^{\mathscr{A}_k}(f_{k+m}) \quad \text{for all } m \geqq 1.$$

But $E^{\mathscr{A}_k}(f_{k+m}) \leqq F_m$ for all $k \geqq 0$ and therefore $\alpha\mu(B(\alpha)) \leqq \sum_{k\geqq n} \int_{B_{n,k}} F_m \leqq \int F_m$ for all $m \geqq 1$. Since $\int F_m \to 0$ we obtain $\alpha\mu(B(\alpha)) = 0$. A similar result holds for the sequence $\{-f_k\}$ and therefore (Remark 1) $f_k \to 0$ a.e.

Remark 3. Suppose $\mu(\Omega)=1$ and the sequence $\{f_k\}_{k\geqq 0}$ satisfies only conditions 2) and 3) of the theorem. Define for each $k \geqq 0$ the function $f_k' = f_k I_{\{|f_k| \leqq 1\}} + I_{\{f_k > 1\}} - I_{\{f_k < -1\}}$ where I_A is the indicator function of $A \in \mathscr{A}$, and put $h_k = f_k' - \int f_k'$. Then the sequence $\{h_k\}_{k\geqq 0}$ is uniformly bounded, $h_k \in \mathscr{A}_k$, and $\int h_k = 0$ for all $k \geqq 0$. Now it is easy to see that $\{h_k\}$ satisfies conditions 1), 2) and 3) of the theorem. Moreover, $h_k \to 0$ a.e. if and only if $f_k \to 0$ a.e. Hence, if $\int \sup_k |E^{\mathscr{A}_k}(h_{k+m})| \to 0$ as $m \to \infty$, then $f_k \to 0$ a.e. In particular, if $\{X_k\}_{k\geqq 0}$ is a sequence of independent random variables which converge to zero in mean, then $X_k \to 0$ almost surely if and only if

$$X_k - X_{k-1} \to 0$$

almost surely.

We consider now sums of integrable functions. More precisely, let $\{g_i\}_{i \geqq 0}$ be a sequence of integrable functions and for each $k \geqq 0$ let $f_k = \sum_{i=0}^{k} g_i$. Denote by $\mathscr{B}_k = \sigma(g_i : i \leqq k)$. If $E^{\mathscr{B}_k}(g_{k+1}) = 0$ for all $k \geqq 0$ then the sequence $\{f_k\}$ forms a martingale, and every martingale can be written in this form. Moreover, if f_k converges in $L^1(\mu)$ then f_k converges a.e. [5]. The question arises as to whether one can relax the condition $E^{\mathscr{B}_k}(g_{k+1}) = 0$ and obtain a.e. convergence given convergence in $L^1(\mu)$. We assume that $|g_i| \leqq g$ integrable for all $i \geqq 0$ and $g_i \to 0$ a.e. Since $f_k - f_{k-1} = g_k$ this implies that $\{f_k\}$ satisfies conditions 1) and 2) of the theorem. Define, for $m \geqq 1$, $G_m = \sup_k |E^{\mathscr{B}_k}(g_{k+m})|$. Under these conditions we have the following

Proposition. *Suppose $f_k = \sum_{i=0}^{k} g_i$ converges to f in $L^1(\mu)$. If $\sum_m \int G_m \, d\mu < \infty$, then f_k converges to f a.e.*

Proof. We are going to verify a condition found in [4] (cf. p. 402 and Corollary 1, p. 405). That is, we show that if α is real and $\alpha_n > \alpha$ with $\alpha_n \downarrow \alpha$, then for all $A \in \bigcup_k \mathscr{B}_k$,

$$\lim_{n \to \infty} \sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} f_k = \int_{AB'(\alpha)} f$$

where $B'(\alpha) = \{\overline{\lim} f_k \geqq \alpha\}$. A similar result will hold for the sequence $\{-f_k\}$. Note that from the Lemma the sums are absolutely convergent. Now

$$\Big|\sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} f_k - \int_{AB'(\alpha)} f\Big| \leqq \Big|\sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} (f_k - f)\Big| + \Big|\int_{AB'(\alpha)} f - \int_{AB_n(\alpha_n)} f\Big|.$$

Since $B_n(\alpha_n) \to B'(\alpha)$ as $n \to \infty$, the second term converge to zero and therefore it suffices to show that

$$\overline{\lim_n} \Big|\sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} (f_k - f)\Big| = 0.$$

Since $|(f_{k+m} - f) - (f_k - f)| = |f_{k+m} - f_k| \leqq mg$ and $f_{k+m} - f_k \to 0$ a.e. for all $m \geqq 1$ we have, as in the proof of the theorem,

$$\overline{\lim_n} \Big|\sum_{k \geqq n} \int_{AB_{n,k}} (f_k - f)\Big| = \overline{\lim_n} \Big|\sum_{k \geqq n} \int_{AB_{n,k}} (f_{k+m} - f)\Big| \quad \text{for all } m \geqq 1.$$

Now $f = \sum_{i \geqq 0} g_i$ where the convergence holds in $L^1(\mu)$, hence $f - f_{k+m} = \sum_{i > k+m} g_i$ in $L^1(\mu)$. If $A \in \mathscr{B}_{k_0}$ then for $k \geqq n \geqq k_0$ we have $AB_{n,k}(\alpha_n) \in \mathscr{B}_k$ and therefore

$$\sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} (f_{k+m} - f) = \sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} \sum_{i > k+m} g_i = \sum_{k \geqq n} \sum_{i > k+m} \int_{AB_{n,k}(\alpha_n)} g_i$$

$$= \sum_{k \geqq n} \sum_{i > k+m} \int_{AB_{n,k}(\alpha_n)} E^{\mathscr{B}_k}(g_i) = \sum_{k \geqq n} \sum_{i \geqq 1} \int_{AB_{n,k}(\alpha_n)} E^{\mathscr{B}_k}(g_{k+m+i}).$$

But $|E^{\mathscr{B}_k} g_{k+m+i}| \leqq G_{m+i}$ for all k, m, i. Hence

$$\overline{\lim_n} \Big|\sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} (f_k - f)\Big| \leqq \overline{\lim_n} \sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} \sum_{i \geqq 1} G_{m+i}$$

$$= \overline{\lim_n} \int_{AB_n(\alpha_n)} \sum_{i \geqq 1} G_{m+i} \leqq \int \sum_{i > m+1} G_i = \sum_{i \geqq m+1} \int G_i \quad \text{for all } m \geqq 1.$$

Since $\lim_{m \to \infty} \sum_{i \geqq m+1} \int G_i = 0$, it follows that $\overline{\lim_n} \Big|\sum_{k \geqq n} \int_{AB_{n,k}(\alpha_n)} (f_k - f)\Big| = 0$ and the proof is complete.

As an application of the above proposition we assume that $\{g_i\}_{i\geqq 0}$ is a uniformly bounded orthonormal sequence of functions on Ω so that for all real sequences $\{a_i\}$ satisfying $\sum_{i\geqq 0} a_i^2<\infty$, $\sum_{i\geqq 0} a_i g_i$ converges in $L^2(\mu)$, hence in $L^1(\mu)$. The case where the sequence $f_k=\sum_{i=0}^k a_i g_i$ forms a martingale has been studied by Gundy [2] (see also [1]). Note that $f_k-f_{k-1}=a_k g_k\to 0$ a.e. and boundedly so that the hypotheses of the proposition are satisfied. Therefore if $\{a_i\}_{i\geqq 0}$ is a square summable sequence of numbers such that $G_m=\sup_{k\geqq 0}|a_{k+m}||E^{\mathscr{B}_k}(g_{k+m})|$ satisfies $\sum_m \int G_m<\infty$, then $\sum_{i\geqq 0} a_i g_i$ converges a.e. On the other hand, a slight modification of the above proof using the Cauchy-Schwarz inequality shows that if $\sup_{k\geqq 0}(\sum_{i\geqq 0} E^{\mathscr{B}_k}(g_{k+m+i})^2)^{\frac{1}{2}}$ is integrable for some large m, then $\sum_{i\geqq 0} a_i g_i$ converges a.e. for all sequences $\{a_i\}$ such that $\sum_i a_i^2<\infty$.

We turn now to the $L^p(\mu)$ ergodic theorem for $1<p<\infty$. Let $T\colon L^p(\mu)\to L^p(\mu)$ be a linear contraction, i.e. $\int|Tf|^p\,d\mu\leqq\int|f|^p\,d\mu$ for all $f\in L^p(\mu)$. Define, for $f\in L^p(\mu)$ and $k\geqq 1$, $S_k f=(1/k)\sum_{j=0}^{k-1} T^j f$. Then S_k is a contraction on $L^p(\mu)$ and it is well-known that $S_k f\to Pf$ in $L^p(\mu)$ as $k\to\infty$ where $P^2=P$ is a projection satisfying $Pf=PTf=TPf$ for all $f\in L^p(\mu)$ ([6, p. 213]). If T is an invertible L^p-isometry and $p\neq 2$, then $S_k f\to Pf$ a.e. for all $f\in L^p(\mu)$ ([3]). Let $f_k=S_k f-Pf=S_k(f-Pf)$, $k\geqq 1$. It is easily verified that $f_k-f_{k-1}=T^{k-1}f/(k-1)-S_k f/(k-1)$ and therefore $|f_k-f_{k-1}|\leqq|T^{k-1}f/(k-1)|+|S_k f/(k-1)|$ for all $k\geqq 2$. Clearly

$$|T^k f/k|^p\leqq\sum_{n\geqq 1}|T^n f/n|^p\equiv g$$

and

$$\int g=\sum_{n\geqq 1}\int|T^n f/n|^p=\sum_{n\geqq 1}1/n^p\int|T^n f|^p\leqq\int|f|^p\sum_{n\geqq 1}1/n^p<\infty.$$

It follows that $T^k f/k\to 0$ a.e. and $|T^k f/k|\leqq g^{1/p}\in L^1(\mu)$ for all $k\geqq 1$. A similar proof shows that $|S_k f/(k-1)|\to 0$ a.e. and there is a function $g_1\in L^1(\mu)$ such that $S_k f/(k-1)\leqq g_1$ for all $k\geqq 2$. Hence $|f_k-f_{k-1}|\leqq g^{1/p}+g_1$ integrable and $f_k-f_{k-1}\to 0$ a.e. In other words, the hypotheses of the theorem are satisfied. On the other hand, $S_k(f-Pf)-S_k(Tf-Pf)=f/k-T^k f/k$. The sequence $\{S_k(Tf-Pf)\}_k$ satisfies the conditions of remark 2, and so too does the sequence $\{S_k(T^m f-Pf)\}_k$ for any integer $m\geqq 1$. Hence, from the theorem

$$\alpha\mu(B(\alpha))=\lim_{n\to\infty}\sum_{k\geqq n}\int_{B_{n,k}(\alpha)} S_k(T^m f-Pf)\,d\mu \qquad \text{for all } m\geqq 1.$$

It follows that if $c(f)=\sum_{i=0}^N \beta_i T^{m_i}f$, $\sum_{i=0}^N\beta_i=1$, $\beta_i\geqq 0$ is any convex combination of elements of the sequence $\{T^m f\}_{m\geqq 0}$, then

$$\alpha\mu(B(\alpha))=\lim_{n\to\infty}\sum_{k\geqq n}\int_{B_{n,k}} S_k(c(f)-Pf)\,d\mu.$$

Denoting by $S_k^*\colon L^q(\mu)\to L^q(\mu)$ the adjoint operator of $S_k (1/p+1/q=1)$ we have then the following result.

Corollary 2. *For all $\alpha\geqq 0$ and for every convex combination $c(f)$ of the sequence $\{T^m f\}_{m\geqq 0}$,*

$$\alpha\mu(B(\alpha))=\lim_{n\to\infty}\sum_{k\geqq n}\int_{B_{n,k}(\alpha)} S_k(c(f)-Pf)\,d\mu=\lim_{n\to\infty}\sum_{k\geqq n}\int(c(f)-Pf)S_k^*(I_{B_{n,k}})\,d\mu.$$

Hence $\mu\{\overline{\lim_k}\, S_k f - Pf > 0\} = 0$ iff for each $\alpha > 0$ the above limit is zero for some, hence all, convex combinations $c(f)$ of $\{T^m f\}_{m \geqq 0}$.

We consider now the case where T is a positive $L^p(\mu)$-contraction, i.e. $f \geqq 0$ a.e. implies $Tf \geqq 0$ a.e. Then the operators S_k as well as S_k^* are also positive operators. Assume furthermore that $h \equiv \sup_{k \geqq 1} S_k 1 \in L^1(\mu)$ where 1 is the constant function. Then for any sequence of disjoint sets $\{B_k\}_{k \geqq 1}$, the positive function $g = \sum_{k \geqq 1} S_k^*(I_{B_k})$ satisfies

$$\int g = \sum_{k \geqq 1} \int S_k^*(I_{B_k}) = \sum_{k \geqq 1} \int I_{B_k} S_k(1) \leqq \sum_{k \geqq 1} \int_{B_k} h \leqq \int h < \infty.$$

In other words, functions of the above form constitute a bounded set in $L^1(\mu)$. If $f \in L^p(\mu)$ is a positive function, $c(f)$ is also positive and

$$\sum_{k \geqq n}^{\infty} \int_{B_{n,k}(\alpha)} S_k(c(f) - Pf)\, d\mu = \sum_{k \geqq n} \int_{B_{n,k}} S_k(c(f)) - \int_{B_{n,k}} Pf$$

converges. Since $\sum_{k \geqq n} \int_{B_{n,k}} Pf = \int_{B_n(\alpha)} Pf$ it follows that $\sum_{k \geqq n}^{\infty} \int_{B_{n,k}} S_k(c(f)) < \infty$. But

$$\sum_{k \geqq n} \int_{B_{n,k}} S_k(c(f)) = \sum_{k \geqq n} \int c(f) S_k^*(I_{B_{n,k}}) = \int c(f) \sum_{k \geqq n} S_k^*(I_{B_{n,k}}) = \int c(f) g_n^\alpha < \infty$$

where

$$g_n^\alpha \equiv \sum_{k \geqq n} S_k^*(I_{B_{n,k}(\alpha)}) \in L^1(\mu).$$

Therefore

$$\sum_{k \geqq n} \int_{B_{n,k}} S_k(c(f) - Pf) = \int c(f) g_n^\alpha - \int Pf g_n^\alpha = \int (c(f) - Pf) g_n^\alpha\, d\mu.$$

It follows that for all $\alpha \geqq 0$, $\alpha\mu(B(\alpha)) = \lim_{n \to \infty} \int (c(f) - Pf) g_n^\alpha$ where the functions $\{g_n^\alpha\}_n$ form a bounded set in $L^1(\mu)$ for each $\alpha \geqq 0$. Note that since $S_k f \to Pf$ in $L^p(\mu)$, a subsequence converges a.e. so that we can always find convex combinations $c_m(f) \to Pf$ a.e. Unfortunately, without stronger conditions on the functions $\{g_n^\alpha\}$ this is not enough to assert that $\mu(B(\alpha)) = 0$.

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Note Added in Proof. It has recently been proven by M.A. Akcoglu, using techniques entirely different from those described above, that the ergodic theorem holds for positive contractions on $L^p(\mu)$, $1 < p < \infty$.