Z. Wahrscheinlichkeitstheorie verw. Gebiete 31, 79–88 (1975)

Doeblin's and Harris' Theory of Markov Processes

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Our notation and definitions are taken from (Chung, K.L.: The general theory of Markov processes according to Doeblin. Z. Wahrscheinlichkeitstheorie und verw. Gebiete **2**, 230–254 (1964)). A closed set H is called *recurrent in the sense of Harris* if there exists a σ-finite measure φ such that for $E = H$, $\varphi(E) > 0$ implies $Q(x, E) = 1$ for all $x \in H$. **Theorem 1.** *Let X be absolutely essential and indecomposable. Then there exists a closed set $B \subseteq X$ such that B contains no uncountable disjoint collection of perpetuable sets if and only if $X = H + I$ where H is recurrent in the sense of Harris and I is either inessential or improperly essential.* **Theorem 2.** *If there exists no uncountable disjoint collection of closed sets, then there exists a countable disjoint collection $\{D_n\}_{n=1}^{\infty}$ of absolutely essential and indecomposable closed sets such that $I = X - \sum_{n=1}^{\infty} D_n$ is either inessential or improperly essential.* Under the additional assumption that Suslin's Conjecture holds, Theorem 2 was proved by Jamison (Jamison, B.: A Result in Doeblin's Theory of Markov Chains implied by Suslin's Conjecture. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24**, 287–293 (1972)).

0. Introduction

In a recent paper Jamison [6] formulated a set-theoretic condition with which he was able to obtain some results in Doeblin's theory of Markov chains using Suslin's conjecture. In this paper we prove the same results without using Suslin's conjecture. In addition we formulate another set-theoretic condition in order to obtain further results. These results show that a major portion of Doeblin's and Harris' theory may be derived without making the standard assumptions about the reference measure, which have characterized this theory (see [8] p. 4, [1, 2, 4–6, 3]).

We begin by reviewing the general terminology of Markov processes and the specific terminology of Doeblin's theory. Let X be an abstract space and $\mathscr{B}$ a Borel field of subsets of X. Let $P(x, B)$, $x \in X$, $B \in \mathscr{B}$, be a transition probability. $P(x, B)$ has the following properties:

i) for each $x \in X$, $P(x, \cdot)$ is a probability measure on $\mathscr{B}$,
ii) for each $B \in \mathscr{B}$, $P(\cdot, B)$ is a $\mathscr{B}$-measurable function of x.

For each probability measure μ on $\mathscr{B}$ there is a probability space $(\Lambda, \mathscr{I}, P_\mu)$ and a sequence of functions $X_0, X_1, X_2, \ldots$ from Λ to X which are $(\mathscr{I}, \mathscr{B})$ measurable, that is $X_j^{-1}(\mathscr{I}) \subseteq \mathscr{B}$ for $j = 0, 1, \ldots$. $X_0, X_1, X_2, \ldots$ satisfy

a) $P_\mu(X_0 \in B) = \mu(B)$ and
b) $P_\mu(X_{n+1} \in B \mid X_0, X_1, \ldots, X_n) = P_\mu(X_n, B)\, P_\mu$-a.s.

for each $B \in \mathscr{B}$ and $n = 0, 1, 2, \ldots$. In the particular case in which $\mu(\cdot) = \delta(x, \cdot)$ where for every $B \in \mathscr{B}$:

$$\delta(x, B) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

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we write P_x for P_μ. For each $x \in X$ and $B \in \mathscr{B}$ we write $L(x, B)$ for $P_x\left(\bigcup_{n=1}^{\infty}\{X_n \in B\}\right)$ and $Q(x, B)$ for $P_x\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k \in B\}\right)$.

Definition 1. A nonempty set $B \in \mathscr{B}$ is called *stochastically closed*, or *closed*, if $P(x, B)=1$ for $x \in B$. A closed set $B \in \mathscr{B}$ is called *indecomposable* if it does not contain a disjoint pair of closed sets; otherwise it is called *decomposable.*

Definition 2. A set $E \in \mathscr{B}$ such that $Q(x, E)=0$ for all $x \in X$ is called *inessential;* otherwise it is called *essential.* An essential set which is the union of countably many inessential is called *improperly essential;* otherwise it is called *absolutely essential.*

Definition 3. Let E be a closed set and φ be a σ-finite measure on $(X, \mathscr{B})$ with $\varphi(E)>0$. Then E is called *φ-recurrent* if $Q(x, F)=1$ for all $x \in E$ whenever $F \subseteqq E$ and $\varphi(F)>0$. If E is φ-recurrent for some such φ, then E is called *recurrent in the sense of Harris* (see [5, 8]).

Definition 4. We say that X is *normal* if it is indecomposable and absolutely essential and if there is a closed set $F \subseteqq X$ which contains no improperly essential subsets. Such an F is called a *final set.*

Definition 5. If $Q(x, E^c)<1$ for some $x \in X$, then E is called *perpetuable* (here $E^c = X \setminus E$).

Chung ([1] p. 254) proved the following:

Proposition 1. *Let X be indecomposable and absolutely essential. If φ is a σ-finite measure such that if A is perpetuable then $\varphi(A)>0$, then X is normal.*

If we assume that there exists a σ-finite measure φ such that if A is perpetuable then $\varphi(A)>0$, then we have the following condition:

($\mathscr{E}$) There exists no uncountable disjoint class of perpetuable sets.

Noting that every closed set is perpetuable (since E closed and $x \in E$ imply $Q(x, E^c)=0$) we have that condition ($\mathscr{E}$) implies the following condition:

($\mathscr{C}$) There exists no uncountable disjoint class of closed sets (see [6] p. 288).

Consider the following conditions:

($\mathscr{D}$) There exists a countable disjoint family $\{D_n : n \geqq 1\}$ of closed, indecomposable, and absolutely essential sets such that $X = I + \sum_{n=1}^{\infty} D_n$ where I is either inessential or improperly essential.

($\mathscr{M}$) There exists a finite measure m such that if C is closed then $m(C)>0$.

Doeblin [2] proved (see [2], p. 74; also [1, 3]):

Proposition 2. $(\mathscr{M}) \Rightarrow (\mathscr{D})$.

We note that in [1] *the measure considered in condition* ($\mathscr{M}$) *is σ-finite rather than finite. It is easy to see that* $(\mathscr{M}) \Rightarrow (\mathscr{C})$; *hence, it is natural to ask: Does* ($\mathscr{C}$) *imply* ($\mathscr{D}$)?

In Section 1 we prove a theorem (Theorem 1) similar to Proposition 1, using condition ($\mathscr{E}$). In [6] Jamison proved that ($\mathscr{C}$) implies ($\mathscr{D}$) under the assumption that Suslin's Conjecture holds. In Section 2 we prove that $(\mathscr{C}) \Rightarrow (\mathscr{D})$ without using Suslin's Conjecture.

1. Perpetuable Sets

We say that condition $(\mathscr{H})$ holds if $X = H + I$ where H is recurrent in the sense of Harris and I is either inessential or improperly essential. We now prove the following theorem.

Theorem 1. *Let X be indecomposable and absolutely essential. Then the following are equivalent:*

i) *$(\mathscr{E})$ holds on some closed subset of X.*

ii) *X is normal.*

iii) *$(\mathscr{H})$ holds.*

Proof. i)$\Rightarrow$ii). Assume that $(\mathscr{E})$ holds on some closed subset B of X. Since X is absolutely essential and indecomposable, B is absolutely essential and indecomposable by Proposition 19 of [1]. If we show that B is normal, then X is normal, hence, for convenience, we assume that $B = X$ (that is, $(\mathscr{E})$ holds on X). We assume that X is not normal and show that $(\mathscr{E})$ does not hold on X.

If X is not normal, then X contains an improperly essential set E_1. By Proposition 23.1 of [1], there exists an improperly essential and perpetuable set F_1 such that $E_1 \subseteqq F_1 \subseteqq X$ and $G_1 = X \setminus F_1$ is closed. If G_1 were improperly essential, then $X = F_1 \cup G_1$ would be improperly essential which is a contradiction. Since X is indecomposable, F_1 contains no closed sets. If G_1 does not contain an improperly essential set, then X is normal. Thus there exists an improperly essential set $E_2 \subseteqq G_1$. Again, by Proposition 23.1 of [1], there exists an improperly essential and perpetuable set F_2 such that $E_2 \subseteqq F_2 \subseteqq G_1$ and $G_2 =_{\text{def}} G_1 \setminus F_2$ is closed. If G_2 were improperly essential, then $G_1 = G_2 \cup F_2$ would be improperly essential which is a contradiction. Since X is indecomposable $G_2{}^c$ contains no closed sets and is improperly essential by Proposition 14.1 of [1]. We continue our reasoning by transfinite induction.

Let Ω be the first uncountable ordinal. Let $\alpha < \Omega$ be a nonlimit ordinal; then $\alpha = \beta + 1$ where β is an ordinal $< \Omega$. Assume that G_β has been defined (here we only need that G_β be absolutely essential and closed). If G_β contains no improperly essential sets, then X is normal. Hence we assume that G_β contains an improperly essential set E_α. By Proposition 23.1 of [1] there exists an improperly essential and perpetuable set F_α such that $E_\alpha \subseteqq F_\alpha \subseteqq G_\beta$ and $G_\alpha =_{\text{def}} G_\beta \setminus F_\alpha$ is closed. If G_α were improperly essential, then $G_\beta = G_\alpha \cup F_\alpha$ would be improperly essential which is a contradiction. Hence G_α is absolutely essential. Also since $G_\alpha{}^c$ contains no closed sets it is improperly essential by Proposition 14.1 of [1].

Assume that α is a limit ordinal $< \Omega$ and that E_β, F_β, and G_β have been defined for all $\beta < \alpha$. Assume that $K =_{\text{def}} \bigcap_{\beta < \alpha} G_\beta$ is empty or improperly essential. Since for each $\beta < \alpha$, $G_\beta{}^c$ is improperly essential, we have that $X = \bigcup_{\beta < \alpha} G_\beta{}^c \cup (\bigcap_{\beta < \alpha} G_\beta)$ is improperly essential which is a contradiction. Hence K is absolutely essential and closed. If K does not contain an improperly essential set, then X is normal which is a contradiction. Hence, we see that K contains an improperly essential set E_α. By Proposition 23.1 of [1] there exists an improperly essential and perpetuable set F_α such that $E_\alpha \subseteqq F_\alpha \subseteqq K$ and $G_\alpha =_{\text{def}} K \setminus F_\alpha$ is closed. If G_α were improperly essential, the $K = F_\alpha \cup G_\alpha$ would be improperly essential which is a contradiction. Hence G_α is an absolutely essential closed set. Then

$$\{E_\alpha\colon \alpha < \Omega\},\ \{F_\alpha\colon \alpha < \Omega\}, \quad \text{and} \quad \{G_\alpha\colon \alpha < \Omega\}$$

are defined by transfinite induction and have the properties described above. But $\{F_\alpha\colon \alpha < \Omega\}$ is an uncountable disjoint collection of perpetuable sets. Hence $(\mathscr{E})$ does not hold.

ii) ⇒ iii). By Theorem 3 of [5] we have that condition $(\mathscr{H})$ holds.

iii) ⇒ i). Let $X = H + I$ where H is recurrent in the sense of Harris and I is either inessential or improperly essential. We will show that $(\mathscr{E})$ holds on H. By Lemma 2.5 of [4] there exists a σ-finite invariant measure π on H such that if $E \subseteqq H$ then $\pi(E) = 0 \Leftrightarrow E$ is inessential in H (that is, $Q(x, E) = 0$ for all $x \in H$). Let $E \subseteqq H$ where $\pi(E) = 0$. We want to show that $Q(x, E) = 0$ for all $x \in X$. Fix $x \in X$. Since $\sup_{x \in E} Q(x, E) \leqq \sup_{x \in H} Q(x, E) = 0$, we have, using Proposition 6 of [1], that

$$Q(x, E) = Q(x, E, E) =_{\text{def}} P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{X_k \in E\} \cap \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{X_k \in E\} \,\middle|\, X_0 = x\right) = 0$$

for all $x \in X$. Hence E is inessential. Thus we see that every essential set must have positive π-measure. Proposition 21 of [1] asserts that every perpetuable set is essential, which implies that condition $(\mathscr{E})$ holds on H.

Corollary 1. *Let X be indecomposable and absolutely essential. Then $(\mathscr{H})$ holds if and only if there exists a closed subset which contains no uncountable disjoint collection of improperly essential sets.*

Proof. If $(\mathscr{H})$ holds, then, by the fact that essential subsets of H have positive π-measure, we immediately have that H can contain no uncountable disjoint collection of improperly essential sets. From above, in i) ⇒ ii), we saw that if X is not normal, then there exists an uncountable disjoint collection of improperly essential and perpetuable sets. Hence, if condition $(\mathscr{H})$ does not hold, we have that there exists a closed set which contains an uncountable disjoint collection of improperly essential sets.

Remark. If X is absolutely essential and indecomposable and $\mathscr{B}$ is separable, then, by Theorem 3 of [5], condition $(\mathscr{H})$ holds, hence, by Theorem 1, condition $(\mathscr{E})$ holds on H.

Let $X = H + I$ and let φ be the Harris measure on H. If I is inessential, extend φ to $\mathscr{B}$ by $\varphi(B) = \varphi(B \cap H)$, $B \in \mathscr{B}$. Then, for $B \in \mathscr{B}$, $\varphi(B) > 0$ implies $Q(x, B) = 1$ for all $x \in H$ since $Q(x, B \cap H) = 1$ for all $x \in H$. Also, $L(x, B \cap H) = 1$ for all $x \in H$. If $x \in I$, then, since $Q(x, B \cap H) =$

$$1 - \sum_{n=0}^{\infty} \int_{B \cap H} P^n(x, dy)(1 - L(y, B \cap H))$$

(see [1], p. 232), we have that $Q(x, B \cap H) = 1$. Hence $Q(x, B) = 1$ for all $x \in X$ and φ is the Harris measure on X.

If I, as above, were improperly essential, and there existed a Harris measure $\bar{\varphi}$ on X, then $\bar{\varphi}(I)>0$. This follows from the fact that $\bar{\varphi}(E)=0$ if and only if E is inessential (see Jain [4], p. 209). But I is a countable union of inessential sets which all must have measure zero. Hence, if I is improperly essential, there exists no Harris measure on X.

The following example shows that there exists a decomposition $X=H+I$ where I' is improperly essential. Let $X=\mathbb{Z}$, the set of integers. Define a transition probability as follows:

$$P(X_1=2(j-1)\,|\,X_0=2j)=\frac{1}{2} \qquad j=0, \pm 1, \pm 2, \ldots,$$

$$P(X_1=2(j+1)\,|\,X_0=2j)=\frac{1}{2} \qquad j=0, \pm 1, \pm 2, \ldots,$$

$$P(X_1=2j+1\,|\,X_0=2j-1)=1-2^{-|2j|} \qquad j=0, \pm 1, \ldots,$$

$$P(X_1=2j\,|\,X_0=2j-1)=2^{-|2j|} \qquad j=0, \pm 1, \ldots.$$

Let $A=\{2j: j=0, \pm 1, \ldots\}$. Then A is closed. The Markov process on A is a symmetric random walk in one dimension. It is well-known that such a random walk is recurrent; i.e. $Q(x, \{y\})=1$ for all x and y in A. Hence, if φ is taken to be the counting measure on A, we have that $A=H$ is φ-recurrent. Let $x=-1$. Then

$$L(x, A)=\sum_{k=1}^{\infty} P(X_k \in A, X_i \in A^c\ 1 \leqq i \leqq k-1\,|\,X_0=x)$$

$$=\sum_{k=1}^{\infty}\left(\prod_{i=0}^{k-1}(1-2^{-|2i|})(2^{-|2k|}\right) \leqq \sum_{k=1}^{\infty} 2^{-|2k|}<1.$$

Hence, $Q(x, A)<1$ and A^c is perpetuable. By Proposition 21 of [1], A^c is essential. Consider $j_1<j_2$. For $n>j_2-j_1$, $P(X_n=2j_2-1 \setminus X_0=2j_1-1)=0$. Also, for all k, $P(X_k=2j_2-1 \setminus X_0=x)=0$ if $x=2j_1-1$ for $j_1 \geqq j_2$ or $x \in A$. Hence, if $y \in A^c$, $Q(x, \{y\})=0$ for all $x \in X$ and A^c is improperly essential.

2. Doeblin's Decomposition

We now state our main result.

Theorem 2. *($\mathscr{C}$) implies ($\mathscr{D}$).*

Lemma 1. (see [6], p. 288, [2], p. 71). *If ($\mathscr{C}$) holds, and every closed subset contains a further subset which is either indecomposable or improperly essential, then ($\mathscr{D}$) holds.*

Proof. Call a collection $\mathscr{A} \subseteqq \mathscr{B}$ admissable if it is a non-void, countable, disjoint collection whose elements are either indecomposable or improperly essential closed sets. By hypothesis, admissable collections exist. Order the class of all such collections by inclusion. A collection $\mathscr{S}$ of admissable sets is called a chain if and only if $\mathscr{A}_1$ and $\mathscr{A}_2 \in \mathscr{S}$ imply that either $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ or $\mathscr{A}_2 \subseteqq \mathscr{A}_1$. Using condition ($\mathscr{C}$), we have that the union of a chain of admissable collections is itself an admissable collection. Applying Zorn's lemma to the set of admissable collections yields a maximal collection $\mathscr{L}$. Let L be the union of all the members of $\mathscr{L}$. Then L is closed. If $X \setminus L$ contained a closed set, then $X \setminus L$ would contain a closed set C which is either indecomposable or improperly essential. But then $\mathscr{L} \cup \{C\}$ would be an admissable collection which contradicts the maximality of $\mathscr{L}$. Since $X \setminus L$ contains no closed sets, we have, using Proposition 14.1 [1], that $X \setminus L$ is either inessential or improperly essential. Let $D_1, D_2, \ldots$ be a possibly void or finite enumeration of the members of $\mathscr{L}$ which are indecomposable and absolutely essential. Then $I = (X \setminus L) \cup (L - \sum_{n=1}^{\infty} D_n) = X - \sum_{n=1}^{\infty} D_n$ is either inessential or improperly essential. Hence ($\mathscr{D}$) holds.

Proof of Theorem 2. Assume ($\mathscr{C}$) holds, but that ($\mathscr{D}$) does not. From Lemma 1 we can assume that there exists a closed subset which contains no indecomposable or improperly essential closed sets. For convenience we assume that X is this closed subset. Hence, every closed subset is assumed to be decomposable and absolutely essential.

Adhering to the mathematical framework used by Jamison in [6] we now define a collection $\mathscr{R} = \mathscr{R}(\mathrm{X})$ of closed sets in X. This definition is taken verbatim from [6] (p. 289). On p. 70 of [2], Doeblin shows that if a closed set E is not indecomposable then there are two disjoint closed subsets A and B of E such that $E \setminus (A \cup B)$ has no closed subset. We say that (A, B) is a *maximal* pair of closed subsets of E. It follows from Proposition 14.1 that $E \setminus (A \cup B)$ is either inessential or improperly essential. Now assume that ($\mathscr{C}$) holds and that every closed subset of X is decomposable and absolutely essential. Let Ω denote the first uncountable ordinal and let the lower case Greek letters denote the ordinals less than Ω. Our set theory is that of [7] (also see [9]), in which every ordinal is the set of all ordinals strictly less than it. A function s into $\{0, 1\}$ is called a *binary sequence* if its domain is equal to an ordinal $\alpha < \Omega$, s is then said to be of *order* α. If s is a binary sequence of order α, then the binary sequences $s \cup \{(\alpha, 0)\}$ and $s \cup \{(\alpha, 1)\}$ are denoted by $s0$ and $s1$ respectively, they are of order $\alpha + 1$. We now use transfinite induction to define on the class of all binary sequences a function $C(\cdot)$ such that

(i) $C(s)$ is either empty or a closed set,
(ii) if $s \subseteqq t$ then $C(t) \subseteqq C(s)$,
(iii) if neither $s \subseteqq t$ nor $t \subseteqq s$ then $C(s) \cap C(t) = \emptyset$ (the latter holding, in particular, if s and t are distinct and of the same order).

The unique binary sequence of order 0 is $\emptyset$. We define $C(\emptyset) = X$. Suppose $C(s)$ has been defined for all binary sequences of order less than α. Suppose α is not a limit ordinal, that is, $\alpha = \beta + 1$ for some ordinal β. Any sequence of order α is equal to $s0$ or $s1$ where s is a sequence of order β. If $C(s) = \emptyset$, we define $C(s0) = C(s1) = \emptyset$. Otherwise $C(s)$ is closed, and by assumption, decomposable. Let (A, B) be a maximal pair of closed subsets of $C(s)$, and $C(s0) = A$ and $C(s1) = B$. Suppose, on the other hand, that α is a limit ordinal. Let s be of order α. For any $\beta < \alpha$ let s_β be the restriction of s to β. We define $C(s)$ to be $\bigcap_{\beta < \alpha} C(s_\beta)$. Since the intersection is countable, $C(s)$ is either empty or closed. The definition of $C(s)$ for all binary sequences is now complete by virtue of the principle of transfinite induction. It is clear that (i) and (ii) hold.

We now show that (iii) holds. Let $\alpha_1 = \text{order } s$ and let $\alpha_2 = \text{order } t$. For convenience we assume that $\alpha_2 \geqq \alpha_1$. If neither $s \subseteqq t$ nor $t \subseteqq s$, then $t_{\alpha_1} \neq s$. Let α_3

be the first ordinal such that $t_{\alpha_3} \neq s_{\alpha_3}$. If α_3 is a limit ordinal, then, for all $\delta < \alpha_3$, we have that $s_\delta = t_\delta$. But using the fact that $\bigcup_{\delta < \alpha_3} = \alpha_3$, we have that $s_{\alpha_3} = t_{\alpha_3}$. This contradiction implies that α_3 cannot be a limit ordinal and, hence, we have that $\alpha_3 = \beta_3 + 1$ for some ordinal β_3. By the construction of $C(\cdot)$, $(C(t_{\alpha_3}), C(s_{\alpha_3}))$ is a maximal pair in $C(s_{\beta_3}) = C(t_{\beta_3})$. By property (ii) we have that $C(s) \subseteqq C(s_{\alpha_3})$ and $C(t) \subseteqq C(t_{\alpha_3})$. Thus $C(s) \cap C(t) \subseteqq C(s_{\alpha_3}) \cap C(t_{\alpha_3}) = \emptyset$.

Let $\mathscr{R} = \{C(s): s \text{ is a binary sequence}\} - \{\emptyset\}$, that is, $\mathscr{R}$ is the range of the function $C(\cdot)$ just defined but with $\emptyset$ thrown out. It follows from (i), (ii), and (iii) that the members of $\mathscr{R}$ are closed and that (iv) $E \in \mathscr{R}$ and $F \in \mathscr{R}$ imply $E \subseteqq F$ or $F \subseteqq E$ or $E \cap F = \emptyset$. From our definition of $C(\cdot)$, it is clear that for each closed set $B \subseteqq X$ we can define a function $C_B(\cdot)$ on the class of all binary sequences such that $C_B(\emptyset) = B$ and $\mathscr{R}(B)$ is a collection of closed sets, formed in the same manner as $\mathscr{R}$, descending from B.

We state several definitions and prove several propositions which will be used in the proof of Theorem 2. By the α-th *level* we mean $\{C(s): \text{order } s = \alpha\} =_{\text{def}} \mathscr{S}_\alpha$. Let $D \in \mathscr{R}$. Let $\alpha < \Omega$ be arbitrary but large (by large we mean $\alpha >$ order s_D where $C(s_D) = D$). By a *successor* of D on the α-th level, we mean a closed set (if it exists) $E = C(s)$ where order $s = \alpha$ and $E \subseteqq D$.

Let $K(\beta)$ be the union of the sets on the β-th level. As previously stated any two sets on the β-th level must be disjoint, and hence, using condition ($\mathscr{C}$), there can exist at most a countable collection of closed sets on the β-th level. Thus $K(\beta)$ is either empty or closed.

Proposition 3 ([6], p. 290). *For each $\beta < \Omega$, $X \setminus K(\beta)$ is either inessential or improperly essential.*

Proof. For each s of order α for which $C(s) \neq \emptyset$, $(C(s0), C(s1))$ is a maximal pair in $C(s)$, hence $C(s) \setminus (C(s0) \cup C(s1))$ is either inessential or improperly essential. This immediately implies that $K(\alpha) \setminus K(\alpha+1) = \bigcup \{C(s) \setminus (C(s0) \cup C(s1))$: order $s = \alpha\}$ is either inessential or improperly essential for all $\alpha < \Omega$. If we show that $X \setminus K(\beta) = \bigcup_{\alpha < \beta} (K(\alpha) \setminus K(\alpha+1))$, then the proposition follows. Let $x \in \bigcup_{\alpha < \beta} (K(\alpha) \setminus K(\alpha+1))$. Then $x \in K(\alpha_0) \setminus K(\alpha_0 + 1)$ for some $\alpha_0 < \beta$ which implies that $x \notin K(\beta)$ since $K(\beta) \subseteqq K(\alpha_0 + 1)$. If $x \notin \bigcup_{\alpha < \beta} (K(\alpha) \setminus K(\alpha+1))$, then, since $x \in K(0) = X$, we have that $x \in K(\alpha+1)$ for all $\alpha < \beta$. Hence $x \in K(\delta)$ for some $\delta \geqq \beta$ $x \in K(\beta)$.

Proposition 4. *Let $\alpha < \Omega$. Then X has a successor on the α-th level.*

Proof. If X has no successor on the α-th level, then $K(\alpha) = \emptyset$. By Proposition 3, $X = X \setminus K(\alpha)$ is either inessential or improperly essential which is a contradiction.

It follows immediately from Proposition 4 that $\mathscr{R}$ must necessarily uncountable. Call a collection $\mathscr{C}$ a *chain* if C_1 and $C_2 \in \mathscr{C}$ imply that either $C_1 \subseteqq C_2$ or $C_2 \subseteqq C_1$.

Proposition 5 ([6], p. 290). *$\mathscr{R}$ contains no uncountable chains.*

Proof. Assume that $\{C(s^\gamma): \gamma \in \Gamma\}$ is an uncountable chain where Γ is uncountable, and where s^{γ_1} and s^{γ_2} are distinct binary sequences if γ_1 and γ_2 are distinct members of Γ. By (ii) and (iii), the set $\{s^\gamma: \gamma \in \Gamma\}$ is a chain of binary sequences relative to the inclusion relation. Letting $\sigma = \bigcup\{s^\gamma : \gamma \in \Gamma\}$, we see that σ is a function from Ω to $\{0, 1\}$ such that, for each $\gamma \in \Gamma$, the restriction of σ to the ordinal which is the order of s^γ is s^γ. For each $\alpha < \Omega$, let σ_α be the restriction of σ to α. Then each set $C(\sigma_\alpha)$ is closed and $C(\sigma_{\alpha+1})$ is one member of a maximal pair of closed subsets of $C(\sigma_\alpha)$. Denote the other member of this pair by $D(\alpha+1)$. If $\beta > \alpha$, then $C(\sigma_\beta) \subseteqq C(\sigma_\alpha)$, hence, $C(\sigma_\beta) \cap D(\alpha+1) = \emptyset$. Thus $\{D(\alpha+1) : \alpha < \Omega\}$ is a collection of pairwise disjoint closed sets which contradicts condition $(\mathscr{C})$.

Lemma 2. *Assume that every closed subset of X is absolutely essential and decomposable and that $(\mathscr{C})$ holds. Let B be a closed set and $\mathscr{R}(B)$ be the collection of closed sets formed as above. Then there exists no σ-finite measure m such that $C \in \mathscr{R}(B)$ implies $m(C) > 0$.*

Proof. Assume that there exists a σ-finite measure m supported on B such that $C \in \mathscr{R}(B)$ implies that $m(C) > 0$. For convenience we may assume that m is a probability measure by passing to an equivalent finite measure and normalizing. The following argument is a modification of arguments due to Doeblin (see [2]) and Harris (see [1, 3]).

We have assumed that $m(B) = 1$. For each $x \in B$ set $C_x = \bigcap\{C \in \mathscr{R}(B) : x \in C\}$. We will show that for each $x \in B$, there exists a binary sequence s_x such that $C_x = C(s_x)$. Let $\mathscr{C}_x =_{\text{def}} \{C \in \mathscr{R}(B) : x \in C\}$. If C_1 and $C_2 \in \mathscr{C}_x$, then $C_1 \cap C_2 \neq \emptyset$, and, using property (iv), either $C_1 \subseteqq C_2$ or $C_2 \subseteqq C_1$. Hence $\mathscr{C}_x$ is a chain and must be countable by Proposition 5. Then $\mathscr{C}_x = \{C(s^\gamma) : \gamma \in \Gamma\}$ where Γ is a countable set. Since $\{s^\gamma : \gamma \in \Gamma\}$ is a chain relative to the inclusion relation, $r =_{\text{def}} \bigcup_{\gamma \in \Gamma} s^\gamma$ is a binary sequence of order $< \Omega$, say $\beta < \Omega$. Let δ_γ be the order of s^γ for all $\gamma \in \Gamma$. Then $r_{\delta_\gamma} = s^\gamma$ (here r_{δ_γ} denotes the restriction of r to δ_γ). By the definition of $\mathscr{R}(B)$, $C(r) = \bigcap_{\alpha \leqq \beta} C(r_\alpha)$. It is immediate that $C(r) \subseteqq \bigcap_{\gamma \in \Gamma} C(s^\gamma)$. Let $y \in \bigcap_{\gamma \in \Gamma} C(s^\gamma)$. Then $y \in C(s^\gamma) = C(r_{\delta_\gamma})$ for $\gamma \in \Gamma$. Let $\alpha \leqq \delta_\gamma$ for some $\gamma \in \Gamma$. Then $y \in C(r_{\delta_\gamma}) \subseteqq C(r_\alpha)$. Hence $y \in C(r_\alpha)$ for all $\alpha \leqq \sup_{\gamma \in \Gamma} \delta_\gamma = \beta$ which implies that $\bigcap_{\alpha \leqq \beta} C(r_\alpha) \subseteqq \bigcap_{\gamma \in \Gamma} C(s^\gamma)$. Set $s_x = r$. Then $C(s_x) = \bigcap_{\alpha \leqq \beta} C(r_\alpha) = \bigcap_{\gamma \in \Gamma} C(s^\gamma) = C_x$.

For each $n \geqq 1$, let $E_n = \{x \in B : m(C_x) \leqq 1/n\}$. If $y \in C_x$, then $m(C_y) \leqq m(C_x)$ since $C_y \subseteqq C_x$. It follows that if $x \in E_n$, then $C_x \subseteqq E_n$ so that $E_n = \bigcup\{C_x : x \in E_n\}$ is the union of closed sets for each $n \geqq 1$. We must show that each E_n is closed. In order to do this it is sufficient to show that each E_n is a countable union of closed sets in $\mathscr{R}(B)$.

Fix $n \geqq 1$. Call a closed set $C \in \mathscr{R}(B)$ maximal in E_n if $m(C) \leqq 1/n$ and C is not properly contained in any subset $D \in \mathscr{R}(B)$ such that $m(D) \leqq 1/n$. Maximal subsets exist by the following argument. Let $C = C(s) \in \mathscr{R}(B)$ where s has order α. Let β_0 be the first element in the set $\{\beta \leqq \alpha : m(C(s_\beta)) \leqq 1/n\}$ (here s_β again denotes the restriction of s to β). Then $C(s_{\beta_0})$ is maximal in E_n. Let C_1 and C_2 be two distinct maximal subsets in E_n. Then either $C_1 \subseteqq C_2$ or $C_2 \subseteqq C_1$ or $C_1 \cap C_2 = \emptyset$ by property (iv). By the definition of maximal in E_n, neither $C_1 \subseteqq C_2$ nor $C_2 \subseteqq C_1$ can hold. Hence $C_1 \cap C_2 = \emptyset$.

Let $\mathscr{C}_n$ be the collection of maximal subsets in E_n. Since distinct maximal subsets in E_n are disjoint, by $(\mathscr{C})$ $\mathscr{C}_n$ can contain at most a countable number of distinct sets. If $x \in E_n$, then $C_x \subseteqq E_n$ and $m(C_x) \leqq 1/n$. Then, by the main argument used to establish the existence of maximal sets in E_n, C_x is contained in a set D which is maximal in E_n. Hence $x \in \bigcup\{C \in \mathscr{C}_n\}$ which implies that $E_n \subseteqq \bigcup\{C \in \mathscr{C}_n\}$. Let $y \in \bigcup\{C \in \mathscr{C}_n\}$. Then $y \in C_0$ for some $C_0 \in \mathscr{C}_n$ which implies that $m(C_y) \leqq m(C_0) \leqq 1/n$ since $C_y \subseteqq C_0$. Hence $y \in E_n$ and $E_n = \bigcup\{C \in \mathscr{C}_n\}$.

Clearly $B = E_1 \supseteqq E_2 \supseteqq \cdots$. Let $x \in B$, then there exists a first n_0 such that $m(C_x) > 1/n_0$. Hence $x \in E_{n_0-1} \setminus E_{n_0}$ and $B = \bigcup_{n \geqq 1} (E_n \setminus E_{n-1})$. We now show that B is improperly essential which will contradict the hypothesis that B is absolutely essential. In order to do this it is sufficient to show that each $E_n \setminus E_{n+1}$ is improperly essential. We know that for each $n \geqq 1$, E_n is the union of closed sets $C \in \mathscr{R}(B)$ where each C is maximal in E_n. Fix $n \geqq 1$ and let $x \subseteqq E_n \setminus E_{n+1}$. Let $\mathscr{D} =_{\mathrm{def}} \{D_\alpha = C(s^\alpha)\}$ be the collection of closed sets $D_\alpha \in \mathscr{R}(B)$ which are maximal in E_{n+1}. Let $\beta = \sup_\alpha$ order s^α. Since $\mathscr{D}$ is countable we have that $\beta < \Omega$.

We now will show that $x \in \bigcup_{\alpha \leqq \beta+1} (X \setminus K(\alpha+1)) =_{\mathrm{def}} G(\beta+1)$. By Proposition 3 each $G(\beta+1)$ is either inessential or improperly essential. If $x \notin G(\beta+1)$, then $x \in K(\alpha+1)$ for all $\alpha \leqq \beta+1$; in particular, $x \in K(\beta+1)$. Hence $x \in C$ for some $C \in \mathscr{R}(B)$ on the $(\beta+1)$-st level and $C_x = C(s)$ for some sequence s of order greater than or equal to $\beta+1$. From the fact that $x \in E_n \setminus E_{n+1}$ we have that $1/(n+1) < m(C(s)) \leqq 1/n$. Since $C(s1)$ and $C(s0)$ are disjoint, $m(C(s1)) > (1/2)m(C(s))$ and $m(C(s0)) > (1/2)m(C(s))$ cannot both hold. Assume that $m(C(s1)) \leqq (1/2)m(C(s))$. Then $m(C(s1)) \leqq 1/2n \leqq 1/(n+1)$. Since any set $C \in \mathscr{R}(\mathrm{B})$ properly containing $C(s1)$ must contain $C(s)$, we have that $C(s1)$ is maximal in E_{n+1}. But the order of $s1$ is strictly greater β which is a contradiction to the definition of β. Hence $x \in G(\beta+1)$ which implies that $E_n \setminus E_{n+1} \subseteqq G(\beta+1)$. Since $G(\beta+1)$ is improperly essential $E_n \setminus E_{n+1}$ is improperly essential.

Conclusion of the Proof of Theorem 2. Assume that every closed set is absolutely essential and decomposable and that $(\mathscr{C})$ holds. Then, as remarked following Proposition 4, $\mathscr{R}$ is uncountable. Assume that there exists a σ-finite measure m such that $m(K(\beta)) > 0$ for all $\beta < \Omega$. Call a set $C \in \mathscr{R}$ *maximal m-null* if $m(C) = 0$ and C is not properly contained in $D \in \mathscr{R}$ such that $m(D) = 0$. By Lemma 2 there exists $C \in \mathscr{R}$ such that $m(C) = 0$. Then $C = C(s)$ where s is of order α. Let β_0 be the first element in the set $\{\beta \leqq \alpha: m(C(s_\beta)) = 0\}$. Then $C(s_{\beta_0})$ is a maximal m-null set.

Let $\mathscr{M}(\beta)$ be the collection of maximal m-null sets on the β-th level; $\mathscr{M}(\beta)$ is countable, but possibly empty. Let $\beta_1 < \beta_2$. If $C_1 \in \mathscr{M}(\beta_1)$ and $C_2 \in \mathscr{M}(\beta_2)$, then C_2 must necessarily be a successor of a set on the β_1-st level with positive m-measure. Hence $C_1 \cap C_2 = \emptyset$. Assume $\sup\{\beta: \mathscr{M}(\beta) \text{ nonempty}\} =_{\mathrm{def}} \alpha_0 < \Omega$. If there exists a set C on the (α_0+1)-st level such that $m(C) > 0$, then C has a successor D on some level, say the δ-th level, such that $m(D) = 0$. If not, then using Lemma 2 with $\mathscr{R}(C) = \{D \in \mathscr{R}: D \subseteqq C\}$, we can show that C is improperly essential which is a contradiction. Let $D = C(r)$ and let δ_0 be the first element in the set $\{\beta \leqq \text{order } r: m(C(r_\beta)) = 0\}$. Then $\alpha_0 + 1 < \delta_0 \leqq \delta$ and $\mathscr{M}(\delta_0)$ is nonempty which contradicts the definition of α_0. Hence $m(C) = 0$ which implies that $m(K(\alpha_0+1)) = 0$, which contradicts the assumption that $(K(\beta)) > 0$ for all $\beta < \Omega$. Thus $\sup\{\beta: \mathscr{M}(\beta) \text{ nonempty}\} = \Omega$. But then there must exist uncountably many β such that $\mathscr{M}(\beta)$ is nonempty. This follows from the fact that $\sup\{\alpha < \Omega: \alpha \in A\} = \Omega$ if and only if A contains an uncountable number of countable ordinals. By choosing one C in each nonempty $\mathscr{M}(\beta)$, we have an uncountable disjoint collection of closed sets which contradicts ($\mathscr{C}$). Hence we have proved the theorem if we show that there exists a σ-finite measure m such that $m(K(\beta))>0$ for all $\beta<\Omega$.

For each $E\in\mathscr{B}$ we set $E^0=\{x: L(x,E)=0\}$. If E^0 is nonempty, then E^0 is closed by Proposition 1 of [7], and if, in addition, E is closed, then $E^0\subseteqq E^c$. Let $\beta<\Omega$. If $K(\beta)^0$ is nonempty, then $K(\beta)^0$ is a closed set contained in $X\setminus K(\beta)$. Since, by Proposition 3, $X\setminus K(\beta)$ is improperly essential, we have that $K(\beta)^0$ is improperly essential which contradicts our assumption that every closed set is absolutely essential. Hence $K(\beta)^0$ is empty for all $\beta<\Omega$. Let $x\in X$. Set $m(\cdot)=\sum_{k=1}^{\infty}(1/2^k)P^k(x,\cdot)$. Since $m(B)>0$ if and only if $L(x,B)>0$, we have that $m(K(\beta))>0$ for all $\beta<\Omega$.

Corollary 2. *If ($\mathscr{E}$) holds, the $X=I+\sum_{n=1}^{\infty}H_n$ where each H_n is recurrent in the sense of Harris and I is either inessential or improperly essential.*

Proof. Since ($\mathscr{E}$) implies ($\mathscr{C}$) we have that $X=J+\sum_{n=1}^{\infty}D_n$ where J is either inessential or improperly essential and each D_n is an absolutely essential and indecomposable closed set. By Theorem 1 each $D_n=H_n+J_n$ where H_n is recurrent in the sense of Harris and J_n is either inessential or improperly essential. Setting $I=J+\sum_{n=1}^{\infty}J_n$ yields the desired result.

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