Z. Wahrscheinlichkeitstheorie verw. Geb. 22, 25–32 (1972)

Changes of Time, Stochastic Integrals, and Weak Martingales

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1. Time Changes Transformations

We need first a series of definitions.

Let Ω be a set, $\mathfrak{F}$ a Borel field of subsets of Ω, P a probability measure defined on $(\Omega, \mathfrak{F})$. We are given a family $(\mathfrak{F}_t)$ of Borel subfields of $\mathfrak{F}$, increasing and right continuous ($t \in [0, \infty[$). We may, and do, assume that $\mathfrak{F}$ has been completed with respect to P, and that each $\mathfrak{F}_t$ contains all $\mathfrak{F}$-sets of measure 0.

We assume that the reader knows the usual definitions: stopping times, martingales, etc. A notation such that "let $M=(M_t, \mathfrak{F}_t)$ be a martingale" means that the martingale property is relative to the $\mathfrak{F}_t$ family. All martingales below are assumed to be right continuous.

By a *change of time* $T=(\mathfrak{F}_t, \tau_t)$ we mean a family of stopping times of the $\mathfrak{F}_t$ family, *finite valued*, such that for $\omega \in \Omega$ the sample function $\tau_.(\omega)$ is increasing[1] and right continuous. We say that the change of time is *continuous* if these sample functions are continuous. Of course change of time $T=(\mathfrak{G}_t, \tau_t)$ can be defined with respect to some other right continuous family $\mathfrak{G}_t$ of sub-σ-fields of $\mathfrak{F}$. If then $X=(X_t, \mathfrak{G}_t)$[2] is a right continuous process, we denote by TX the process $(X_{\tau_t}, \mathfrak{G}_{\tau_t})$, called the *time changed process* of X. We say that the change of time T is X-*continuous* if X is constant on all intervals $[\tau_{t-0}, \tau_t]$, and on $[0, \tau_0]$.

As usual, we do not distinguish two processes X and Y such that for a.e. ω $X_\bullet(\omega) \equiv Y_\bullet(\omega)$. This is important for the understanding of uniqueness statements below.

Lemma 1. *Let $T=(\mathfrak{F}_t, \tau_t)$ be a change of time. Then the $\mathfrak{F}_{\tau_t}$ family is right continuous.*

Proof. Clearly this family is increasing, and so it suffices to prove that $\mathfrak{F}_{\tau_t+0} \subset \mathfrak{F}_{\tau_t}$. For every $A \in \mathfrak{F}_{\tau_t+0}$ and every $u \geqq 0$ we have

$$A \cap \{\tau_{t+h} < u\} \in \mathfrak{F}_u \qquad (h>0)$$

from the right continuity of T it follows that $A \cap \{\tau_t < u\} \in \mathfrak{F}_u$. Thus $A \in \mathfrak{F}_{\tau_t}$.

Lemma 2. *Let $T=(\mathfrak{F}_t, \tau_t)$ and $S=(\mathfrak{F}_{\tau_t}, s_t)$ be two changes of time. Then $ST=(\mathfrak{F}_t, \tau_{s_t})$ is a change of time.*

Proof. Lemma 2 amounts to the following fact: if R is a stopping time of the $\mathfrak{F}_{\tau_t}$ family, then τ_R is a stopping time of the $\mathfrak{F}_t$ family. Otherwise stated, we want

[1] By *increasing* we mean "non decreasing", by *positive* "non negative".

[2] This notation means that X_t is adapted to the $\mathfrak{G}_t$ family.

to prove that $\{\tau_R < t\} \in \mathfrak{F}_t$ for all t. It follows from right continuity that this event is the union, over all rationals r, of the events $\{R<r, \tau_r<t\}$. Since R is a stopping time of the $\mathfrak{F}_{\tau_t}$ family, $\{R<r\}$ belongs to $\mathfrak{F}_{\tau_r}$, and therefore $\{R<r\} \cap \{\tau_r<t\}$ belongs to $\mathfrak{F}_t$. This completes the proof.

The following definition of a local martingale is slightly different from that of [3], but the authors themselves agree that it is the most convenient one.

Definition 1. A stochastic process $M=(M_t, \mathfrak{F}_t)$ is said to be a local martingale (resp. a locally square integrable martingale) if there exists an increasing sequence (T_n) of stopping times of $(\mathfrak{F}_t)$ such that $\lim_n T_n = \infty$, and for each n the process $(M_{t \wedge T_n} I_{\{T_n>0\}}, \mathfrak{F}_t)$ is an uniformly integrable martingale (resp. an L^2-bounded one).

Note that if M_0 is integrable, the process $(M_{t \wedge T_n}, \mathfrak{F}_t)$ also is a uniformly integrable martingale.

Proposition 1. *Let $M=(M_t, \mathfrak{F}_t)$ be a local martingale. If $T=(\mathfrak{F}_t, \tau_t)$ is a finite M-continuous change of time, then TM is a local martingale. This applies in particular to continuous changes of time.*

Proof. Set $N_t = M_{\tau_t}$, and denote by (T_n) a sequence of stopping times satisfying the conditions of Definition 1. Set $J_n = \inf\{u \geqq 0: \tau_u \geqq T_n\}$. It is immediately seen that for each n J_n is a stopping time of the $\mathfrak{F}_{\tau_t}$ family, that $J_n \uparrow \infty$, and that $\{0<J_n\} \subset \{0<T_n\}$. The theorem therefore will be proved if we show that for each n $(N_{t \wedge J_n}, \mathfrak{F}_{\tau_t})$ is an uniformly integrable martingale on $\{0<T_n\}$. Now define

$$D_n = \tau_{J_n - 0}, \qquad E_n = \tau_{J_n}.$$

For $t<J_n$, we have $N_{t \wedge J_n} = M_{\tau_t \wedge T_n}$, and for $t \geqq J_n$ we have $N_{t \wedge J_n} = M_{E_n}$. Since T is M-continuous, M is constant and equal to M_{T_n} on the interval $[D_n, E_n]$. Therefore we have $N_{t \wedge J_n} = M_{\tau_t \wedge T_n}$. Now the process $(M_{t \wedge T_n} I_{\{T_n>0\}})$ is a uniformly integrable martingale, and so is $(M_{\tau_t \wedge T_n} I_{\{T_n>0\}})$ by Doob's optional sampling theorem.

Lemma 3. *Let U and V be stopping times of the $\mathfrak{F}_t$ family, such that $U \leqq V$. Then there exists a continuous change of time $T=(\mathfrak{F}_t, \tau_t)$ such that $\tau_0 = U$, $\tau_\infty \, (= \lim_{t \to \infty} \tau_t) = V$.*

Proof. Put $\tau_t = U \vee (V \wedge t)$. Then $(\mathfrak{F}_t, \tau_t)$ is a change of time possessing the desired properties.

Remark. The sample functions $\tau_\bullet(\omega)$ aren't strictly increasing. This is impossible in general, since the stopping time V can be totally inaccessible.

Theorem 1. *Let $M=(M_t, \mathfrak{F}_t)$ be a local martingale (resp. a locally square integrable martingale) such that $M_0=0$. Then there exists a continuous change of time $S=(\mathfrak{F}_t, s_t)$, increasing from 0 to $+\infty$, such that SM is a martingale (resp. a square integrable martingale[3]).*

Proof. Let $(T_n)_{n \geqq 1}$ be a sequence of stopping times satisfying the conditions of Definition 1. Without loss of generality we may suppose $T_n \leqq n$ for all n, and set $T_0=0$. According to Lemma 3, for each $n \geqq 1$ there exists a continuous change of time $S^n = (\mathfrak{F}_t, s_t^n)$ such that $s_0^n = T_{n-1}$, $s_\infty^n = T_n$. For each n, let q_n be an increasing

[3] A martingale N is said to be square integrable if $E[N_t^2] < \infty$ for all t.

bijection of $[n-1, n]$ onto $[0, \infty]$. We have

$$s^n_{q_n(n-1)} = s^n_0 = T_{n-1} = s^{n-1}_\infty = s^{n-1}_{q_{n-1}(n-1)}.$$

The formula

$$s_t = s^n_{q_n(t)} \quad \text{if } n-1 \leqq t \leqq n$$

therefore defines a continuous change of time. By Doob's optional sampling theorem $(M_{s_t \wedge T_n}, \mathfrak{F}_{s_t})$ is a uniformly integrable martingale (resp. an L^2-bounded one), and this means, since $s_n = T_n$, that $SM = (M_{s_t}, \mathfrak{F}_{s_t})$ is a uniformly integrable (resp. an L^2-bounded) martingale when restricted to $[0, n]$. This completes the proof.

Remark. Set $\mathfrak{G}_t = \mathfrak{F}_{s_t}$, $N_t = M_{s_t}$, and $a_t = \inf\{u\colon s_u > t\}$. Then it is very easy to check that $A = (\mathfrak{G}_t, a_t)$ is a change of time, and AN is equal to M.

We are going now to investigate the behaviour of stochastic integrals with respect to a local martingale M under *continuous* changes of time. The definitions of $\langle M, M\rangle$, $[M, M]$, predictable processes, M^c and M^d (the continuous and discontinuous parts of M), and the stochastic integral $C \cdot M$ are taken from [3]. However, we shall summarily recall them during the proof, for the reader's convenience.

Proposition 2. *Let $M = (M_t, \mathfrak{F}_t)$ be a local martingale such that $M_0 = 0$, and let $T = (\mathfrak{F}_t, \tau_t)$ be a continuous change of time such that $\tau_0 = 0$. Then we have*

1) $(TM)^c = T(M^c)$; $(TM)^d = T(M^d)$.

2) $[TM, TM] = T([M, M])$, *and if M is locally square integrable* $\langle TM, TM\rangle = T(\langle M, M\rangle)$.

3) *If $C = (C_t, \mathfrak{F}_t)$ is predictable and locally bounded, then TC is predictable and locally bounded with respect to the family $(\mathfrak{F}_{\tau_t})$, and we have $TC \cdot TM = T(C \cdot M)$.*

Proof. We set $N_t = M_{\tau_t}$, $\mathfrak{G}_t = \mathfrak{F}_{\tau_t}$.

We first remark that if $C = (C_t, \mathfrak{F}_t)$ is an adapted and left continuous process, then TC is also adapted and left continuous with respect to the $\mathfrak{G}_t$ family. Therefore, since predictable processes are those which (as functions of (t, ω)) are measurable with respect to the σ-field generated by all adapted left continuous processes, it follows that if C is predictable, so is TC with respect to the $\mathfrak{G}_t$ family.

We start with the case of an L^2-*bounded* martingale M. Then $\langle M, M\rangle$ is defined as the only predictable increasing process such that

$$\langle M, M\rangle_0 = 0, \qquad \langle M, M\rangle - M^2 \qquad \text{is a martingale}.$$

In this case, this martingale is even uniformly integrable. Therefore $T(\langle M, M\rangle)$ is 0 at time 0, and $T(\langle M, M\rangle - M^2)$ is a martingale (uniformly integrable). This implies $T(\langle M, M\rangle) = \langle TM, TM\rangle$.

Next, we recall that the L^2-norm $\|M\|_2$ is defined as the norm in L^2 of the random variable $M_\infty = \lim_{t \to t\infty} M_t$. It is clear that $\|TM\|_2 \leqq \|M\|_2$. M is said to be *purely discontinuous* if M is orthogonal to every L^2-bounded, continuous martingale. This amounts to saying that M is the limit (in the L^2-norm) of L^2-bounded martingales M^i the sample functions of which have bounded variation on $[0, \infty]$. The martingales TM^i possess the same property, and therefore if M is purely discontinuous, so is TM.

M can be uniquely decomposed as $M=M^c+M^d$, where M^c and M^d are L^2-bounded, M^c is continuous and M^d purely discontinuous. We obviously have $TM=T(M^c)+T(M^d)$, $T(M^c)$ is continuous and $T(M^d)$ purely discontinuous from the above. Therefore this must be the unique decomposition of TM.

The increasing process $[M, M]$ is defined as

$$[M, M]_t=\langle M^c, M^c\rangle_t+\sum_{s\leqq t}(\Delta M_s)^2$$

where ΔM_s is the jump of M at s. This formula shows at once that $[TM, TM]=T([M, M])$.

If C is an "elementary" predictable process, i.e. if there exist times

$$0=t_0<t_1\ldots<t_n=\infty,$$

and random variables $c_0, \ldots, c_n$ such that c_k is $\mathfrak{F}_{t_k}$-measurable for $k=0, \ldots, n$, and $C_t=c_k$ for $t_k<t\leqq t_{k+1}$, then an elementary computation shows immediately that $T(C\cdot M)=TC\cdot TM$. If we call $L^2(M)$ the set of all predictable processes C such that the norm

$$\left(E\left[\int_0^\infty C_s^2\, d\langle M, M\rangle_s\right]\right)^{\frac{1}{2}}$$

is finite, then the stochastic integral operator $C\mapsto C\cdot M$ is a continuous operator from $L^2(M)$ to the space of L^2-bounded martingales, and the "elementary" predictable processes $C\in L^2(M)$ are dense in $L^2(M)$. The norm of TC in $L^2(TM)$ is at most that of C in $L^2(M)$, and the relation $T(C\cdot M)=TC\cdot TM$ extends by continuity to all $C\in L^2(M)$. This completes the proof of the L^2-bounded case.

Next we assume that M is a local martingale, and we consider a sequence (T_n) of stopping times satisfying the conditions of Definition 1. As in Proposition 1, we set $J_n=\inf\{u\geqq 0: \tau_u\geqq T_n\}$, and we denote by M^n the martingale M stopped at T_n, by N^n the martingale N stopped at J_n. We have $N^n=TM^n$, because T is continuous.

The process M^n is an uniformly integrable martingale, but not an L^2-bounded one. However, it is shown in [3] that the stopping times T_n can be chosen in such a way that one can write

$$M^n=H^n+V^n$$

where H^n is an L^2-bounded martingale, and V^n an uniformly integrable martingale, the sample functions of which have bounded variation on $[0, \infty]$, and even $E\left[\int_0^\infty |dV_s|\right]<\infty$ ([3], Proposition 4). We set $TH^n=K^n$, $TV^n=W^n$.

The martingales H^n and V^n aren't uniquely determined, but the *continuous* part of H^n is uniquely determined, since V^n has bounded variation. Let us denote it by M^{nc}. We define M^c as the unique continuous local martingale which coincides with M^{nc} for $0\leqq t\leqq T_n$. We have from the above $T(M^{nc})=T(H^{nc})=K^{nc}=N^{nc}$, and therefore $T(M^c)=N^c$. Defining M^d as $M-M^c$, we have $T(M^d)=N^d$. We have, also from the results on L^2-bounded martingales, $T(\langle M^c, M^c\rangle)=\langle N^c, N^c\rangle$, if we define $\langle M^c, M^c\rangle$ as the process which coincides with $\langle M^{nc}, M^{nc}\rangle$ for $0\leqq t\leqq T_n$.

When M is locally square integrable, we can take $V^n=0$ for every n, and define $\langle M, M\rangle$ as the process which coincides with $\langle M^n, M^n\rangle$ for $0 \leqq t \leqq T_n$. Then we have $T(\langle M, M\rangle)=\langle N, N\rangle$.

We define $[M, M]_t$ as $\langle M^c, M^c\rangle_t + \sum_{s \leqq t} (\Delta M_s)^2$. It is obvious that $T([M, M]) = [N, N]$.

Let C be a predictable locally bounded process: this means that the stopping times T_n can be chosen in such a way that the processes $C^n=(C_{t \wedge T_n} I_{\{T_n>0\}}, \mathfrak{F}_t)$ are uniformly bounded in absolute value (by constants which may depend on n). Then if we set $D=TC$, the processes $D^n=(D_{t \wedge J_n} I_{\{J_n>0\}}, \mathfrak{G}_t)$ are smaller in absolute value than TC^n, and therefore D is locally bounded. The stochastic integral $C \cdot M$ is defined as the local martingale which coincides for $0 \leqq t \leqq T_n$ with $C^n \cdot M^n = C^n \cdot H^n + C^n \cdot V^n$, where $C^n \cdot H^n$ has been defined above, and $C^n \cdot V^n$ is an ordinary Stieltjes integral. It can be shown that it depends neither on the decomposition $M^n = H^n + V^n$, nor on the stopping times T_n. It follows from the results on L^2-bounded martingales that $T(C^n \cdot H^n) = TC^n \cdot TH^n$, and it is obvious that $T(C^n \cdot V^n) = TC^n \cdot TV^n$. Therefore $T(C^n \cdot M^n) = TC^n \cdot TM^n$, and letting n tend to infinity, that $T(C \cdot M) = TC \cdot TM$. This completes the proof.

2. Weak Martingales

Definition 2. A stochastic process $M=(M_t, \mathfrak{F}_t)$ is said to be a *weak martingale* if there exists an increasing sequence (T_n) of stopping times of the $\mathfrak{F}_t$ family such that

i) for each n there exists a right continuous, uniformly integrable martingale $M^n=(M_t^n, \mathfrak{F}_t)$ such that $M_t = M_t^n$ for $0 \leqq t < T_n$.

ii) $\lim_n T_n = +\infty$.

For shortness, we shall say in this paper that a stopping time T *reduces* a right continuous process $M=(M_t, \mathfrak{F}_t)$ if there exists a uniformly integrable martingale $H=(H_t, \mathfrak{F}_t)$ such that $H_t = M_t$ for $0 \leqq t < T$. Since $(H_{t \wedge T})$ still is a uniformly integrable martingale, we can always assume that H is stopped at time T. Note that the word "reduces" isn't used here in the same sense as in [3], where it was demanded that $H_t = M_t$ also for $t = T$ on $\{T>0\}$. If T reduces M, so does any stopping time $S \leqq T$.

It is obvious that a local martingale is a weak martingale. Also, if $(M_t, \mathfrak{F}_t)$ is a weak martingale, so is $(M_{t \wedge T}, \mathfrak{F}_t)$ for any stopping time T.

The following theorem is the reason for considering weak martingales.

Theorem 2. *If $M=(M_t, \mathfrak{F}_t)$ is a weak martingale, then for every change of time $T=(\mathfrak{F}_t, \tau_t)$, TM is also a weak martingale.*

Proof. Denote by T_n, M^n, stopping times and martingales satisfying the conditions of Definition 2. As in the proof of Proposition 1, set $N_t = M_{\tau_t}$,

$$J_n = \inf\{u : \tau_u \geqq T_n\}.$$

We have seen in that proof that, $t < J_n$ implying $\tau_t < T_n$, for $t < J_n$, we have

$$N_t = M_{\tau_t} = M_{\tau_t}^n = N_t^n$$

where the process $(N_s^n, \mathfrak{F}_{\tau_s}) = (M_{\tau_s}^n, \mathfrak{F}_{\tau_s})$ is an uniformly integrable martingale according to Doob's optional sampling theorem. This means that the stopping times J_n reduce $(N_s, \mathfrak{F}_{\tau_s})$, which is therefore a weak martingale.

We are going now to investigate the properties of weak martingales.

Lemma 4. *Let $M = (M_t, \mathfrak{F}_t)$ be a right continuous process. Assume there exists a sequence (S_n) of stopping times reducing M (not necessarily increasing) such that $\lim_n S_n = +\infty$ a.e. Then M is a weak martingale.*

Proof. The stopping times $T_n' = \inf_{m \geqq n} S_m$ increase, reduce M, and converge a.e. to $+\infty$. To get stopping times which increase everywhere to $+\infty$ (as in Definition 2), remark that any positive random variable equal a.e. to a stopping time is itself a stopping time (since all null sets belong to $\mathfrak{F}_0$). Then set $T_n = T_n'$ for all n if $\lim_k T_k' = +\infty$, and $T_n = +\infty$ for all n otherwise. These stopping times satisfy the conditions of Definition 2.

Lemma 5. *Assume $M = (M_t, \mathfrak{F}_t)$ be a right continuous process. Assume there exists an increasing sequence (R_n) of stopping times, such that $\lim_n R_n = +\infty$ and that for each n there exists a weak martingale M^n equal to M for $0 \leqq t < R_n$. Then M is a weak martingale.*

Proof. Replacing if necessary R_n by $R_n \wedge n$ for all n, we may assume that the stopping times R_n are finite. Let then R_n' be a stopping time reducing M^n, such that $P\{R_n' < R_n\} < 2^{-n}$, and let S_n be $R_n' \wedge R_n$: S_n reduces M^n, and since $M_t^n = M_t$ for all $t < S_n$ it also reduces M. On the other hand, according to the Borel-Cantelli lemma we have $\lim_n S_n = +\infty$ a.s. We conclude the proof with an application of Lemma 4.

The following result is useful for many purposes. Semimartingales are defined in [3], but the meaning of this word will anyway be clear from the proof.

Proposition 3. *Let T reduce the weak martingale $(M_t, \mathfrak{F}_t)$, and let M^T be the process $(M_{t \wedge T}, \mathfrak{F}_t)$. Then M^T is a semimartingale.*

Proof. Let $(H_t, \mathfrak{F}_t) = H$ be an uniformly integrable martingale, such that $H_t = M_t$ for $0 \leqq t < T$. We may assume that H is stopped at time T. Since H is uniformly integrable, we have $H_t = E[H_\infty | \mathfrak{F}_t]$. Set

$$H_t^+ = E[H_\infty^+ | \mathfrak{F}_t], \qquad H_t^- = E[H_\infty^- | \mathfrak{F}_t].$$

These are two positive martingales, uniformly integrable. Next, set

$$Y_t^+ = H_t^+ I_{\{t<T\}}, \qquad Y_t^- = H_t^- I_{\{t<T\}}.$$

These are now two positive supermartingales, belonging to the class (D), which therefore have decompositions in Doob's sense

$$Y_t^+ = U_t^+ - V_t^+, \qquad Y_t^- = U_t^- - V_t^-.$$

Here U^+ and U^- are two uniformly integrable martingales, and V^+, V^- two predictable, integrable increasing processes. Set $U = U^+ - U^-$, an uniformly integrable martingale; $V = V^- - V^+$, a predictable process whose sample functions have bounded variation on $[0, \infty]$, and remark that $Y_t^+ - Y_t^- = M_t I_{\{t<T\}}$. There

comes:

$$M_t^T = U_t + V_t + M_T I_{\{t \geqq T\}} = U_t + W_t.$$

The sample functions of W have bounded variation on $[0, \infty]$, but their total variation in general isn't integrable. This decomposition shows that M^T is indeed a semimartingale in the sense of [3].

Proposition 4. *Let $M = (M_t, \mathfrak{F}_t)$ be a weak martingale. For each stopping time T, denote by M^T the weak martingale $(M_{t \wedge T}, \mathfrak{F}_t)$. Let $C = (C_t, \mathfrak{F}_t)$ be a locally bounded predictable process. There is an unique weak martingale $C \cdot M$ such that, for any stopping time T reducing M, $C \cdot M$ is equal for $0 \leqq t \leqq T$ to the stochastic integral $C \cdot M^T$ of C with respect to the semimartingale M^T.*

Proof. Let us assume that T reduces M, and keep the notation of the preceding proof. We first recall that $C \cdot M^T$ is defined as $C \cdot U + C \cdot W$ (the second one, an ordinary Stieltjes integral), which depends only on M^T, not on the decomposition $M^T = U + W$ (see [3]). It follows from the behaviour of stochastic integrals under stopping that, if (T_n) is an increasing sequence of stopping times reducing T, such that $T_n \to \infty$, the processes $C \cdot M^{T_n}$ and $C \cdot M^{T_{n+1}}$ agree for $0 \leqq t \leqq T_n$, and therefore can be pasted together into one single process $C \cdot M$. The only non obvious remaining point is the fact that $C \cdot M$ is a weak martingale. To see that, we come back to T reducing M and to the preceding notations. We have

$$H_t = U_t + V_t + H_T I_{\{t \geqq T\}}.$$

Therefore, $C \cdot M^T$ and $C \cdot H$ agree for $0 \leqq t < T$. The latter one being a local martingale, the result follows from Lemma 5.

The decomposition of $M^T = U + W$ in Proposition 3 allows the definition of other intrinsic elements connected with M. For instance, the *continuous part* of U doesn't depend on the decomposition, only on M^T. and therefore one can show the existence of a unique continuous local martingale M^c, such that for any stopping time T reducing M we have $M_t^c = U_t^c$ for $t \leqq T$. We can also define

$$[M, M]_s = \langle M^c, M^c \rangle_s + \sum_{s \leqq t} (\Delta M_s)^2.$$

However, as we shall see below, there exist sample continuous weak martingales such that $M^c = 0$, $[M, M] = 0$, and thus we cannot characterize the stochastic integrals $C \cdot M$ by an identity involving the brackets [,], as in [3] for the local martingale case.

A counter-example. After the rather satisfactory Theorem 2, and Propositions 3 and 4, which mean that the process deduced from a local martingale by means of a change of time still has some pleasant properties, we shall investigate another problem, with rather disquieting results.

The problem is the following: how far reaching is the generalization we have done of the notion of a local martingale? For instance, is every weak martingale, which belongs to the class (D), a true martingale (as in the case of local martingales)? We shall see that the answer is negative.

We start with the following elementary result: let (X_t) a Poisson process of parameter λ, with respect to its natural family of σ-fields $\mathfrak{F}_t$, such that $X_0 = 0$.

Let S be its first jump time (an exponential r.v. of parameter λ). Then an easy computation shows

$$E\left[S-\frac{1}{\lambda}\,|\,\mathfrak{F}_t\right]=\begin{cases} t & \text{if } t<S \\ S-\frac{1}{\lambda} & \text{if } t\geqq S. \end{cases}$$

This is an uniformly integrable martingale. Consider now on a suitable probability space a sequence of such independent Poisson processes (X_t^n), with parameters λ_n tending to 0. Let S_n be the first jump time of the process (X_t^n). Let $\mathfrak{F}_t$ the σ-field generated by the random variables X_s^n for all n and $s\leqq t$, and all sets of measure 0. Set $M_t=t$, $M=(M_t, \mathfrak{F}_t)$. It isn't difficult to see that the family $(\mathfrak{F}_t)$ is right continuous. We also find, using independence, that

$$E\left[S_n-\frac{1}{\lambda_n}\,|\,\mathfrak{F}_t\right]=t \quad \text{if } t<S_n, \qquad S_n-\frac{1}{\lambda_n} \quad \text{if } t\geqq S_n.$$

Otherwise stated, S_n reduces M. Now take for instance $\lambda_n=n^{-3}$; then

$$\sum_n P\{S_n\leqq n\}=\sum_n (1-e^{-n\lambda_n})$$

converges, and the Borel-Cantelli lemma implies that $S_n\to\infty$ a.s. According to Lemma 4, M is a weak martingale, a result which doesn't correspond to our expectation. Stopping M at a fixed time u, we get an example of a *bounded* weak martingale, sample continuous, which isn't a martingale. This example is due to P. A. Meyer.

Acknowledgement. The author is indebted to Professor P. A. Meyer for suggesting a detailed study of the weak martingales, after reading a first draft of this paper.

References

1. Dambis, K. E.: On the decomposition of continuous submartingales. Theor. Probab. Appl. **10**, 438–448 (1965).
2. Doob, J. L.: Stochastic processes. New York: Wiley 1953.
3. Dade, C. D., Meyer, P. A.: Intégrales stochastiques par rapport aux martingales localles, Université de Strasbourg, Lecture notes in Mathematics, vol. 124. Berlin-Heidelberg-New York: Springer 1970.
4. Kazamaki, N.: Some properties of martingale integrals. Ann. Inst. Henri Poincaré (to appear).
5. Meyer, P. A.: Probabilités et potentiel. Paris: Hermann 1966.

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