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On Barrier Problems for the Vibrating String

E. M. Cabaña*

Barrier problems for the vibrating string forced by plane white noise have lead us to derive a generalization of D. André Reflection Principle, and to apply it to solve some particular barrier problems for sums of independent symmetric random variables, and for processes with independent symmetric increments.

The main results, concerning the solution of the problems about the vibrating string, are stated and proved in § 3. The generalization of the Reflection Principle is Theorem 1.1, and § 2 is devoted to obtain some steps for the proofs in § 3, by applying Theorem 1.1.

No attempt is made in this paper to describe further applications of the generalized Reflection Principle.

1. Reflection

On a partially ordered finite set $(A, \leqq)$ a family of independent symmetric random variables $X = \{X_\alpha | \alpha \in A\}$ is defined. The partial sums $S = \{S_\alpha | \alpha \in A\}$ are defined by

$$S_\alpha = \sum_{\beta \leqq \alpha} X_\beta \tag{1}$$

and any function $\varphi: A \to R$ defines a functional

$$\varphi(S) = \sum_{\alpha \in A} \varphi(\alpha) S_\alpha \quad \Big(\text{or } \varphi(X) = \sum_{\alpha \in A} \varphi(\alpha) X_\alpha\Big).$$

To each φ we associate its *support* $\mathscr{S}(\varphi) = \{\alpha | \varphi(\alpha) \neq 0\}$ and its *past* $\mathscr{P}(\varphi) = \{\alpha | \text{there exists some } \beta \text{ such that } \alpha \leqq \beta \in \mathscr{S}(\varphi)\}$.

Let $\mathscr{F}$ be a family of real functions on A. The collection $(\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_m)$ of subfamilies of $\mathscr{F}$ is said to be an *ordered partition of $\mathscr{F}$ bounded by the family $\mathscr{H}$* of real functions on A, when

(i) $\mathscr{H}_i \cap \mathscr{H}_j = \emptyset$ for $i \neq j$ $(i, j = 1, 2, \ldots, m)$ and $\bigcup_{i=1}^{m} \mathscr{H}_i = \mathscr{F}$, and

(ii) there is a function $T: \mathscr{F} \to \mathscr{H}$ such that for each $i = 1, 2, \ldots, m$ and each $\varphi \in \mathscr{H}_i$, the functional χ defined by

$$\chi(X) = T\varphi(S) - \varphi(S)$$

satisfies

$$\mathscr{S}(\chi) \cap \mathscr{P}_i = \emptyset \tag{2}$$

where

$$\mathscr{P}_i = \bigcup \{\mathscr{P}(\varphi) | \varphi \in \bigcup_{h \leqq i} \mathscr{H}_h\}$$

* Instituto de Matemática y Estadística, Universidad de la República, Montevideo, Uruguay. This work was finished while visiting the Inter American Statistical Training Center, in Santiago, Chile.

Theorem 1.1. *Let X be a family of independent symmetric random variables on the partially ordered set $(A, \leqq)$, and let $\mathscr{F}$ be a finite family of real functions on A with an ordered partition $(\mathscr{H}_1, \ldots, \mathscr{H}_m)$ bounded by $\mathscr{H}$. Then, for each real constant a,*

$$P\{\max_{\varphi \in \mathscr{F}} \varphi(S) > a\} \leqq 2P\{\max_{\varphi \in \mathscr{H}} \varphi(S) > a\}. \tag{3}$$

Proof. Set $E = \{\max_{\varphi \in \mathscr{F}} \varphi(S) > a\}$, $E_h = \bigcap_{j=1}^{h-1} \{\max_{\varphi \in \mathscr{H}_j} \varphi(S) \leqq a\} \cap \{\max_{\varphi \in \mathscr{H}_h} \varphi(S) > a\}$ $(h = 1, 2, \ldots, m)$, $F = \{\max_{\varphi \in \mathscr{H}} \varphi(S) > a\}$, and call $\psi_{h,\omega}$ a random function in $\mathscr{H}_h$ such that, for $\omega \in E_h$, the inequality

$$\psi_{h,\omega}(S) > a \tag{4}$$

holds. (The function ψ_h has to be choosed in such a way that the sets appearing below are events. But this is easy to obtain because $\mathscr{F}$ is finite; for instance, if the functions φ in $\mathscr{F}$ are arbitrarily ordered, one can define ψ_h as the first function in $\mathscr{H}_h$ for which the condition (4) holds, if any, and, say, the last function in $\mathscr{H}_h$ when no one satisfies (4).)

Then

$$P(F) = P(F \cap E) = P\left(F \cap \bigcup_{h=1}^{m} E_h\right) = \sum_{h=1}^{m} P(E_h)\, P(F \mid E_h),$$

and $P(F \mid E_h) \geqq P(T\psi_h(S) > a \mid E_h)$; this requires the probability $P(E_h)$ to be positive, in order to be defined, but if $P(E_h) = 0$ the corresponding term in the sum may be omitted.

Now $T\psi_h(S) = \psi_h(S) + \chi(X)$ with χ satisfying (2). Then, if $\mathscr{A}_i$ is the σ-field generated by the variables $\{X_\alpha \mid \alpha \in \mathscr{P}_i\}$, $\chi(X)$ is a symmetric random variable independent of $\mathscr{A}_h$. On the other hand, the event E_h is $\mathscr{A}_h$-measurable, hence

$$P(T\psi_h(S) > a \mid E_h) \geqq P(\chi(X) \geqq 0)\, P(\psi_h(S) > a \mid E_h) \geqq \tfrac{1}{2},$$

and combining this with the previous relations we obtain

$$P(F) \geqq \tfrac{1}{2} \sum_{h=1}^{m} P(E_h)$$

and (3) follows.

2. Some Barrier Problems

Theorem 2.1. *If $A = (1, \ldots, m_1) \times (1, \ldots, m_2) \times \cdots \times (1, \ldots, m_\nu)$ with the partial order $(i_1, \ldots, i_\nu) \leqq (j_1, \ldots, j_\nu)$ if and only if $i_h \leqq j_h$ $(h = 1, \ldots, \nu)$, and $\{X_\alpha \mid \alpha \in A\}$ is a family of independent symmetric random variables with partial sums $\{S_\alpha \mid \alpha \in A\}$, then*

$$P\{\max_{\alpha \in A} S_\alpha > a\} \leqq 2^\nu P\{S_{(m_1, \ldots, m_\nu)} > a\}. \tag{1}$$

Proof. When $\nu = 1$, the present theorem reduces to D. André Reflection Principle for sums of independent symmetric random variables. The conclusion follows from Theorem 1.1 with $\mathscr{F} = \{\varphi_1, \ldots, \varphi_{m_1}\}$, $\varphi_i(S) = S_i$, $\mathscr{H}_i = \{\varphi_i\}$ $(i = 1, 2, \ldots, m_1)$ and $\mathscr{H} = \mathscr{H}_{m_1}$.

The case $\nu = 2$ has already been proved directly and applied to solve barrier problems for stationary Gaussian processes with non-negative convex covariances in [2].

In order to prove the general case, set $B(h)=\{\alpha=(i_1,\ldots,i_\nu)|\alpha\in A,\ i_\nu=h\}$ $(h=1,\ldots,m_\nu)$; it suffices to show that

$$P\{\max_{\alpha\in A} S_\alpha>a\}\leqq 2P\{\max_{\alpha\in B(m_\nu)} S_\alpha>a\} \tag{2}$$

and to apply finite induction. The inequality (2) follows from Theorem 1.1 with $\mathscr{F}=\{\varphi_\alpha|\alpha\in A\}$, $\varphi_\alpha(S)=S_\alpha(\alpha\in A)$, $\mathscr{H}_h=\{\varphi_\alpha|\alpha\in B(h)\}$ $(h=1,\ldots,m_\nu)$ and $\mathscr{H}=\mathscr{H}_{m_\nu}$; this ends the proof of the Theorem.

Given the linear array of independent symmetric random variables $X_1,\ldots,X_n$, let us extend it to the whole integer line by defining $X_i=0$ for $i\leqq 0$ or $i>n$. The partial sums $S_h=\sum_{i\leqq h} X_i$ remain unchanged on $(1,\ldots,n)$ and constant on $(\ldots,-1,0)$ and $(n,n+1,\ldots)$, vanishing on the former set.

A functional of the form

$$\varphi(S)=\sum_{\nu=-\infty}^{\infty}(S_{i-\nu k}-S_{i-h-\nu k}) \qquad (0<h<k) \tag{3}$$

will be called a $[k]$-*accumulated doublet of length* h.

For a plane array $\{X_{ij}|i=1,\ldots,m;\ j=1,\ldots,n\}$, we make the analogous extension by defining $X_{ij}=0$ for $(i,j)\notin(1,\ldots,m)\times(1,\ldots,n)$ and the partial sums $S_{hk}=\sum_{i\leqq h}\sum_{j\leqq k} X_{ij}$ are now defined on the whole integer plane.

A functional of the form

$$\varphi(S)=\sum_{\nu=-\infty}^{\infty}(S_{i-\nu k,\,j+\nu k}-S_{i-h-\nu k,\,j+h+\nu k}) \qquad (0<|h|<k) \tag{4}$$

will be called a $[k]$-*accumulated plane doublet of length* $|h|$. The sum $i+j$ will be called the *index* of the accumulated plane doublet (4).

It will be noticed that for sufficiently big $|\nu|$, the corresponding term in (3) or (4) becomes irrelevant, that is, the accumulated doublets (3) and (4) can be considered as finite sums.

The following theorems solve barrier problems for accumulated doublets. The restriction to be imposed on the length could be relaxed, but it was chosen in view of the applications in § 3.

Theorem 2.2. *If $\mathscr{F}$ is the family of all $[k]$-accumulated doublets of length not greater than $k/2$, corresponding to the array $X_1, X_2,\ldots,X_n$ of independent symmetric random variables with partial sums $\{S_i\}$, then for each $a>0$,*

$$P\{\max_{\varphi\in\mathscr{F}}\varphi(S)>a\}\leqq 8P\{S_n>a\}.$$

Proof. On the square $(1,\ldots,k)\times(1,\ldots,k)$, we define the arrays Y_{ij}, Z_{ij} by $Y_{ij}=Z_{ij}=0$ for $i+j\neq k+1$, and

$$Y_{i,k+1-i}=\sum_{\nu=-\infty}^{\infty}X_{i+\nu k}$$

$$Z_{i,k+1-i}=\sum_{\nu=-\infty}^{\infty}X_{i+\left[\frac{k}{2}\right]+\nu k}.$$

It is easily seen that every accumulated doublet appearing in F is a partial sum in one at least of the arrays, hence

$$P\{\max_{\varphi\in\mathscr{F}}\varphi(S)>a\}\leqq P\left\{\max_{i,j\leqq k}\sum_{i'=1}^{i}\sum_{j'=1}^{j}Y_{i'j'}>a\right\}+P\left\{\max_{i,j\leqq k}\sum_{i'=1}^{i}\sum_{j'=1}^{j}Z_{i'j'}>a\right\}$$

and since

$$\sum_{i=1}^{k}\sum_{j=1}^{k}Y_{ij}=\sum_{i=1}^{k}\sum_{j=1}^{k}Z_{ij}=\sum_{i=1}^{n}X_i=S_n,$$

the inequality (1) with $\nu=2$ leads to the required conclusion.

Theorem 2.3. *If $\mathscr{F}$ is the family of all $[k]$-accumulated plane doublets corresponding to the array $\{X_{ij}\}$ $(1\leqq i\leqq m,\ 1\leqq j\leqq n)$ of independent symmetric random variables with partial sums $\{S_{ij}\}$, then for each $a>0$,*

$$P\{\max_{\varphi\in\mathscr{F}}\varphi(S)>a\}\leqq 64\,P\left\{S_{mn}>\frac{a}{2}\right\}.$$

Proof. It will be assumed without loss of generality, that the terms in the accumulated doublets are paired in such a way that the lengths are always not greater than $k/2$. Let us notice first that the family F is finite, because the subfamilies $\mathscr{H}_p=\{\varphi\in\mathscr{F}\mid \text{the index of } \varphi \text{ is } p\}$ are obviously finite, and for $p\geqq m+n$, $\mathscr{H}_{p+k}\subset\mathscr{H}_p$. Then $\mathscr{F}=\bigcup_{p=1}^{m+n+k}\mathscr{H}_p$[1] and we shall prove first that the collection $(\mathscr{H}_p)_{p=1,\ldots,m+n+k}$ is an ordered partition of $\mathscr{F}$ bounded by the family $\mathscr{H}$ of all differences $\pm\varphi'\mp\varphi''$ of $[k]$-accumulated doublets of the same length not greater than $k/2$, where φ' is associated to the array $\{S_{in}\mid i=1,\ldots,m\}$ and φ'' is associated to the array $\{S_{mj}\mid j=1,\ldots,n\}$.

The sets $\mathscr{P}_p$ appearing in the definition of ordered partition are now

$$\mathscr{P}_p=\{(i,j)\mid i+j\leqq p\}\qquad (p=1,\ldots,m+n+k).$$

For each (i,j), let us define $\varphi_{ij}(S)=S_{ij}$; then the transformation

$$T\varphi_{ij}=\varphi_{in}+\varphi_{mj}-\varphi_{mn}$$

has the property

$$\mathscr{S}(T\varphi_{ij}-\varphi_{ij})\cap\mathscr{P}_{i+j}=\emptyset,$$

even when i (or j) is greater or equal than m (or n) because in that case $T\varphi_{ij}-\varphi_{ij}$ vanishes and it may be assumed $\mathscr{S}(T\varphi_{ij}-\varphi_{ij})=\emptyset$. Now any member φ of $\mathscr{H}_p$ is a linear combination $\varphi=\sum\gamma_{ij}\varphi_{ij}$ of elements of $\{\varphi_{ij}\mid i+j=p\}$ and we extend T by

$$T(\sum\gamma_{ij}\varphi_{ij})=\sum\gamma_{ij}T\varphi_{ij}$$

hence the family $\{T\varphi\mid\varphi\in\mathscr{F}\}$ closes the ordered partition $\mathscr{H}_1,\ldots,\mathscr{H}_{m+n}$. But it is readily verified that $\{T\varphi\mid\varphi\in\mathscr{F}\}$ is precisely $\mathscr{H}$, therefore Theorem 1.1 gives the estimate

$$P\{\max_{\varphi\in\mathscr{F}}\varphi(S)>a\}\leqq 2P\{\max_{\varphi\in\mathscr{H}}\varphi(S)>a\}.$$

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[1] Indeed, $\mathscr{H}_{p+k}=\mathscr{H}_p$ for $p\geqq m+n-1$ and therefore $\mathscr{F}=\bigcup_{p=1}^{m+n+k-2}\mathscr{H}_p$, but we do not need here such a precise description.

Let us call now $\mathscr{F}'$ and $\mathscr{F}''$ the families of $[k]$-accumulated doublets associated to $\{S_{in}\}$ and $\{S_{mj}\}$ respectively. Then

$$P\{\max_{\varphi\in\mathscr{H}} \varphi(S)>a\} \leqq P\left\{\max_{\varphi'\in\mathscr{F}'} |\varphi'(S)|>\frac{a}{2}\right\}+P\left\{\max_{\varphi''\in\mathscr{F}''} |\varphi''(S)|>\frac{a}{2}\right\}$$
$$\leqq 2P\left\{\max_{\varphi'\in\mathscr{F}'} \varphi'(S)>\frac{a}{2}\right\}+2P\left\{\max_{\varphi''\in\mathscr{F}''} \varphi''(S)>\frac{a}{2}\right\},$$

and from Theorem 2.2 we have

$$P\left\{\max_{\varphi'\in\mathscr{F}'} \varphi'(S)>\frac{a}{2}\right\} \leqq 8P\left\{S_{mn}>\frac{a}{2}\right\}$$
$$P\left\{\max_{\varphi''\in\mathscr{F}''} \varphi''(S)>\frac{a}{2}\right\} \leqq 8P\left\{S_{mn}>\frac{a}{2}\right\},$$

hence

$$P\{\max_{\varphi\in\mathscr{H}} \varphi(S)>a\} \leqq 32P\left\{S_{mn}>\frac{a}{2}\right\}$$

and the conclusion follows.

The conclusions of Theorem 2.1, 2.2 and 2.3 can be extended to processes with independent symmetric increments, depending on a continuous parameter, provided that the information about finite or denumerable sets of values of the process describes the behavior of the process itself in regard with barrier problems. This is the case for separable processes and for processes with continuous paths. The following corollaries will extend the above conclusions to the latter restricted case, but similar proofs could be carried out for separable processes.

In what follows, processes $X(t)$ $(0\leqq t\leqq T)$ with continuous paths and independent and symmetric increments will be considered. It is well known that the assumptions of continuity and independence imply that the increments are Gaussian, so that it is enough to assume that the increments are centered (around a zero mean) in order to obtain the symmetry.

We shall also consider processes $X(x,y)$ $(0\leqq x\leqq M, 0\leqq y\leqq N)$ with continuous paths and independent and symmetric double increments $X(x_2,y_2)-X(x_1,y_2)-X(x_2,y_1)+X(x_1,y_1)$ associated to intervals $(x_1<x\leqq x_2, y_1<y\leqq y_2)$. In this context, independence and continuity also imply that increments are Gaussian.

In both cases, the increments correspond to the variables X_i or X_{ij} in the discrete case, and the values of the process correspond to the partial sums S_i or S_{ij}. Since we want the values of the process to be precisely the sum of the increments, we shall impose initial conditions $X(0)=0$ or $X(0,y)=X(x,0)=0$ for $0\leqq x\leqq M$, $0\leqq y\leqq N$. An important example of such processes are Wiener integrals

$$X(t)=\int_0^t f(\tau)\,db(\tau), \qquad X(x,y)=\iint_{(0,x)\times(0,y)} f(\xi,\eta)\,d\beta(\xi,\eta)$$

with respect to a Brownian motion b or a *plane Brownian motion* β[2].

[2] Plane Brownian motion is defined in [1] and will be referred in § 3.

The next corollary is the continuous analogue of Theorem 2.1 for $\nu=2$. The formulation could be done as well for any ν, and it is restricted to $\nu=2$ to simplify the notations. It should be observed that the analogue for $\nu=1$ is D. André Reflection Principle for Brownian motion, and the inequality can in this case be replaced by equality; but this cannot be extended for other values of ν, as it is easily verified.

Corollary 2.1. *Let $\{X(x,y) \mid 0 \leqq x \leqq M,\ 0 \leqq y \leqq N\}$ be a (Gaussian) process with continuous paths and independent centered (and symmetric) increments, with initial values $X(0,y)=X(x,0)=0$ for all $0 \leqq x \leqq M$, $0 \leqq y \leqq N$. Then*

$$P\{\sup_{\substack{0 \leqq x \leqq M \\ 0 \leqq y \leqq N}} X(x,y)>a\} \leqq 4P\{X(M,N)>a\}.$$

Proof. Let A_n be the set of all vertices of the partition of $[0,M]\times[0,N]$ obtained as the product of the partitions into 2^n equal parts of the sides. The restriction of X to A_n gives a set of partial sums of the increments of X, hence Theorem 2.1 for $\nu=2$ gives

$$P\{\max_{(x,y)\in A_n} X(x,y)>a\} \leqq 4P\{X(M,N)>a\} \tag{5}$$

and since

$$\lim_{n\to\infty}\{\max_{(x,y)\in A_n} X(x,y)>a\}=\{\sup_{\substack{0 \leqq x \leqq M \\ 0 \leqq y \leqq N}} X(x,y)>a\},$$

we obtain the required result passing to the limit in (5).

The next statements require the analogue of accumulated doublets for the continuous case. Given $X(t)$ $(0 \leqq t \leqq T)$ we shall extend it to the whole line by setting $X(t)=0$ for $t<0$ and $X(t)=X(T)$ for $T<t$. Then *$[k]$-accumulated doublets of length h* are defined to be the functionals of the form

$$\varphi(X)=\sum_{\nu=-\infty}^{\infty} X(t-\nu k)-X(t-h-\nu k)$$

where h,k are real numbers satisfying $0<h<k$. In the same way, given $X(x,y)$ $(0 \leqq x \leqq M,\ 0 \leqq y \leqq N)$ we shall extend the domain to the whole plane by setting $X(x,y)=X(x\wedge M, y\wedge N)$ for $x, y \geqq 0$, and $X=0$ otherwise. Then *$[k]$-accumulated plane doublets of length h* are defined to be the functionals

$$\varphi(X)=\sum_{\nu=-\infty}^{\infty} X(x-\nu k, y+\nu k)-X(x-h-\nu k, y+h+\nu k)$$

where h,k are real numbers such that $0<|h|<k$, and $x+y$ will be said to be the *index*.

We are in position now to formulate the analogues of Theorems 2.2 and 2.3.

Corollary 2.2. *Given a (Gaussian) process $\{X(t) \mid t\in[0,T]\}$ with continuous paths and independent centered (and symmetric) increments, starting from $X(0)=0$, the inequality*

$$P\{\sup_{\varphi\in\mathscr{F}} \varphi(X)>a\} \leqq 8P\{X(T)>a\}$$

holds for each $a>0$ when $\mathscr{F}$ is the family of all $[k]$-accumulated doublets of length not greater than $k/2$.

Corollary 2.3. *Given a (Gaussian) process* $\{X(x,y)|0 \leqq x \leqq M,\ 0 \leqq y \leqq N\}$ *with continuous paths and independent centered (and symmetric) increments, starting from* $X(0,y)=X(x,0)=0$, *the inequality*

$$P\{\sup_{\varphi \in \mathscr{F}} \varphi(X) > a\} \leqq 64\, P\left\{X(M,N) > \frac{a}{2}\right\}$$

holds for each $a>0$ *when* $\mathscr{F}$ *is the family of all* $[k]$*-accumulated plane doublets.*

As it was observed above, the assumptions indicated in the parentheses are superflous. We omit the proofs of both corollaries since they use the same ideas than the proof of Corollary 2.1.

3. The Vibrating String

The equation of the position $u(t,z)$ at time t of an undamped vibrating string of length L, at the point of abscissa z, is

$$\partial^2 u(t,z)/\partial t^2 = \partial^2 u(t,z)/\partial z^2 + F(t,z) \tag{1}$$

when the external force is F. It is well known that when the string starts from rest at $t=0\,(u(0,z)=\partial u(0,z)/\partial t=0,\ 0 \leqq z \leqq L)$ and is tied at both ends $(u(t,0)=u(t,L)=0,\ t \geqq 0)$, the solution of (1) is

$$u(t,z)=\iint \chi_{t,z}(\tau,\zeta)\, F(\tau,\zeta)\, d\mu(\tau,\zeta) \tag{2}$$

where $\chi_{t,z}$ denotes the function described in Fig. 1, defined by

$$\chi_{t,z} = \sum_{k=-\infty} \psi_{t,z+2kL} - \psi_{t,-z+2kL}$$

$$\psi_{t,z}(\tau,\zeta) = \begin{cases} 1 & \text{for } 0 \leqq \tau,\ 0 \leqq \zeta \leqq L,\ |\zeta - z| \leqq t-\tau \\ 0 & \text{otherwise,} \end{cases}$$

μ is plane Lebesgue measure, and the integral is extended to the whole plane.

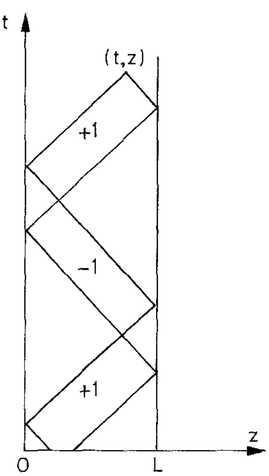

Fig. 1. Diagram of the function $\chi_{t,z}$

If the forcing term is a *plane white noise*

$$F(t, z) = \partial^2 \beta(t, z)/\partial t\, \partial z = \partial \beta(t, z)/\partial \mu,$$

(2) must be replaced by the stochastic integral

$$u(t, z) = \iint \chi_{t,z}\, d\beta. \tag{3}$$

A better description of (3) is given in [1], but in order to perform our calculations, it is enough to state here that for bounded measurable sets A, B, the integrals $\iint_A d\beta$, $\iint_B d\beta$ are centered Gaussian variables with variances $\mu(A)$, $\mu(B)$ and covariance $\mu(A \cap B)$, and that the resulting process $u(t, z)$ given by (3) has continuous paths (a.s.). We shall use this occasion to emphasize that plane and linear Brownian motions can be defined with respect to measures other than the Lebesgue one, and, except perhaps for the continuity of paths, the properties remain the same, taking the new measure the place of Lebesgue measure in the computation of variances and covariances.

Let us consider now two barrier problems for the vibrating string (3) forced by white noise.

Problem 1. *To find an upper bound for the probability*

$$P\{\sup_{0 \leqq z \leqq L} u(t, z) > a\} \tag{4}$$

for each $a > 0$, $t > 0$.

Problem 2. *To find an upper bound for the probability*

$$P\{\sup_{0 \leqq z \leqq L,\, 0 \leqq t \leqq T} u(t, z) > a\} \tag{5}$$

for each $a > 0$.

Our first step in solving Problem 1 will be to represent the process $u(t, \cdot)$ in a suitable form.

Lemma 3.1. *Let us define the function* f_t *on* $[0, L] \times [0, L]$ *by*

$$\begin{aligned} 2f_t(x, y) = {} & \left[\frac{x+y+2t+L}{2L}\right] + \left[\frac{x-y+2t+L}{2L}\right] \\ & + \left[\frac{-x+y+2t+L}{2L}\right] + \left[\frac{-x-y+2t+L}{2L}\right] \end{aligned}$$

for $0 \leqq x + y \leqq L$, *the brackets denoting the integral part, and*

$$f_t(x, y) = 0 \quad \textit{for } x + y > L.$$

Then, if db *is a plane white noise on* $[0, L] \times [0, L]$ *with respect to the measure with density* f_t *(instead of Lebesgue measure), the process* $\int_{[0,z] \times [0, L-z]} db$ $(0 \leqq z \leqq L)$ *is equivalent to the process* $u(t, z)$ *given by* (3), *that is, both processes have the same distributions.*

The proof, which we omit here, is made by direct computation of the covariances.

Lemma 3.2. *When t is an integer multiple $kL/2$ of the half-length the distribution of the $u(t,z)$ is the same than the conditional distribution of $\sqrt{k}\,L\,w(z/L)$ given that the* Wiener process *w satisfies* $w(1)=0$.

Proof. When $t=kL/2$, $f_t(x,y)=k$ for $0\leqq x+y\leqq L$, hence $E\big(u(t,z_1)\,u(t,z_2)\big)=k(z_1\wedge z_2)(L-z_1\vee z_2)$. On the other hand, given the Wiener process w we set

$$w(y)=y\,w(1)+\big(w(y)-y\,w(1)\big) \tag{6}$$

and since $E\big(w(1)(w(y)-y\,w(1))\big)=0$, the parenthesis in (6) is independent of $w(1)$. Therefore

$$\begin{aligned}&E\big\{(\sqrt{k}\,L)^2\big(w(z_1/L)-z_1\,w(1)/L\big)\big(w(z_2/L)-z_2\,w(1)/L\big)\big\}\\&=k\,L^2\left(\frac{z_1\wedge z_2}{L}-\frac{z_1\,z_2}{L^2}\right)=k(z_1\wedge z_2)(L-z_1\vee z_2),\end{aligned}$$

as it was to be shown.

Theorem 3.1. *Let $\Phi_a(\sigma^2)$ be the probability*

$$\int_a^\infty \frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{1}{2}\frac{u^2}{\sigma^2}}\,du$$

that a centered Gaussian variable with variance σ^2 be greater than a. Then:

(i) $4\Phi_a(L\,t)$ *is an upper bound of* (4);

(ii) *given any $\delta>0$, there exists a constant A_δ such that $A_\delta\,\Phi_a\big(g(t)+\delta\big)$ is an upper bound of* (4), *where*

$$g(t)=L\,t/2+\pi(t)$$

and $\pi(t)$ is the periodic function of period L, defined by

$$\pi(t)=t(|t|-L/2)\qquad(-L/2\leqq t\leqq L/2);$$

(iii) *for any positive integer k,*

$$P\Big\{\sup_{0\leqq z\leqq L}u(k\,L/2,z)>a\Big\}=P\Big\{\sup_{0\leqq\tau\leqq 1}w(\tau)>a/\sqrt{k}\,L\,|\,w(1)=0\Big\},$$

where w is a Wiener process.

Proof. Using the representation of Lemma 3.1, part (i) follows from Corollary 2.1, and part (iii) is a direct consequence of Lemma 3.2.

In order to prove (ii) let us consider the continuous function

$$F_t(x,y)=\int_0^x d\xi\int_0^y f_t(\xi,\eta)\,d\eta$$

whose maximum for $x+y=L$ is precisely $g(t)$. Then the interval $[0,L]$ can be partitioned by points $0=z_0<z_1<\cdots<z_p=L$ in such a way that for $i=1,2,\ldots,p$, $F_t(z_i,L-z_{i-1})\leqq g(t)+\delta$, and therefore, using Corollary 2.1, it follows that

$$P\Big\{\sup_{z_{i-1}\leqq z\leqq z_i}u(t,z)>a\Big\}\leqq 4\,\Phi_a\big(F_t(z_i,L-z_{i-1})\big)$$

hence

$$P\{\sup_{0\leqq z\leqq L} u(t,z)>a\}=P\left(\bigcup_{i=1}^{p}\{\sup_{z_{i-1}\leqq z\leqq z_i} u(t,z)>a\}\right)$$

$$\leqq 4\sum_{i=1}^{p}\Phi_a\big(F_t(z_i, L-z_{i-1})\big)\leqq 4p\,\Phi_a\big(g(t)+\delta\big).$$

The partition depends on δ, and so does in particular the coefficient $A_\delta=4p$. This ends the proof of the Theorem.

It should be noticed that, since $g(t)$ is the maximum of $\mathrm{Var}\big(u(t,z)\big)$ for $0\leqq z\leqq L$ (which is attained at $z=L/2$), there is no bound of (4) *holding for all* $a>0$ such as the one obtained in (ii) but with $\delta<0$.

Theorem 3.2. *Let* Φ_a *be defined as in Theorem* 3.1. *Then*

(i) $64\,\Phi_{a/2}(LT)$ *is an upper bound of* (5), *and*

(ii) *given any* $\delta>0$, *there exists a constant* B_δ *such that* $B_\delta\Phi_a\big(g(T)+\delta\big)$ *is an upper bound of* (5), *where g is the same function appearing in Theorem* 3.1.

Proof. Let us define in

$$0\leqq x\leqq\frac{T+L}{\sqrt{2}},\qquad 0\leqq y\leqq\frac{T+L}{\sqrt{2}}$$

the set

$$S=\left\{(x,y)\,\middle|\,\frac{L}{\sqrt{2}}\leqq x+y\leqq\frac{L+2T}{\sqrt{2}},\quad |x-y|\leqq\frac{L}{\sqrt{2}}\right\}$$

and construct the measure $\mu(A)=\int_{A\cap S} dx\,dy$.

A plane Brownian motion β with respect to the measure μ is introduced, and the stochastic process $X(x,y)=\int_{[0,x]\times[0,y]} d\beta$ is considered. Then our process $u(t,z)$

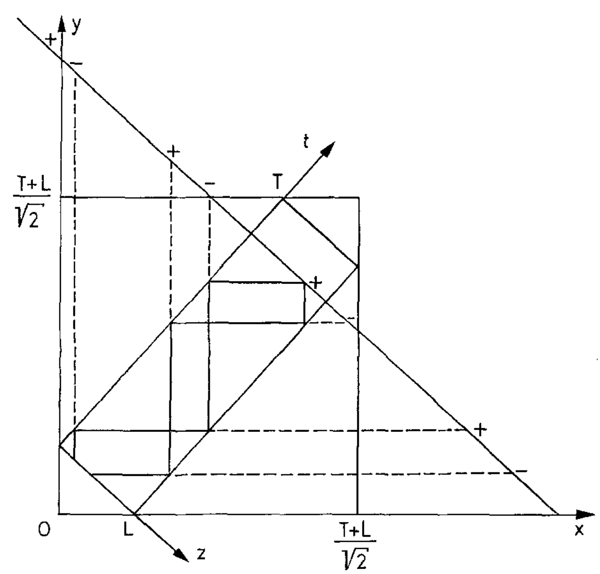

Fig. 2. Representation of $u(t,z)$ as an accumulated plane doublet

can be obtained as a $[\sqrt{2}L]$-accumulated plane doublet for the process X, as is shown in Fig. 2, and the terms can obviously be paired so to have a length not greater than $\sqrt{2}L/2$. Hence part (i) follows readily from Corollary 2.3.

As to the second part, let us divide the rectangle

$$S=\{(t,z)|0\leqq z\leqq L,\ 0\leqq t\leqq T\}$$

into the small regions $\{R_h|h=1,2,\ldots,p\}$ obtained by intersecting with each of the squares

$$\{(t,z)|\,i\,L/v\leqq t+z\leqq(i+1)\,L/v;\,j\,L/v\leqq t-z\leqq(j+1)\,L/v\}\qquad(i,\ \pm j=0,1,2,\ldots),$$

hence (5) is bounded by

$$\sum_{h=1}^{p} P\{\sup_{(t,z)\in R_h} u(t,z)>a\}.$$

Given a fixed R_h, the sets S_h, S'_h defined by

$$S_h=\bigcap_{(t,z)\in R_h}\{(\tau,\zeta)|\chi_{t,z}(\tau,\zeta)\neq 0\},$$

and

$$S'_h=\bigcup_{(t,z)\in R_h}\{(\tau,\zeta)|\chi_{t,z}(\tau,\zeta)\neq 0\}\setminus S_h,$$

induce a decomposition of $u(t,z)$ on R_h as the sum of a constant term

$$w=\iint_{S_h}\chi_{t,z}\,d\beta$$

and a remainder

$$v(t,z)=u(t,z)-\iint_{S_h}\chi_{t,z}\,d\beta=\iint_{S'_h}\chi_{t,z}\,d\beta.$$

Since $g(T)$ is the maximum of the variance of $u(t,z)$ for $(t,z)\in S$, then $\mathrm{Var}(w)\leqq g(T)$, hence

$$P\{w>\lambda a\}\leqq\Phi_{\lambda a}(g(T)).\tag{7}$$

As to the process $v(t,z)$, the same reasoning used to derive the part (i) of the present theorem, now leads to the analogue result

$$P\{\sup_{(t,z)\in R_h} v(t,z)>\mu a\}\leqq 64\,\Phi_{\mu a/2}(\sigma^2)\tag{8}$$

where $\sigma^2\left(\leqq\dfrac{2LT}{v}\right)$ is the area of S'_h.

Combining (7) and (8),

$$P\{\sup_{(t,z)\in R_h} u(t,z)>a\}\leqq\Phi_a(g(T))+64\,\Phi_{(1-\lambda)a/2}(\sigma^2)$$

is obtained $(0<\lambda<1)$, and since

$$\Phi_{\lambda a}(g(T))+64\,\Phi_{(1-\lambda)a/2}(\sigma^2)\leqq 65\,\Phi_a(g(T)+\delta)$$

for

$$g(T)/\lambda^2\leqq g(T)+\delta\tag{9}$$

and

$$4\sigma^2/(1-\lambda)^2\leqq g(T)+\delta\tag{10}$$

we choose λ such that (9) holds and then σ^2 in order that (10) also holds (this last condition imposed on the area of S'_h is obtained with a partition sufficiently fine, namely, for $\nu \geqq 8LT/(g(T)+\delta)(1-\lambda)^2$).

This construction leads to the bound $65p\,\Phi_a(g(T)+\delta)$ for (5), with the coefficient $B_\delta = 65p$ depending on δ.

As in the case of Theorem 3.1, this bound cannot be substantially improved for all a simultaneously.

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Prof. E. M. Cabaña
Instituto de Matemática y Estadística
Universidad de la República
Av. J. Herrera y Reissig 565
Montevideo, Uruguay

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