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Existence of Proper Conditional Probabilities

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Introduction

Let $(X, \mathscr{A}, P)$ be a probability space and let $\mathscr{B}$ be a sub-σ-algebra of $\mathscr{A}$. It is well known that even if there exists a regular conditional probability (r.c.p.) on $\mathscr{A}$, given $\mathscr{B}$, it is not necessarily proper (Blackwell and Ryll-Nardzewski [2]).

It is shown in this paper that under some assumptions on P-null sets in $\mathscr{B}$ there exist σ-algebras $\mathscr{A}_0 \subset \mathscr{A}$ and $\mathscr{B}_0 \subset \mathscr{A}_0 \cap \mathscr{B}$ such that $\mathscr{A}_0 \sim \mathscr{A}$ $(P/\mathscr{A})$ and $\mathscr{B}_0 \sim \mathscr{B}(P/\mathscr{A})$ and there exists a proper r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}_0$.

The theorems proved here generalize the ones of Pfanzagl's paper [7] and the proofs seem to be simpler.

Moreover under some assumptions about separability of σ-algebras under consideration there is given a full condition for the existence of a r.c.p. and a proper r.c.p.

Besides, there is given a simple characterisation of Blackwell spaces (Rao) which employs the proper r.c.p.-ies.

1. Definitions and Basic Lemmas

Let $(X, \mathscr{A}, P)$ be a probability space. For given two subsystems $\mathscr{B} \subset \mathscr{A}$ and $\mathscr{C} \subset \mathscr{A}$ we shall write $\mathscr{B} \subset \mathscr{C}(P/\mathscr{A})$ if for any $B \in \mathscr{B}$ there exists $C \in \mathscr{C}$ such that $P(B \triangle C) = 0$. We shall write $\mathscr{B} \sim \mathscr{C}(P/\mathscr{A})$ if $\mathscr{B} \subset \mathscr{C}(P/\mathscr{A})$ and $\mathscr{C} \subset \mathscr{B}(P/\mathscr{A})$. In this paper by an *algebra* we shall always understand a σ-algebra and by a measure the probability measure. If $\mathscr{B} \subset \mathscr{A}$ is an algebra then a set $D \in \mathscr{B}$ with $P(D) > 0$ is a $P/\mathscr{B}$-atom if $B \in \mathscr{B}$ and $B \subset D$ together imply $P(B) = 0$ or $P(B) = P(D)$. $P/\mathscr{B}$ is purely atomic if the union of all $P/\mathscr{B}$-atoms has measure 1, $P/\mathscr{B}$ is nonatomic if $P/\mathscr{B}$-atoms do not exist.

Let $\mathscr{A}_0 \subset \mathscr{A}$ and $\mathscr{B} \subset \mathscr{A}$ be arbitrary algebras. A *regular conditional probability* (r.c.p.) on $\mathscr{A}_0$, given $\mathscr{B}$, is a function $P(\cdot, \cdot/\mathscr{B})$: $\mathscr{A}_0 \times X \to [0, 1]$ such that

(i) $P(\cdot, x/\mathscr{B})$ is a measure on $\mathscr{A}_0$ for all $x \in X$,

(ii) $P(A, \cdot/\mathscr{B})$ is $\mathscr{B}$-measurable for all $A \in \mathscr{A}_0$,

(iii) $\int_B P(A, x/\mathscr{B})\, P(dx) = P(A \cap B)$ for all $A \in \mathscr{A}_0$ and $B \in \mathscr{B}$.

A r.c.p. $P(\cdot, \cdot/\mathscr{B})$ on $\mathscr{A}_0$ is proper if $P(B, x/\mathscr{B}) = \chi_B(x)$ for all $B \in \mathscr{B} \cap \mathscr{A}_0$, $x \in X$.

For any $A \subset X$ let $A^1 = A$ and $A^0 = X - A$. If $\{A_\gamma\}_{\gamma \in \Gamma}$ is a family of subsets of X, then we shall call them σ-independent if for any sequence $\{\delta_n\}_{n=1}^{\infty} \in \{0, 1\}^{\aleph_0}$ and for any sequence $\{A_{\gamma_n}\}_{\gamma_n \in \Gamma}$ we have $\bigcap_{n=1}^{\infty} A_{\gamma_n}^{\delta_{\gamma_n}} \neq \emptyset$.

Lemma 1. *Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of σ-independent sets which belong to $\mathscr{A}$. For any $\Gamma_0 \subset \Gamma$ there exists a proper r.c.p. on $\mathscr{A}_0 = \sigma(\{A_\gamma\}_{\gamma\in\Gamma})$, given $\mathscr{B} = \sigma(\{A_\gamma\}_{\gamma\in\Gamma_0})$.*

Proof. For $\Gamma_0 = \Gamma$ this is obvious. If $\Gamma_0 \neq \Gamma$ we may assume that the set $\Gamma - \Gamma_0$ is ordered by ordinals $\gamma < \gamma_0$.

We put $P_0(B, x/\mathscr{B}) = X_B(x)$ for $B \in \mathscr{B}$; clearly P_0 is a proper r.c.p. on $\mathscr{B}$, given $\mathscr{B}$.

Let us make inductive assumption that for a certain $\gamma_* < \gamma_0$ a family of consistent proper r.c.p.-ies $P_\varepsilon(\cdot, \cdot/\mathscr{B})$ on $\mathscr{A}_\varepsilon = \sigma(\{A_\gamma\}_{\gamma<\varepsilon}, \mathscr{B})$, for all $\varepsilon < \gamma_*$ is already constructed (consistency is used here in the meaning that for every $x \in X$ the measures $\{P_\varepsilon(\cdot, x/\mathscr{B})\}_{\varepsilon<\gamma_*}$ are consistent in the sense of Kolmogorov).

It is our aim now to define a proper r.c.p. $P_{\gamma_*}(\cdot, \cdot/\mathscr{B})$ on $\mathscr{A}_{\gamma_*}$, being an extension of all P_ε for $\varepsilon < \gamma_*$.

Case 1. γ_* is a limit ordinal. Then we take as P_{γ_*} the common extension of all P_ε for $\varepsilon < \gamma_*$. It is easy to see, that P_{γ_*} has all required properties.

Case 2. $\gamma_* = \gamma_1 + 1$. We have already built a proper r.c.p. $P_{\gamma_1}(\cdot, \cdot/\mathscr{B})$ on $\mathscr{A}_{\gamma_1}$. Let $f(x)$ be any conditional probability of the set A_{γ_1}, given $\mathscr{A}_{\gamma_1}$ (let us observe, that $A_{\gamma_1} \notin \mathscr{A}_{\gamma_1}$). We can choose $f(x)$ such that $0 \leq f(x) \leq 1$.

In view of the independence of algebras $\mathscr{A}_{\gamma_1}$ and $\sigma(A_{\gamma_1})$, the function $\bar{P}(\cdot, \cdot/\mathscr{A}_{\gamma_1})$ defined on $\mathscr{A}_{\gamma_*} \times X$ by

$$\bar{P}(E \cap A_{\gamma_1} \cup F \cap A^0_{\gamma_1}, x/\mathscr{A}) = X_E(x) f(x) + X_F(x)(1 - f(x))$$

for $E \in \mathscr{A}_{\gamma_1}$ and $F \in \mathscr{A}_{\gamma_1}$, is a proper r.c.p. on $\mathscr{A}_{\gamma_*}$, given $\mathscr{A}_{\gamma_1}$.

Now, the function $P_{\gamma_*}(\cdot, \cdot/\mathscr{B})$ defined for arbitrary $A \in \mathscr{A}_{\gamma_*}$ by

$$P_{\gamma_*}(A, x/\mathscr{B}) = \int \bar{P}(A, u/\mathscr{A}_{\gamma_1})\, P_{\gamma_1}(du, x/\mathscr{B})$$

is a proper r.c.p. on $\mathscr{A}_{\gamma_*}$, given $\mathscr{B}$. It is easy to see that P_{γ_*} is an extension of P_{γ_1}.

Having the described procedure accomplished we obtain a proper r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}$.

Proposition 1 (Tarski). *If N is a set of cardinality $\mathfrak{m}$, such that $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$ then there exist $2^{\mathfrak{m}}$ of σ-independent subsets of N.*

This is a particular case of Hilfssatz 3.16 from [10].

Proposition 2. *Let $(N, \mathscr{C})$ be any measurable space, such that $\mathscr{C}$ contains $\mathfrak{m} \geq \mathfrak{c}$ disjoint sets. If the spaces $(N, \mathscr{C})$ and $(N, \mathscr{C})^{\aleph_0}$ (the product symbol has the usual meaning) are isomorphic, then there exists in $(N, \mathscr{C})$ a family of cardinality $\mathfrak{m}$ of σ-independent and measurable subsets.*

The idea of the proof is analogous to that from [5].

2. Main Theorems

Theorem 1. *Let $P/\mathscr{A}$ be a measure on an algebra $\mathscr{A}$. Let us assume that $\mathfrak{m} \geq \aleph_0$ is the smallest cardinality of a dense subset of $\mathscr{A}$ in Frechet-Nikodym's metric. Let $\mathfrak{n}$ be the smallest cardinal number such that $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$ and $\mathfrak{m} \leq 2^{\mathfrak{n}}$, and let $\mathscr{B}$ be any subalgebra of $\mathscr{A}$. If $P/\mathscr{B}$ is either purely atomic, or $\mathscr{B}$ contains a P-null set N of cardinality $\mathfrak{n}$, such that all its subsets belong to $\mathscr{B}$ then for some algebra $\mathscr{A}_0 \subset \mathscr{A}$ with $\mathscr{A}_0 \sim \mathscr{A}(P/\mathscr{A})$ and for some algebra $\mathscr{B}_0 \subset \mathscr{B} \cap \mathscr{A}_0$ with $\mathscr{B}_0 \sim \mathscr{B}(P/\mathscr{A})$ there exists a proper r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}_0$.*

Proof. If $P/\mathscr{B}$ is purely atomic, this is trivial; the function $P(\cdot, \cdot/\mathscr{B})$ defined by

$$P(A, x/\mathscr{B}) = \sum_{n=1}^{\infty} \frac{P(A \cap E_n)}{P(E_n)} \chi_{E_n}(x),$$

where E_n, $n = 1, 2, \ldots$ are $P/\mathscr{B}$-atoms and $\bigcup_{n=1}^{\infty} E_n = X$, is a proper r.c.p. on $\mathscr{A}$, given $\sigma(\{E_n\}_{n=1}^{\infty}) \sim \mathscr{B}(P/\mathscr{A})$.

The rest of the proof is devoted to the case, when $\mathscr{B}$ contains a P-null set N such that card $N = \mathfrak{n}$, and all subsets of N are in $\mathscr{B}$.

Suppose the family $\{A_\gamma\}_{\gamma \in \Gamma}$ with card $\Gamma = \mathfrak{m}$ is dense in $\mathscr{A}$. We may assume that there exists $\Gamma_0 \subset \Gamma$ such that $A_\gamma \in \mathscr{B}$ for $\gamma \in \Gamma_0$, and $\{A_\gamma\}_{\gamma \in \Gamma_0}$ is dense in $\mathscr{B}$.

By Proposition 1 there is a family $\{N_\gamma\}_{\gamma \in \Gamma}$ of σ-independent subsets of N.

For any $\gamma \in \Gamma$ let

$$C_\gamma = A_\gamma \cap N^0 \cup N_\gamma \cap N.$$

Of course for every $\gamma \in \Gamma$ we have $C_\gamma \in \mathscr{A}_\gamma$ and $C_\gamma^{\delta_\gamma} \cap N \neq \emptyset$, where $\delta_\gamma = 0$ or 1. Moreover, for $\gamma \in \Gamma_0$ we have $C_\gamma \in \mathscr{B}$.

From this remark and the σ-independence of N_γ, $\gamma \in \Gamma$, it follows that all C_γ, $\gamma \in \Gamma$ are σ-independent.

Let us notice that because of $P(N) = 0$, we have

$$\sigma(\{A_\gamma\}_{\gamma \in \Gamma_0}) \sim \sigma(\{C_\gamma\}_{\gamma \in \Gamma_0})(P/\mathscr{A}).$$

The last relation together with the assumed fact that

$$\sigma(\{A_\gamma\}_{\gamma \in \Gamma_0}) \sim \mathscr{B}(P/\mathscr{A})$$

gives us the equivalence

$$\sigma(\{C_\gamma\}_{\gamma \in \Gamma_0}) \sim \mathscr{B}(P/\mathscr{A}).$$

In the similar way we can prove that

$$\sigma(\{C_\gamma)_{\gamma \in \Gamma}) \sim \mathscr{A}(P/\mathscr{A}).$$

Putting $\mathscr{A}_0 = \sigma(\{C_\gamma\}_{\gamma \in \Gamma})$ and $\mathscr{B}_0 = \sigma(\{C_\gamma\}_{\gamma \in \Gamma_0})$, we have, by Lemma 1, the desired result.

Theorem 2. *Let $P/\mathscr{A}$ be a measure on an algebra $\mathscr{A}$. Let us assume that $\mathfrak{m} \geqq \aleph_0$ is the smallest cardinality of a dense subset of $\mathscr{A}$ in Frechet-Nikodym's metric. Let $\mathfrak{n} \geqq \mathfrak{m}$ be the smallest cardinal number such that $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$, and let $\mathscr{B}$ be any subalgebra of $\mathscr{A}$. If $P/\mathscr{B}$ is either purely atomic or $\mathscr{B}$ contains a P-null set N of cardinality $\mathfrak{n}$ and there is a subalgebra $\mathscr{C} \subset \mathscr{B}$ such that the measurable spaces $(N, \mathscr{C} \cap N)$ and $(N, \mathscr{C} \cap N)^{\aleph_0}$ are isomorphic and $\mathscr{C} \cap N$ contains $\mathfrak{n}$ disjoint sets then for some algebra $\mathscr{A}_0 \subset \mathscr{A}$ with $\mathscr{A}_0 \sim \mathscr{A}(P/\mathscr{A})$ and for some algebra $\mathscr{B}_0 \subset \mathscr{B} \cap \mathscr{A}_0$ with $\mathscr{B}_0 \sim \mathscr{B}(P/\mathscr{A})$ there exists a proper r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}_0$.*

Proof. Using Proposition 2 we can prove this theorem in the same manner as the first one.

3. The Separable Case

In this section by a *separable* algebra we shall mean an algebra which is σ-generated and contains points. If $\mathscr{A}$ is a separable algebra, then $A\in\mathscr{A}$ is an $\mathscr{A}$-*atom* if $A_1\in\mathscr{A}$ and $A_1\subset A$ implies $A_1=\emptyset$ or $A_1=A$.

As a special case of Theorem 1 we obtain the theorem proved by Pfanzagl in [7].

Theorem 3. *Let $P/\mathscr{A}$ be a measure on an algebra $\mathscr{A}$, which is the completion of a separable algebra. Let $\mathscr{B}$ be a subalgebra of $\mathscr{A}$ contining all P-null sets from $\mathscr{A}$. If $P/\mathscr{B}$ is either purely atomic or $\mathscr{A}$ contains a P-null set of the cardinality $\mathfrak{c}$ then for some algebra $\mathscr{A}_0\subset\mathscr{A}$ with $\mathscr{A}_0\sim\mathscr{A}\ P/\mathscr{A}$ and for some algebra $\mathscr{B}_0\subset\mathscr{A}_0\cap\mathscr{B}$ with $\mathscr{B}_0\sim\mathscr{B}\ P/\mathscr{A}$ there exists a proper r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}_0$.*

Assuming the separability of $\mathscr{A}$ in the sense of Frechet-Nikodym, we obtain such a theorem:

Theorem 4. *Let $P/\mathscr{A}$ be a measure on a separable in the sense of Frechet-Nikodym algebra $\mathscr{A}$. Let $\mathscr{B}$ be any subalgebra of $\mathscr{A}$. If $P/\mathscr{B}$ is either purely atomic or $\mathscr{B}$ contains a P-null set N of the cardinality $\mathfrak{c}$ and an algebra $\mathscr{C}$ such that the measurable space $(N,\mathscr{C}\cap N)$ is standard* (cf. Parthasarathy [6]) *then for some algebra $\mathscr{A}_0\subset\mathscr{A}$ with $\mathscr{A}_0\sim\mathscr{A}\ P/\mathscr{A}$ and for some algebra $\mathscr{B}_0\subset\mathscr{A}_0\cap\mathscr{B}$ with $\mathscr{B}_0\sim\mathscr{B}\ P/\mathscr{A}$ there exists a proper r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}_0$.*

Of course, this theorem holds for any $\mathscr{A}$ containing a dense subfamily of cardinality not greater than continuum.

Now we shall generalize another theorem of the paper [7] (Proposition 3 in [7]).

We begin with an easy observation:

Proposition 3. *Let $P/\mathscr{A}$ be a measure on an algebra $\mathscr{A}$. If there exists a r.c.p. $P(\cdot,\cdot/\mathscr{B})$ on $\mathscr{A}_0$, then for every σ-generated algebra $\mathscr{C}\subset\mathscr{A}_0\cap\mathscr{B}$ there exists a set $N_{\mathscr{C}}\in\mathscr{B}$ such that $P(N)=0$ and $P(C,x/\mathscr{B})=\chi_C(x)$ for every $C\in\mathscr{C}$ and $x\in X-N_{\mathscr{C}}$.*

Theorem 5. *Let $\mathscr{A}$ be arbitrary algebra and let $\mathscr{A}_0\subset\mathscr{A}$ and $\mathscr{B}\subset\mathscr{A}$ be such that there exists a σ-generated algebra $\mathscr{C}\subset\mathscr{A}_0\cap\mathscr{B}$ such that every set from $\mathscr{A}_0$ is a union of atoms of $\mathscr{C}$. Then a r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}$ exists iff there exists a set $N\in\mathscr{B}$ such that $P(N)=0$, and $\mathscr{A}_0\cap N^0\subset\mathscr{B}\cap N^0$.*

Proof. Suppose $P(\cdot,\cdot/\mathscr{B})$ is a r.c.p. on $\mathscr{A}_0$. In virtue of Proposition 3 there exists a set $N\in\mathscr{B}$ such that $P(N)=0$ and $P(C,x/\mathscr{B})=\chi_C(x)$ for all $C\in\mathscr{C}$ and $x\in X-N$. Thus, for every $\mathscr{C}$-atom e_x we have $P(e_x,y/\mathscr{B})=\chi_{e_x}(y)$ for every $x\in X-N$.

It means that

$$P(A,x/\mathscr{B})=\begin{cases}1 & \text{if } x\in A\cap N^0\\ 0 & \text{if } x\in A^0\cap N^0.\end{cases}$$

But from the property of a r.c.p. we have $\{x\colon P(A,x/\mathscr{B})=1\}\in\mathscr{B}$, so that $A-N=\{x\colon P(A,x/\mathscr{B})=1\}-N\in\mathscr{B}$. And so we have proved that $\mathscr{A}_0\cap N^0\subset\mathscr{B}\cap N^0$.

The converse implication is almost trivial. In fact the function $P(\cdot,\cdot/\mathscr{B})$ defined for any $A\in\mathscr{A}_0$ by

$$P(A,x/\mathscr{B})=\begin{cases}\chi_{A-N}(x) & \text{if } x\in X-N\\ P(A) & \text{if } x\in N\end{cases}$$

is a r.c.p. on $\mathscr{A}_0$, given $\mathscr{B}$.

Corollary 1. *Suppose* $\mathscr{A}_0 \cap \mathscr{B}$ *contains a separable algebra. Then a r.c.p. on* $\mathscr{A}_0$, *given* $\mathscr{B}$, *exists iff there exists a set* $N \in \mathscr{B}$ *such that* $P(N)=0$ *and*

$$\mathscr{A}_0 \cap N^0 \subset \mathscr{B} \cap N^0.$$

If $\mathscr{A}_0 \not\subset \mathscr{B}$ *then* N *is an uncountable set.*

Corollary 2. *Let* $\mathscr{A}$ *be arbitrary algebra and let* $\mathscr{B} \subset \mathscr{A}$ *contains a separable algebra. Then a r.c.p. on* $\mathscr{A}$, *given* $\mathscr{B}$, *exists iff there exists a set* $N \in \mathscr{B}$ *such that* $P(N)=0$ *and* $\mathscr{A} \cap N^0 = \mathscr{B} \cap N^0$. *If* $\mathscr{A} \neq \mathscr{B}$ *then* N *is an uncountable set.*

We shall come back now to proper r.c.p.-ies.

Theorem 6. *Let* $\mathscr{A}$ *be arbitrary algebra and let* $\mathscr{A}_0 \subset \mathscr{A}$ *and* $\mathscr{B} \subset \mathscr{A}$ *be such that every set from* $\mathscr{A}_0$ *is a union of sets from* $\mathscr{A}_0 \cap \mathscr{B}$. *Then a proper r.c.p. on* $\mathscr{A}_0$, *given* $\mathscr{B}$, *exists iff* $\mathscr{A}_0 \subset \mathscr{B}$.

Proof. Suppose $P(\cdot, \cdot/\mathscr{B})$ is a proper r.c.p. on $\mathscr{A}_0$. For every $C \in \mathscr{A}_0 \cap \mathscr{B}$ and each $x \in X$ we have $P(C, x/\mathscr{B}) = \chi_C(x)$. Thus for every $A \in \mathscr{A}_0$ we have $P(A, x/\mathscr{B}) = \chi_A(x)$. So it must be $\mathscr{A}_0 \subset \mathscr{B}$.

The converse implication is obvious.

Corollary 3. *If* $\mathscr{A}_0 \cap \mathscr{B}$ *contains all points then a proper r.c.p. on* $\mathscr{A}_0$, *given* $\mathscr{B}$, *exists iff* $\mathscr{A}_0 \subset \mathscr{B}$.

If $\mathscr{A}$ is separable $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{B}$ is arbitrary separable subalgebra of $\mathscr{A}$ then we obtain a characterisation of Blackwell spaces (for definition see Rao).

Corollary 4. *If* $\mathscr{A}$ *is a separable algebra, then* $(X, \mathscr{A})$ *is Blackwell space iff for every separable algebra* $\mathscr{B} \subset \mathscr{A}$ *there exists a proper r.c.p. on* $\mathscr{A}$, *given* $\mathscr{B}$.

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References

1. Blackwell, D.: On a class of probability spaces. Proc. Third Berkeley Sympos. math. Statist. Probab. **2**, 1–6 (1956).
2. – Ryll-Nardzewski, C.: Non-existence of everywhere proper conditional distributions. Ann. math. Statistics **34**, 223–225 (1963).
3. Doob, J. L.: Stochastic processes. New York: Wiley 1953.
4. Halmos, P. R.: Measure theory. Princeton: Van Nostrand 1962.
5. Marczewski, E.: Ensembles indépendants et leurs applications à la théorie de la mesure. Fundamenta Math. **35**, 13–28 (1948).
6. Parthasarathy, K. R.: Probability measures on metric spaces. New York and London: Academic Press 1967.
7. Pfanzagl, J.: On the existence of regular conditional probabilities. Z. Wahrscheinlichkeitstheorie verw. Geb. **11**, 244–256 (1969).
8. – Pierlo, W.: Compact systems of sets. Lecture Notes in Mathematics **16**. Berlin-Heidelberg-New York: Springer 1966.
9. Rao, B.V.: Studies in Borel structures. Doctor thesis. Indian Statistical Institute. Calcutta 1969.
10. Tarski, A.: Ideale in vollständigen Mengenkörpern. I. Fundamenta Math. **32**, 45–63 (1939).

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