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On the Weak Convergence of Superpositions of Point Processes

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1. A Continuity Theorem

Let X be a σ-compact, second countable space and $\mathcal{M}(X)$ the set of non-negative Radon Borel measures on X, i.e., elements of $\mathcal{M}(X)$ are defined on the Borel algebra $\mathcal{B}(X)$ and finite on compact sets. Denote by $\mathcal{C}_K$ the set of continuous functions $X \to R$ with compact support and endow $\mathcal{M}(X)$ with the vague topology, generated by neighborhoods $V(f_1, \ldots, f_n, \varepsilon, \mu) = \{\nu \in \mathcal{M}(X);\ |\nu f_j - \mu f_j| < \varepsilon\ 1 \leqq j \leqq n\}$, $n \in Z_+$, $f_j \in \mathcal{C}_K$, $\varepsilon > 0$. Here and in the sequel $\mu f = \int f d\mu$.

Harris [5, p. 111ff.] has shown that probability measures on cylinder algebras in $\mathcal{M}(X)$ can be defined through projective systems satisfying obvious additivity and continuity conditions. In the present case his proof simplifies since compactness arguments may be used instead of completeness. Also the probability measures may be defined on the vague Borel algebra, $\mathcal{B}(\mathcal{M})$.

For any Borel probability measure on $\mathcal{M}(X)$, define its characteristic functional [7] $\chi(P)$ on $\mathcal{C}_K$ as the function

$$f \to \int_{\mathcal{M}(X)} e^{i\mu f} P(d\mu)$$

If $\chi(P) = \chi(Q)$, $f_1, \ldots, f_n \in \mathcal{C}_K$, $t = (t_1, \ldots, t_n) \in R^n$ and $P_{f_1 \ldots f_n}$ is the distribution of $\mu f_1, \ldots, \mu f_n$:

$$P_{f_1 \ldots f_n}(x_1, \ldots, x_n) = P\{\mu \in \mathcal{M}(X);\ \mu f_j \leqq x_j\ 1 \leqq j \leqq n\};$$

then

$$\chi(Q)\left(\sum_{j=1}^{n} t_j f_j\right) = \chi(P)\left(\sum_{j=1}^{n} t_j f_j\right) = \int e^{i\Sigma t_j \mu f_j} P(d\mu) = \int_{R^n} e^{itx} P_{f_1 \ldots f_n}(dx).$$

Therefore P and Q give the same measure to sets $\{\mu \in \mathcal{M}(X); (\mu f_1, \ldots, \mu f_n) \in E\}$ $n \in Z_+$, $f_j \in \mathcal{C}_K$, $E \in R^n$, and so to all sets in the σ-algebra generated by those sets. By second countability this σ-algebra contains (and equals) $\mathcal{B}(\mathcal{M})$.

On the set of Borel probability measures we study weak convergence: $P_n \xrightarrow{w} P$ iff for any bounded continuous $f: \mathcal{M}(X) \to R$, $\int f dP_n \to \int f dP$.

Theorem 1 [cf. 7, 9]. *$P_n \xrightarrow{w} P$ if and only if $\chi(P_n) \to \chi(P)$.*

Proof. Since, for $f \in \mathcal{C}_K$, $\mu \to e^{i\mu f}$ is continuous and bounded (though complex valued) the necessity is obvious.

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The Sufficiency. By σ-compactness there is a sequence of compact sets $\{K_j\}$ such that $K_j \subset K_{j+1}^0$ (^{0} denotes "interior of") and $X = \bigcup_{j=1}^{\infty} K_j$. By Urysohn's lemma there is to each $j \in Z_+$ a $k_j \in \mathscr{C}_K$ satisfying $0 \leqq k_j \leqq 1$, $k_j | K_j = 1$ and $S(k_j) \subset K_{j+1}^0$ (for any real or complex function f, $S(f)$ is its support). If now $t \in R$, then $\chi(P_n)(t k_j) \to \chi(P)(t k_j)$ by assumption. Therefore, $P_n\{\mu k_j \leqq x\}$ converges to $P\{\mu k_j \leqq x\}$ in all continuity points of the latter.

Let $\varepsilon > 0$ be given. By the convergence there is to each $j \in Z_+$ an A_j such that $P_n\{\mu k_j > A_j\} < \varepsilon 2^{-j}$ for all n.

$$H = \bigcap_{j=1}^{\infty} \{\mu k_j \leqq A_j\}$$

is closed and $P_n(H) > 1 - \varepsilon$. If, for $f \in \mathscr{C}_K$, $n(f) = \inf\{j;\ S(f) \subset K_j\}$, then $n(f) < \infty$, since $\bigcup_j K_j^0$ covers $S(f)$ and therefore so does $\bigcup_{j=1}^{n} K_j^0$ for some n. Hence, for $f \in \mathscr{C}_K$, $\mu \in H$

$$|\mu f| \leqq \|f\|\, \mu k_{n(f)} \leqq \|f\|\, A_{n(f)} < \infty.$$

In other words, H is vaguely bounded and for that reason [1, p. 192] compact. Thus, $\{P_n\}$ is tight. By [1, p. 194] the vague topology on $\mathscr{M}(X)$ is metrizable and so Prohorov's theorem in the form given in [2, p. 37] implies that the sequence is sequentially compact. But it can have only one limit point, P determined by $\chi(P)$. Q.E.D.

2. Convergence to the Poisson Process

Consider now a probability space $(\Omega, \mathscr{S}, P)$ and a triangular array $\{\xi_{1n}, \ldots, \xi_{r_n n}\}$ $n \in Z_+$ of measurable independent functions $(\Omega, \mathscr{S}) \to (\mathscr{M}(X), \mathscr{B}(\mathscr{M}))$, such that the range of each ξ_{jn} is actually contained in the set of integer or infinite valued measures. Such functions $\Omega \to \mathscr{M}(X)$, we call point processes.

A particularly interesting point process is the Poisson process: If $\xi A_1, \ldots, \xi A_n$, $n \in Z_+$, $A_j \in \mathscr{B}(X)$ are independent as soon as $A_1, \ldots, A_n$ are disjoint and

$$P\{\xi A = j\} = \frac{(\lambda A)^j}{j!} e^{-\lambda A}$$

for all bounded (i.e. contained in some compact set), $A \in \mathscr{B}(X)$ and some $\lambda \in \mathscr{M}(X)$, then ξ is a Poisson process with intensity λ. That this process is the only one satisfying different sets of "natural" assumptions is well known [6 and 8].

A triangular array of point processes is called infinitesimal if for any compact K $\lim_{n \to \infty} \max_{1 \leqq j \leqq r_n} P\{\xi_{jn} K \geqq 1\} = 0$.

Assumption 1. $\lim_{n \to \infty} \sum_{j=1}^{r_n} P\{\xi_{jn} K \geqq 2\} = 0$ for all compact sets K.

Assumption 2a. There is a Radon measure λ on X such that for any bounded Borel set A

$$\sum_{j=1}^{r_n} P\{\xi_{jn} A = 1\} \to \lambda A,$$

as $n \to \infty$.

If K is compact and $\lambda_{jn}^K A = P\{\xi_{jn}(A \cap K)=1, \xi_{jn} K=1\}$, then λ_{jn}^K is a finite measure and so is the sum

$$\lambda_n^K = \sum_{j=1}^{r_n} \lambda_{jn}^K$$

and we weaken 2a to:

Assumption 2b. For each compact K the sequence $\{\lambda_n^K\}$ is weakly convergent in the sense that $\lambda_n^K f \to \lambda^K f$ if $f \in \mathscr{C}_K$, $S(f) \subset K$.

Note that $\lambda_{jn}^K / \lambda_{jn}^K(K)$ may be interpreted as the probability distribution of a point placed by the process ξ_{jn} in K.

Now let Assumption 1 be satisfied and assume that $\lambda_n^K \xrightarrow{w} \lambda^K$ for all compacts K. If $C \subset K$, then for any $A \in \mathscr{B}$

$$\begin{aligned}
\lambda_n^K(A \cap C) &= \sum_{j=1}^{r_n} P\{\xi_{jn}(A \cap C \cap K)=1, \xi_{jn} K=1\} \\
&= \sum_{j=1}^{r_n} [P\{\xi_{jn}(A \cap C)=1, \xi_{jn} C=1\} - P\{\xi_{jn}(A \cap C)=1, \xi_{jn} C=1, \xi_{jn} K \geqq 2\}] \\
&= \lambda_n^C(A) - \sum_{j=1}^{r_n} P\{\xi_{jn}(A \cap C)=1, \xi_{jn} C=1, \xi_{jn} K \geqq 2\}.
\end{aligned}$$

Since the sum is a measure, whose total mass tends to zero, it follows that $\lambda^K(\bullet \cap C) = \lambda^C$. And by σ-compactness there is exactly one $\lambda \in \mathscr{M}(X)$ such that $\lambda^K = \lambda(\bullet \cap K)$ for all K.

If X is a topological group and each ξ_{jn} is stationary in the sense that for any $x \in X$, $A, B \in \mathscr{B}(X)$

$$P\{\xi_{jn}(A+x)=1, \xi_{jn}(B+x)=1\} = P\{\xi_{jn}(A)=1, \xi_{jn}(B)=1\},$$

then λ must be Haar measure.

Theorem 2. *Let* $\{\xi_{1n}, \ldots, \xi_{r_n n}\}$ $n \in Z_+$ *be an infinitesimal array of independent point processes in a* σ*-compact, second countable space. Let Assumptions* 1 *and* 2b *be satisfied. Then, and only then, does the distribution of* $\eta_n = \sum_{j=1}^{r_n} \xi_{jn}$ *tend weakly to the Poisson process with intensity* λ, *where* $\lambda = \sup_K \lim_{n \to \infty} \lambda_n^K$ *(weak limit).*

Note. The strengthening of 2b to 2a is necessary and sufficient for setwise convergence in the sense that for each set A, $\eta_n(A) = \sum_{j=1}^{r_n} \xi_{jn} A$ tend in distribution to the number of points in A of a Poisson process with intensity λ [cf. 4]. This may be shown by the methods used in the following proof.

Proof. The Sufficiency. Let Π_λ be Poisson measure on $(\mathscr{M}(X), \mathscr{B}(\mathscr{M}))$ with intensity λ. Set $P_n = P\eta_n^{-1}$ and $P_{jn} = P\xi_{jn}^{-1}$. By Theorem 1 the weak convergence $P_n \xrightarrow{w} \Pi_\lambda$ follows from $\chi(P_n) \to \chi(\Pi_\lambda)$.

If $g = \sum_{j=1}^{m} a_j 1_{A_j}$, A_j bounded and disjoint, then

$$\chi(\Pi_\lambda)(g) = \prod_{j=1}^{m} \exp\{\lambda A_j(e^{ia_j}-1)\} = \exp \int_X (e^{ig(x)}-1)\, \lambda(dx).$$

Since $f \in \mathscr{C}_K$ can be uniformly approximated by simple functions

$$\chi(\Pi_\lambda)(f) = \exp \int_X (e^{if(x)} - 1)\, \lambda(dx) = \exp \int_{-\infty}^{+\infty} (e^{iu} - 1)\, \lambda f^{-1}(du),$$

which is the canonical representation of an infinitely divisible (actually compound Poisson) characteristic function.

Write $\phi_{jn}(t) = \chi(P_{jn})(tf)$, $t \in R$, $\phi_n = \prod_{j=1}^{n} \phi_{jn}$. We show that $\phi_n(t) \to \chi(\Pi_\lambda)(tf)$, $t \in R$:

Take $t_0 > 1$ and $\varepsilon > 0$. By infinitesimalty there is an n_0 such that for $|t| \leqq t_0$ and $n \geqq n_0$

$$|1 - \phi_{jn}(t)| < \varepsilon \qquad j = 1, \ldots, r_n.$$

In the usual way we can take the logarithm of $\phi_n(t)$, $|t| \leqq t_0$, choosing the branch that makes $t \to \log \phi_n(t)$ continuous and vanishing at zero.

$$\log \phi_n(t) = \sum_{j=1}^{r_n} \log \phi_{jn}(t) = \sum_{j=1}^{r_n} [\phi_{jn}(t) - 1] + r_n(t).$$

By direct computations, for some C

$$\begin{aligned} |r_n(t)| &\leqq C\varepsilon \sum_{j=1}^{r_n} |1 - \phi_{jn}(t)| \leqq C\varepsilon \sum_{j=1}^{r_n} \int |1 - e^{it\mu f}|\, P_{jn}(d\mu) \\ &\leqq 2C\varepsilon \sum_{j=1}^{r_n} \int |1 - e^{itf(x)}|\, P\{\xi_{jn}(dx) = 1,\ \xi_{jn} S(f) = 1\} + o(1) \\ &\to 2C\varepsilon \int |1 - e^{itf(x)}|\, \lambda(dx). \end{aligned}$$

Hence $\lim_{n \to \infty} r_n(t) = 0$.

Similarly, for $t > 0$

$$0 \leqq \sum_{j=1}^{r_n} \int_{|\mu f| < t} |\mu f|\, P_{jn}(d\mu) = \int_{|f(x)| < t} |f(x)|\, \lambda_n^{S(f)}(dx) + o(1).$$

And so by classic theory [3, p. 528ff.], it follows that

$$\phi_n(t) \to \chi(\Pi_\lambda)(f) \qquad t \in R$$

if and only if

$$t^2 \sum_{j=1}^{r_n} P\{\xi_{jn} f \in dt\} \to t^2 \lambda f^{-1}(dt) \tag{1}$$

vaguely,

$$\sum_{j=1}^{r_n} P\{\xi_{jn} f > t\} \to \lambda\{f > t\} \qquad \text{if } t > 0 \tag{2}$$

is a continuity point of $u \to \lambda\{f > u\}$,

$$\sum_{j=1}^{r_n} P\{\xi_{jn} f \leqq t\} \to \lambda\{f \leqq t\} \qquad \text{if } t < 0 \tag{3}$$

is a continuity point of $u \to \lambda\{f \leqq u\}$, and $t^2 \lambda f^{-1}(dt)$ is a canonic measure on R, as is evidently the case:

$$\int_{-\infty}^{+\infty} t^2 \lambda f^{-1}(dt) \leqq \|f\|^2 \lambda S(f) < \infty.$$

For (1) let $\phi: R \to R$ be continuous with compact support:

$$\sum_{j=1}^{r_n} \int \phi(t)\, t^2\, P\{\xi_{jn} f \in dt\} = \sum_{j=1}^{r_n} \int \phi(\mu f)(\mu f)^2 P_{jn}(d\mu)$$
$$= \sum_{j=1}^{r_n} \int_{\mu S(f)=1} \phi(\mu f)(\mu f)^2 P_{jn}(d\mu) + \sum_{j=1}^{r_n} \int_{\mu S(f) \geqq 2} \phi(\mu f)(\mu f)^2 P_{jn}(d\mu).$$

The absolute value of the last sum equals

$$\left| \sum_{j=1}^{r_n} \int_{\substack{\mu S(f) \geqq 2 \\ \mu f \in S(\phi)}} \phi(\mu f)(\mu f)^2 P_{jn}(d\mu) \right| \leqq \|\phi\| \, [\sup |S(\phi)|]^2 \sum_{j=1}^{r_n} P\{\xi_{jn} S(f) \geqq 2\} \to 0, \quad \text{as } n \to \infty.$$

And the first sum is

$$\sum_{j=1}^{r_n} \int_X \varphi \circ f(x) [f(x)]^2 P\{\xi_{jn}(dx) = 1, \xi_{jn} S(f) = 1\}$$
$$= \int_X \varphi \circ f(x) [f(x)]^2 \lambda_n^{S(f)}(dx) \to \int \varphi \circ f(x) [f(x)]^2 \lambda(dx).$$

Similarly,

$$\sum_{j=1}^{r_n} P\{\xi_{jn} f > t\} = \sum_{j=1}^{r_n} P\{\xi_{jn} f > t, \xi_{jn} S(f) = 1\} + \sum_{j=1}^{r_n} P\{\xi_{jn} f > t, \xi_{jn} S(f) \geqq 2\},$$

yields (2) and a corresponding relation (3).

The Necessity. If there is weak convergence, then [3] (1), (2), and (3) hold since

$$\lim_{n \to \infty} E[e^{it\eta_n f}] = \exp \int_{-\infty}^{+\infty} (e^{itu} - 1)\, \lambda f^{-1}(du) \qquad t \in R, \ f \in \mathscr{C}_K.$$

Let $K \subset X$ be compact. There is $g \in \mathscr{C}_K$ such that $0 \leqq g \leqq 1$, $g|K = 1$ and so by (2)

$$\limsup_{n \to \infty} \sum_{j=1}^{r_n} P\{\xi_{jn} K \geqq 2\} \leqq \lim_{n \to \infty} \sum_{j=1}^{r_n} P\{\xi_{jn} g > \tfrac{3}{2}\} = \lambda\{g > \tfrac{3}{2}\} = 0.$$

Obviously $\frac{3}{2}$ is a continuity point of $u \to \lambda\{g > u\}$. Now let $f \in \mathscr{C}_K$ with $S(f) \subset K$ and $a = \|f\|$. Then

$$\int f(x)\, \lambda_n^K(dx) = \sum_{j=1}^{r_n} \int f(x)\, P\{\xi_{jn}(dx) = 1, \xi_{jn} K = 1\}$$
$$= \sum_{j=1}^{r_n} \int \mu f\, P\{\xi_{jn} \in d\mu, \xi_{jn} K = 1\}$$
$$= \sum_{j=1}^{r_n} \int_{|u| \leqq a} u P\{\xi_{jn} f \in du\} - \sum_{j=1}^{r_n} \int_{|u| \leqq a} u P\{\xi_{jn} f \in du, \xi_{jn} K \geqq 2\}$$
$$\to \int f(x)\, \lambda^K(dx), \qquad \text{as } n \to \infty,$$

showing the weak convergence of λ_n^K to $\lambda(\bullet \cap K)$.

3. Point Processes on the Line

So far $\mathcal{M}(X)$ has been endowed with the vague topology. If X is the real line or some subset of it, each $\mu \in \mathcal{M}(X)$ corresponds to a non-decreasing right continuous function, which we denote by μ, too, and vague convergence is convergence in the continuity points of the limit function.

To be specific, let us take $X=[0,1]$. Then the Skorohod J_1-topology [2, p. 111] on $D[0,1]$, relativized to $\mathcal{M}(X)$, is another candidate for a topology on $\mathcal{M}(X)$. Obviously, Skorohod convergence implies vague convergence but not vice versa.

However, if the vague limit is a pure step function with all jumps of size one, then a converse holds. Let d denote the Skorohod metric:

$$d(\mu, v)=\inf_{\lambda} \max\{\sup_{0\leqq x\leqq 1} |\mu\circ\lambda(x)-v(x)|, \sup_{0\leqq x\leqq 1} |\lambda(x)-x|\},$$

where the infimum is over all strictly increasing continuous λ with $\lambda(0)=0$, $\lambda(1)=1$, and let ρ be the Lévy metric:

$$\rho(\mu, v)=\inf\{y>0;\ v(x-y)-y\leqq\mu(x)\leqq v(x+y)+y \text{ for all } x\}.$$

It is known that ρ metrizes the vague topology. Now assume that μ is constant except for jumps at $0<x_1<\cdots<x_n<1$, $\mu(x_i)-\mu(x_i-)=1$, that v is an integer valued non-decreasing function and that $\rho(\mu, v)<h$ where $h>0$ is smaller than one and smaller than $\frac{1}{2}\min_{2\leqq i\leqq n}(x_i-x_{i-1})$. Then for $0\leqq x<x_1$, $v(x-h)\leqq\mu(x)+h=h$ or $v(x-h)=0$. Hence, if y_1 is the place of the first jump of v, then $y_1\geqq x-h$. But for $x\geqq x_1$, $v(x+h)\geqq\mu(x_1)-h$, which means that $v(x)\geqq 1$ and so $x_1-h\leqq y_1\leqq x_1+h$. Also, $v(x_1+h)\leqq\mu(x_1+2h)+h=1+h$, implying that $v(y_1)-v(y_1-)=1$. By induction it follows that v has n jumps, too, all of size one, and if y_k is the place of the k^{th} jump $1\leqq k\leqq n$, then $|x_k-y_k|\leqq h$. Hence, $d(\mu, v)<h$: choose λ by $\lambda(x_k)=y_k$ and linear between x_{k-1} and x_k, $1\leqq k\leqq n+1$, $x_0=y_0=0$, $x_{n+1}=y_{n+1}=1$.

It follows that if μ is as above, μ_n is a sequence of integer valued step functions, and $\rho(\mu_n, \mu)\to 0$, then $d(\mu_n, \mu)\to 0$. To proceed we need a simple lemma. Let S be any separable space with two metrics d and ρ. If $\{P_n\}$ is a sequence of probability measures defined on the union of the Borel algebras of (S, d) and (S, ρ), let $P_n \xrightarrow{wd} P$ mean weak convergence on (S, d) and correspondingly for $P_n \xrightarrow{w\rho} P$.

Lemma. *For S, d, ρ and P_n as mentioned, assume that for all $s\in E\subset S$ $\rho(s_n, s)\to 0$ implies $d(s_n, s)\to 0$. Then if $P_n \xrightarrow{w\rho} P$, E is ρ-open, and $P(E)=1$, it holds that $P_n \xrightarrow{wd} P$.*

Proof. The set E is a P-continuity set in (S, ρ) and so $P_n(E)\to P(E)$. Let f be bounded and d-continuous: $S\to R$

$$\int f\,dP_n=\int_E f\,dP_n+\int_{S\setminus E} f\,dP_n\to\int_E f\,dP=\int f\,dP$$

since f is ρ-continuous on E and $P_n(S\setminus E)\to 0$. Now Poisson measures on $\mathcal{M}[0,1]$ give measure one to the set of step functions with jumps only of size one and no jumps at 0 or 1, provided the intensity λ is non-atomic, i.e. $\lambda\{x\}=0$ for all $x\in X$. Point processes on $[0,1]$ are integer valued step functions. Thus, the lemma and the preceding arguments show that Theorem 2 holds also if $\mathcal{M}(X)$ is given the Skorohod topology.

As a simple example, let ζ_{in} $1 \leqq i \leqq n$ be independent Bernoulli variables,

$$1-P\{\zeta_{in}=0\}=P\{\zeta_{in}=1\}=\lambda/n+o(n^{-1}), \qquad \lambda>0.$$

Define

$$\eta_n(t)=\sum_{i=1}^{[nt]}\zeta_{in} \qquad 0\leqq t\leqq 1,\ n=1,2,\ldots.$$

Then η_n converges Skorohod-weakly to the Poisson process on [0, 1]. In our context the demonstration of this is direct: Let

$$\xi_{in}(t)=\begin{cases}\zeta_{in} & i\leqq [nt]\\ 0 & i>[nt].\end{cases}$$

Each ξ_{in} $1 \leqq i \leqq n$ is a point process on [0, 1] and $\eta_n=\sum_{i=1}^{n}\xi_{in}$. The array $\{\xi_{in},\ldots,\xi_{nn}\}$ is infinitesimal:

$$P\{\xi_{in}(1)\geqq 1\}=\frac{\lambda}{n}+o(n^{-1})\to 0.$$

$$\sum_{i=1}^{n}P\{\xi_{in}(1)\geqq 2\}=0$$

for all n and

$$\sum_{i=1}^{n}P\{\xi_{in}(t)=1\}=\sum_{i=1}^{[nt]}P\{\zeta_{in}=1\}=\frac{[nt]}{n}\lambda+[nt]\,o(n^{-1})\to\lambda t, \qquad 0\leqq t\leqq 1.$$

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