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# Maximal Measures for Simple Piecewise Monotonic Transformations

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**Summary.** It is shown that the transformation $x \mapsto \beta x + \alpha \pmod 1$ $(\beta > 1, 0 \leqq \alpha < 1)$ on $[0, 1]$ has unique maximal measure.

## § 0. Introduction

In this paper we consider two classes of piecewise monotonic transformations $f$ on $I = [0, 1]$. The first one is the class of all $(I, f)$ such that $I = J_1 \cup J_2$, $J_1, J_2$ are disjoint intervals and $f/J_1, f/J_2$ are continuous and strictly increasing. Furthermore we assume that $(J_1, J_2)$ is a generator for $(I, f)$ and that $h_{\text{top}}(f) > 0$. The second one is the class of transformations $f: x \mapsto \beta x + \alpha \pmod 1$ on $I$ where $\beta > 1$ and $0 \leqq \alpha < 1$.

An invariant measure $\mu$ on $(I, f)$ is called maximal if its entropy $h(\mu)$ is equal to the topological entropy $h_{\text{top}}(f)$ of $(I, f)$, or equivalently, if $h(\mu)$ is greater than or equal to the entropy of every other invariant measure on $(I, f)$. The two classess of transformations above have always at least one maximal measure, because they are expansive. For definitions see [3].

We show that in the first case above there are at most two ergodic maximal measures and characterize those $f$'s which have unique maximal measure. In the second case, for $f: x \mapsto \beta x + \alpha \pmod 1$, we have always unique maximal measure.

To this end we use the results proved in [2]. For every $(I, f)$ a subshift $\Sigma_M$ of finite type ($M$ is the corresponding transition matrix) over a countable alphabet $D$ is constructed there such that $\Sigma_M$ and $(I, f)$ have isomorphic sets of maximal measures. Hence it suffices to consider the problem of uniqueness of the maximal measure for $\Sigma_M$. A tool for this is the theorem cited in § 1 which contains also a description of other results from [2]. Furthermore, for the above two classes of transformations the transition matrices $M$ are derived which are considered as oriented graphs. The results of this paper are proved in § 2. We determine conditions for $M$ such that $\Sigma_M$ can have more than one maximal measure (Lemmas 2 and 3) and characterize the $(I, f)$ of the first class above which give rise to such an $M$ (Theorem 1). Then we prove that such an $M$ cannot occur for $f: x \mapsto \beta x + \alpha \pmod 1$ (Theorem 2).

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## §1.

We begin with a short description of the methods developped in [2]. We have a piecewise monotonic transformation $f$ on $I$, i.e. $I=\bigcup_{i=1}^{n} J_i$, $J_i$ disjoint intervals, such that $f/J_i$ is continuous and strictly increasing. Furthermore we need that $(J_i)_{1\leqq i\leqq n}$ is a generator for $(I,f)$ and that $h_{\text{top}}(f)>0$. The $f$-expansion $\varphi:(I,f)\to\Sigma_n^+=\{1,2,\ldots,n\}^{\mathbb{N}}$ is defined by $\varphi(x)=i_0 i_1 i_2\ldots$, where $i_k$ is the number $i$ of the interval $J_i$ such that $f^k(x)\in J_i$. If $J_k=(r,s)$, set $\underline{a}^k=\lim_{t\downarrow r}\varphi(t)$ and $\underline{b}^k=\lim_{t\uparrow s}\varphi(t)$. Define

$$\Sigma_f^+=\{\underline{x}\in\Sigma_n^+: \underline{a}^{x_m}\leqq\sigma^m\underline{x}=x_m x_{m+1}x_{m+2}\ldots\leqq\underline{b}^{x_m}\text{ for all } m\geqq 0\},$$

where $\sigma$ denotes the shift transformation and $\leqq$ the lexicographic ordering in $\Sigma_n^+$. Then $\varphi$ is an order preserving isomorphism modulo small sets (cf. §0 of [2]) between $(I,f)$ and $(\Sigma_f^+,\sigma)$. In particular, $(I,f)$ and $(\Sigma_f^+,\sigma)$ have isomorphic sets of measures with maximal entropy.

Now divide $\underline{a}^i$ into initial segments of $\underline{b}^j$'s, which can be done in a unique way, because $\underline{b}^j$ is the only element among the $\underline{b}^k$'s which begins with $j$, and denote the lengths of these segments by $r(i,1), r(i,2),\ldots$, i.e. we have for $m\geqq 0$ $a^i_{r(i,1)+\ldots+r(i,m)+k}=b^j_k$, $\quad 0\leqq k\leqq r(i,m+1)-1$, $\quad j=a^i_{r(i,1)+\ldots+r(i,m)}$ and $a^i_{r(i,1)+\ldots+r(i,m+1)}\neq b^j_{r(i,m+1)}$.

Similary divide $\underline{b}^j$ into initial segments of $\underline{a}^i$'s, and denote their lengths by $s(j,1), s(j,2),\ldots$. Condiser the set

$$D=\{(A,i,k),(B,i,k): 1\leqq i\leqq n, k\geqq 1\}=\{A,B\}\times\{1,\ldots,n\}\times\mathbb{N},$$

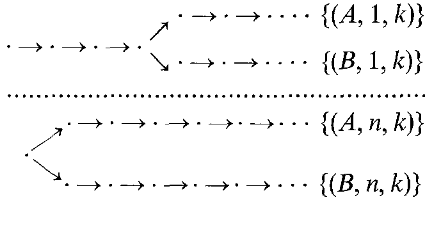
identify $(A,i,k)=(B,i,k)$ for $1\leqq k\leqq r(i,1)=s(i,1)$ and also

$$\begin{aligned}&(A,i,p+k)=(A,j,q+k)\,\forall k\geqq 1,\text{ if }\sigma^p\underline{a}^i=\sigma^q\underline{a}^j, p=\sum_{m=1}^{u} r(i,m), q=\sum_{m=1}^{v} r(j,m)\\ &(B,i,p+k)=(B,j,q+k)\forall k\geqq 1,\text{ if }\sigma^p\underline{b}^i=\sigma^q\underline{b}^j, p=\sum_{m=1}^{u} s(i,m), q=\sum_{m=1}^{v} s(j,m).\end{aligned}\tag{1.1}$$

Together with the following arrows $D$ becomes a graph $M$.

$(A,i,k)\to(A,i,k+1), (B,j,k)\to(B,j,k+1)$.
If $k=r(i,1)+\ldots+r(i,m)$ for some $m$, then
$(A,i,k)\to(B,j,r(i,m)+1)$, where $j=a^i_{r(i,1)+\ldots+r(i,m-1)}$
and $(A,i,k)\to(A,t,1)=(B,t,1)$ for $a^i_k<t<b^j_{r(i,m)}$.  (1.2)
If $k=s(j,1)+\ldots+s(j,m)$ for some $m$, then
$(B,j,k)\to(A,i,s(j,m)+1)$, where $i=b^j_{s(j,1)+\ldots+s(j,m-1)}$
and $(B,j,k)\to(A,t,1)=(B,t,1)$ for $a^i_{s(j,m)}<t<b^j_k$.

The graph looks like this



Only the arrows $(A, i, k) \to (A, i, k+1)$ and $(B, j, k) \to (B, j, k+1)$ are indicated in this picture.

Set $\Sigma_M = \{\underline{y} \in D^{\mathbb{Z}}$: there is an arrow from $y_i$ to $y_{i+1} \forall i \in \mathbb{Z}\}$. Then $(\Sigma_f, \sigma)$, the natural extension of $(\Sigma_f^+, \sigma)$, and $(\Sigma_M, \sigma)$ are isomorphic modulo small sets. In particular $(\Sigma_f, \sigma)$ and $(\Sigma_M, \sigma)$ and hence also $(I, f)$ and $(\Sigma_M, \sigma)$ have isomorphic sets of maximal measures.

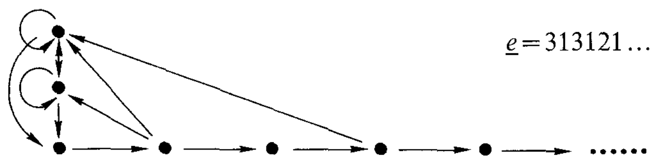
Divide $M$ into irreducible subgraphs $M_1, M_2, \ldots$ and denote the corresponding subshifts of $\Sigma_M$ by $\Sigma_{M_i}$. We consider $M$ as 0–1 matrix with index set $D$, $M_{jk} = 1$ iff there is an arrow $j \to k$.

**Theorem.** (i) $h_{\text{top}}(\Sigma_M) = \log r(M)$ (spectral radius of the $l^1$-operator $M$).

(ii) Every ergodic maximal measure is concentrated on a $\Sigma_{M_i}$ satisfying $r(M_i) = r(M)$ (the ergodic maximal measures are the extremal points of the compact convex set of all maximal measures).

(iii) There is at most one (ergodic) maximal measure on every such $\Sigma_{M_i}$.

Our goal is to apply these results to special piecewise monotonic transformations. The simplest nontrivial example is the $\beta$-transformation $x \mapsto \beta x \pmod 1$ for $\beta > 1$. Let $n$ be so that $\beta \leqq n < \beta + 1$ and $\underline{e}$ the $\beta$-expansion of 1. Then $\underline{a}^i = i1111\ldots$ for $1 \leqq i \leqq n$, $\underline{b}^i = i\underline{e}$ for $1 \leqq i \leqq n-1$ and $\underline{b}^n = \underline{e}$. $\Sigma_f^+$ becomes $\Sigma_\beta^+ = \{\underline{x} \in \Sigma_n^+ : \sigma^k \underline{x} \leqq \underline{e} \forall k\}$. Using the identifications (1.1) we get for $n=3$ the following graph

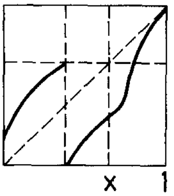


$\underline{e} = 313121\ldots$

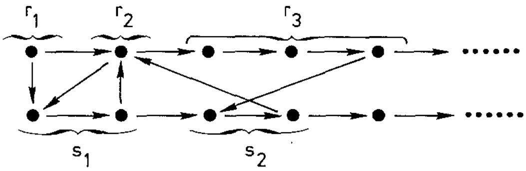
There are $e_i$ arrows starting at the $i$-th point of the row. This gives an irreducible graph (cf. [1]). Hence the $\beta$-shift $\Sigma_\beta^+$ has unique maximal measure.

One sees that, due to the identifications, only one of the $2n$ rows has remained. In the next more complicated case we consider the situation where there are two remaining rows.

Consider any piecewise monotonic transformation with $n=2$. We assume that the end point of $J_1$ is mapped to 1 and that the initial point of $J_2$ is mapped to 0. We can reduce all other cases to this case taking away wandering sets and fixed points. For example consider the following graph of an $f$. We take away $(x, 1]$, where $x = f$(end point of $J_1$). $(x, 1)$ is a wandering set and 1 is a fixed point. $f(x) \leqq x$, otherwise $(J_1, J_2)$ is no generator.

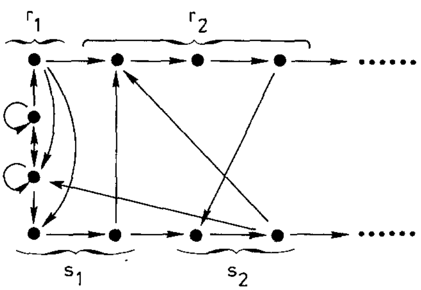


Let $\underline{a}^1 = \underline{a}$ be the expansion of 0 and $\underline{b}^2 = \underline{b}$ that of 1. Then $\underline{b}^1 = 1\underline{b}$ and $a^2 = 2\underline{a}$. Set $r_k = r(1, k)$ and $s_k = s(2, k)$. By (1.1) we identify $(A, 2, k+1) = (A, 1, k)$ for $k \geqq r_1$ and $(B, 1, k+1) = (B, 2, k)$ for $k \geqq s_1$ and get the graph




$\underline{a} = 111211\ldots$ (1.3)
$\underline{b} = 21212\ldots$

Also the transformation $x \mapsto \beta x + \alpha \pmod 1$ for $\beta > 1$ and $0 < \alpha < 1$ gives such diagrams. Again let $\underline{a}^1 = \underline{a}$ be the expansion of 0 and $\underline{b}^n = \underline{b}$ the expansion of 1 ($n$ so that $\alpha + \beta \leqq n < \alpha + \beta + 1$). Then $\underline{a}^i = i\,\underline{a}$ for $2 \leqq i \leqq n$ and $\underline{b}^i = i\,\underline{b}$ for $1 \leqq i \leqq n-1$. Again set $r_k = r(1, k)$ and $s_k = s(n, k)$. By (1.1) we identify $(A, i, k+1) = (A, 1, k)$ for $k \geqq r(i, 1)$ and $2 \leqq i \leqq n$ and $(B, i, k+1) = (B, n, k)$ for $k \geqq s(i, 1)$ and $1 \leqq i \leqq n-1$. We get (for $n = 4$)




$\underline{a} = 12412\ldots$ (1.4)
$\underline{b} = 41314\ldots$

We shall denote the points in the first row by $c_k = (A, 1, k)$, the points in the last row by $d_k = (B, n, k)$, and the remaining points by $e_2, \ldots, e_{n-1}$. Set $A(i, j) = \{c_k : i \leqq k \leqq j\}$ ($A(i, \infty) = \{c_k : i \leqq k\}$) and $B(i, j) = \{d_k : i \leqq k \leqq j\}$.

Hence in the cases we want to consider in this paper we have $D = \{c_k, d_k, e_j : k \geqq 1, 2 \leqq j \leqq n-1\}$ (for $n = 2$ there are no $e_j$) and $M$ has the following arrows

$$\begin{aligned}
&c_k \to c_{k+1},\ d_k \to d_{k+1} \quad (k \geqq 1)\\
&c_{r_1+\ldots+r_k} \to d_{r_k} \text{ and } e_j \text{ for } a_{r_1+\ldots+r_k} < j < b_{r_k - 1} \quad (k \geqq 1)\\
&d_{s_1+\ldots+s_k} \to c_{s_k} \text{ and } e_j \text{ for } a_{s_k - 1} < j < b_{s_1+\ldots+s_k} \quad (k \geqq 1)\\
&e_k \to c_1, d_1 \text{ and } e_j \text{ for } 2 \leqq j \leqq n-1 \quad (2 \leqq k \leqq n-1).
\end{aligned} \tag{1.5}$$

Furthermore by (1.1) we can identify ($u, v \geqq 0$)

$$\begin{aligned}
&c_{p+k} = c_{q+k}\ \forall k \geqq 1, \text{ if } \sigma^p \underline{a} = \sigma^q \underline{a},\ p = r_1 + \ldots + r_u,\ q = r_1 + \ldots + r_v\\
&d_{p+k} = d_{q+k}\ \forall k \geqq 1, \text{ if } \sigma^p \underline{b} = \sigma^q \underline{b},\ p = s_1 + \ldots + s_u,\ q = s_1 + \ldots + s_v.
\end{aligned} \tag{1.6}$$

The two $\Sigma_M$ arising from $M$ with and without identifications are then isomorphic.

For the $r_i$ and $s_i$, which denote the lengths of initial segments of $v\underline{b}$ ($1 \leqq v \leqq n-1$) in $\underline{a}$ and of $u\underline{a}$ ($2 \leqq u \leqq n$) in $\underline{b}$ respectively, we have the following lemma. Remark that

$$\underline{a}, \underline{b} \in \Sigma_f^+, \text{ which becomes } \{\underline{x} \in \Sigma_n^+ : \underline{a} \leqq \sigma^k \underline{x} \leqq \underline{b} \forall k \geqq 0\}. \tag{1.7}$$

**Lemma 1.** *There are maps $P, Q: \mathbb{N} \to \mathbb{N} \cup \{0\}$, such that*

$$\begin{aligned} r_k &= s_1 + \ldots + s_{P(k)} + 1 \quad \text{for } k \geqq 1 \\ s_k &= r_1 + \ldots + r_{Q(k)} + 1 \quad \text{for } k \geqq 1. \end{aligned} \tag{1.8}$$

*Proof.* We prove only the first statement. Suppose that $1 + s_1 + \ldots + s_m < r_k < 1 + s_1 + \ldots + s_{m+1}$ for some $m$. We have $a_{r_1 + \ldots + r_{k-1} + j} = b_{j-1}$ for $1 \leqq j \leqq r_k - 1$, $a_{r_1 + \ldots + r_k} < b_{r_k - 1}$ and $b_{s_1 + \ldots + s_m + l} = a_{l-1}$ for $1 \leqq l \leqq s_{m+1} - 1$ by definition of $r_k$, $s_m$.
Setting $j = s_1 + \ldots + s_m + l + 1$ we have because $r_k - s_1 - \ldots - s_m - 1 \leqq s_{m+1} - 1$ $a_{r_1 + \ldots + r_{k-1} + s_1 + \ldots + s_m + l + 1} = a_{l-1}$ for $1 \leqq l \leqq r_k - s_1 - \ldots - s_m - 2$ ($\Rightarrow j \leqq r_k - 1$) and $a_{r_1 + \ldots + r_k} < a_{r_k - s_1 - \ldots - s_m - 2}$.
If $r_k - s_1 - \ldots - s_m - 2 = 0$, we have $a_{r_1 + \ldots + r_k} < a_0 = 1$, a contradiction, and $\sigma^{r_1 + \ldots + r_{k-1} + s_1 + \ldots + s_m + 2} \underline{a} < \underline{a}$, a contradiction to (1.7), in case of $r_k - s_1 - \ldots - s_m - 2 \geqq 1$. The lemma is proved.

This lemma shows that, if an arrow goes from $B(1, \infty)$ to $c_i$, then $i = s_k = r_1 + \ldots + r_{Q(k)} + 1$ and hence by (1.5) and $i - 1 = r_1 + \ldots + r_{Q(k)}$ there is an arrow from the point $c_{i-1}$ before $c_i$ to $B(1, \infty)$. This will be used several times in the sequel.

We want to describe another thing which will be important, namely how one can regain $\underline{a}$ and $\underline{b}$ from $M$. This can be done by induction. $a_0 = 1$, $b_0 = n$. $r_1 > 1$ and $s_1 > 1$ is a contradiction to (1.8). Hence suppose $r_1 = 1$. Then $a_1$ is the number such that there are arrows $c_1 \to e_j$ for $a_1 < j < b_0$ in (1.5). If $a_0 \ldots a_{p-1}$ and $b_0 \ldots b_{q-1}$ are already determined, where $p = r_1 + \ldots + r_i + 1$ and $q = s_1 + \ldots + s_k + 1$ (the above step is for $i = 0$ and $k = 0$), $r_{i+1} > s_1 + \ldots + s_k + 1$ and $s_{k+1} > r_1 + \ldots + r_i + 1$ is again a contradiction to (1.8). Supposing $r_{i+1} \leqq s_1 + \ldots + s_k + 1$, we have by the definition of $r_{i+1}$ that $a_p \ldots a_{p-2+r_{i+1}} = b_0 \ldots b_{r_{i+1}-2}$, which is already known, and $a_{p-1+r_{i+1}}$ is the number such that there are arrows $c_{r_1 + \ldots + r_{i+1}} \to e_j$ for $a_{p-1+r_{i+1}} < j < b_{r_{i+1}-1}$ in (1.5).

## §2.

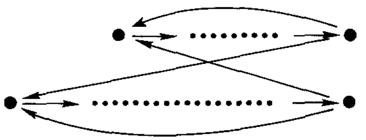
To apply the above theorem we have to divide $M$ into irreducible subgraphs. If $M$ is reducible we can divide $D$ into two disjoint subsets $D_1$ and $D_2$ such that there may be transition from $D_1$ to $D_2$, but not from $D_2$ to $D_1$. $D_2$ is not empty. Suppose $c_l \in D_2$ and $l$ is the smallest integer with this property. Then $A(l, \infty) \subset D_2$, because there is no transition from $D_2$ to $D_1$. If there is no arrow from $A(l, \infty)$ to $B(1, \infty)$, then $D_2 = A(l, \infty)$ has spectral radius 1 and hence we can take it away. We assume that there is such an arrow and hence $a\,d_m \in D_2$. Again let $m$ be the smallest integer with this property. As above $B(m, \infty) \subset D_2$ and $D_2 = A(l, \infty) \cup B(m, \infty)$.

Let $\lambda_0$ be the largest real solution of $\lambda^{l+m}-\lambda^l-\lambda^m=0$ and $M_2=M/D_2$. We have

**Lemma 2.** *$r(M_2)\leqq\lambda_0$. Equality holds, iff there are i, j, such that*

$$\begin{aligned} r_1+\ldots+r_{i-1}=l-1, \quad s_1+\ldots+s_{j-1}=m-1 \quad \textit{and} \\ s_j=s_{j+1}=\ldots=l, \quad r_i=r_{i+1}=\ldots=m. \end{aligned} \tag{2.1}$$

*Proof.* We can assume that there are arrows arriving at $c_l$ and $d_m$. If not, let $l'\geqq l$ and $m'\geqq m$ be the smallest integers, such that there are arrows from $B(m,\infty)$ to $c_{l'}$, and from $A(l,\infty)$ to $d_{m'}$. Then we can never return from $A(l',\infty)\cup B(m',\infty)$ to $c_l,\ldots,c_{l'-1}$ and to $d_m,\ldots,d_{m'-1}$ and hence we can take away this points from $D_2$ without changing the spectral radius of $M_2$. Hence, if we have proved the result with this assumption, $r(M_2)\leqq\lambda_0'<\lambda_0$, where $\lambda_0'$ is the largest solution of $\lambda^{l'+m'}-\lambda^{l'}-\lambda^{m'}=0$. Now we assume that there are arrows arriving at $c_l$ and $d_m$, i.e. there are $i, j$, such that $l=s_u=r_1+\ldots+r_{i-1}+1$ and $m=r_v=s_1+\ldots+s_{j-1}+1$ for some $u, v$ (cf. Lemma 1). If now (2.1) is satisfied, we have $\sigma^{l+m}\underline{a}=\sigma^l\underline{a}$ and $\sigma^{l+m}\underline{b}=\sigma^m\underline{b}$, i.e. $c_{k+m}=c_k$ for $k\geqq l$ and $d_{k+l}=d_k$ for $k\geqq m$ (cf. (1.1)). We get the following graph for $M_2$.

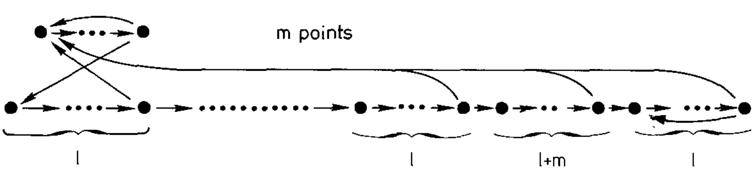



(2.2)

Hence the characteristic equation for $M_2$ is $\lambda^{l+m}-\lambda^l-\lambda^m=0$ and $r(M_2)=\lambda_0$.

If (2.1) is not satisfied we have $r_k\geqq m\,\forall k\geqq i$ and $s_k\geqq l\,\forall k\geqq j$, $i, j$ as above (otherwise there would be transition from $D_2$ to $D_1$) and there is at least one $r_g(g\geqq i)$ or $s_h(h\geqq j)$ with $r_g>m(\Rightarrow r_g\geqq l+m$ because of (1.2)) or $s_h>l(\Rightarrow s_h\geqq l+m$ because of (1.2)). We show that $M_2$ has then a spectral radius strictly less than $\lambda_0$.

First we compute the spectral radius of $M_2$ for the case $r_k=m\,\forall k\geqq i$, $s_k=l\,\forall k\geqq j$, $k\neq h$ and $s_h=m+l$. As above (2.2) we get the diagram (identifications for $\sigma^{m+l}\,\underline{a}=\sigma^l\underline{a}$ and $\sigma^{2m+(h-j+2)l}\underline{b}=\sigma^{2m+(h-j+1)l}\underline{b}$)



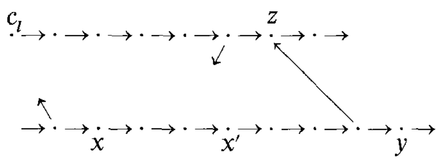

(2.3)

We get the characteristic polynomial

$$\begin{aligned} P(\lambda)&=\lambda^{(h-j+2)l+2m}-\lambda^{(h-j+2)l+m}-\lambda^{(h-j+1)l+2m} \\ &\quad-\sum_{f=1}^{h-j}\lambda^{(h-j-f+2)l+m}-\lambda^l-1+\sum_{f=1}^{h-j}\lambda^{(h-j-f+1)l+m} \\ &\quad+1+\lambda^{(h-j+1)l+2m} \\ &=\lambda^{(h-j+1)l+m}(\lambda^{l+m}-\lambda^l-\lambda^m)+\lambda^{l+m}-\lambda^l. \end{aligned}$$

It is easy to see that $P(\lambda) \geqq \lambda_0^m > 0$ for all $\lambda \geqq \lambda_0$. Hence the largest solution of $P(\lambda)=0$ is less than $\lambda_0$, i.e. $r(M_2) < \lambda_0$. The same argument holds if $r_g = l+m$ instead of $s_h = m+l$. For the general case we can find a minimal $g \geqq i$ or a minimal $h \geqq j$ such that $r_g \geqq l+m$ or $s_h \geqq l+m$ respectively. We consider the second case. The first one is similar. Denote the matrix corresponding to (2.3) by $M_h$ (without identifications). We show $r(M_2) \leqq r(M_h)$. Let $N_k^x$ and $\tilde{N}_k^x$ be the numbers of blocks of length $k$ admitted by $M_2$ and $M_h$ respectively and beginning with $x \in D_2$. It suffices to show $\|M_2^k\|_1 = \sup_x N_k^x \leqq 2\|M_h^k\|_1 = 2 \sup_x \tilde{N}_k^x$.

To this end set $R_u = r_1 + \ldots + r_u$, $S_v = s_1 + \ldots + s_v$ and $D' = \{c_{R_u+1}, d_{S_v+1}: u \geqq i, v \geqq j\}$. We show by induction that $N_k^x \leqq \tilde{N}_k^x$, if $x \in D'$. Let $x = d_{S_v+1}$ (the case $x = c_{R_u+1}$ is similar). Cancelling the first $t := s_{v+1}$ elements, which are equal in all $M_2$-blocks beginning with $x$, we get $N_k^x = N_{k-t}^y + N_{k-t}^z$, where $y = d_{S_{v+1}+1} \in D'$ and $z = c_{s_{v+1}} = c_{R_{Q(v+1)}+1} \in D'$ (Lemma 1). Here is a picture of the relevant part of $M_2$.




If $k \leqq t$, $N_k^x = 1 \leqq \tilde{N}_k^x$. Otherwise, as $y, z \in D'$, we have by induction that $N_{k-t}^y \leqq \tilde{N}_{k-t}^y$ and $N_{k-t}^z \leqq \tilde{N}_{k-t}^z$, hence

$$N_k^x \leqq \tilde{N}_{k-t}^y + \tilde{N}_{k-t}^z. \tag{2.4}$$

Now we consider the case $v+1 \neq h$. The arguments for $v+1=h$ are the same replacing $l$ by $l+m$. Because $s_{v+1} \geqq l$ there must be an arrow in $M_h$ from a point in $B(S_v+1, S_{v+1})$ to $c_l$. Let $x' = d_{S_v+t'}$ be the first such point ($t' \leqq l$). Then $x' c_l c_{l+1} \ldots z$ is an admissible block in $M_h$ of length $t-l+2$. Hence $\tilde{N}_k^x = \tilde{N}_{k-t'+1}^{x'} \geqq \tilde{N}_{k-t'}^{c_l} + \tilde{N}_{k-t}^y$ and $\tilde{N}_{k-t'}^{c_l} \geqq \tilde{N}_{k-t'-t+l}^z \geqq \tilde{N}_{k-t}^z$, because $l-t' \geqq 0$. Together with (2.4) we have $\tilde{N}_k^x \geqq N_k^x$. Now let $x = d_{S_v+1+w} \notin D'$. Then (2.4) is proved as above with $t = s_{v+1} - w$ and the same $y, z \in D'$. Therefore $N_k^x \leqq 2 \sup_z \tilde{N}_k^z$. For $x \in D'$ this follows from $N_k^x \leqq \tilde{N}_k^x$. This implies $\sup_x N_k^x \leqq 2 \sup_x \tilde{N}_k^x$. Hence we have $r(M_2) \leqq r(M_h) < \lambda_0$ and the lemma is proved.

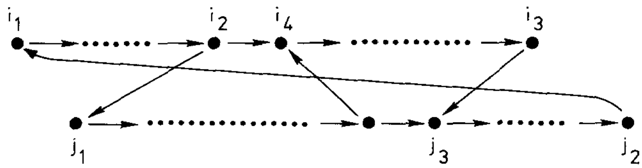
Now we consider the case (1.3) and the case (1.4) only, if $E = \{(A,i,1) = (B,i,1): 2 \leqq i \leqq n-1\} = \{e_2, e_3, \ldots, e_{n-1}\}$ is an irreducible subset of $M$. We can consider the two rows $D \backslash E$ for themselves (there is no transition back to $E$) and $D_1$ and $D_2$ are subsets of $D \backslash E$.

Recall $A(i,j) = \{c_k: i \leqq k \leqq j\}$ and $B(i,j) = \{d_k: i \leqq k \leqq j\}$. In the following we shall identify $c_k$ with its index $k$ and similar for $d_k$. It will be always clear, whether $k$ is $c_k$ or $d_k$. We have $D_1 = A(1, l-1) \cup B(1, m-1)$. Set $M_1 = M/D_1$.

**Lemma 3.** $r(M_1) \geqq \lambda_0$ *or* $=1$. $r(M_1) = \lambda_0$ *iff there is an integer* $t$ *with* $(t+2)m > l > (t+1)m$ *and, setting* $q = l-(t+1)m$ *and* $p = (t+2)m - l$ *(i.e.* $p+q=m$, $(p+q)(t+1)+q=l$*), iff there are* $i, j$ *such that* $r_1 + \ldots + r_{i-1} = p-1$, $s_1 + \ldots + s_{j-1} = q-1$ *and* $r_i = r_{i+t+1} = q$, $r_{i+1} = r_{i+2} = \ldots = r_{i+t} = p+q$, $s_j = p$ *and* $r(M_1/A(1,p-1) \cup B(1,q-1)) = 1$ *(or the same with the roles of* $m$ *and* $l$, $r_k$ *and* $s_k$ *exchanged).*

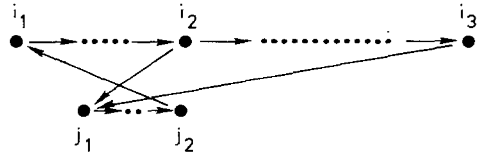
*Proof.* Let $i_1 \in A(1, l-1)$ be the smallest point, where an arrow from $B(1, m-1)$ arrives, and $j_1 \in B(1, m-1)$ the smallest point, where an arrow from $A(1, l-1)$ arrives. Let $j_2 \in B(1, m-1)$ and $i_2 \in A(1, l-1)$ be the smallest points, where these arrows start to $i_1$ and $j_1$ respectively. We can take away $A(1, i_1-1)$ and $B(1, j_1-1)$, because we can never return to these points. If $i_2 < i_1$ ($j_2 < j_1$) we have taken away $i_2$ ($j_2$) and also the arrow starting there. We take now the earliest point in $B(j_1, m-1)$ ($A(i_1, l-1)$), where an arrow from $A(1, l-1)$ ($B(1, m-1)$) arrives, and call it again $j_1$ ($i_1$). We get also a new $i_2$ ($j_2$). If $i_1 \leqq i_2$ and $j_1 \leqq j_2$ and there is no other arrow arriving in $A(i_1, i_2)$ from $B(1, m-1)$ or in $B(j_1, j_2)$ from $A(1, m-1)$, then $A(i_1, i_2) \cup B(j_1, j_2)$ is an irreducible subset of $D_1$ with spectral radius 1 (the corresponding shift space is a periodic orbit). Hence we can take away this set. Now take for $i_1, j_1$ the earliest points in $A(i_2+1, l-1)$ and $B(j_2+1, m-1)$ respectively, where arrows arrive as above and repeat this procedure. There are two cases. We reach the points $c_{l-1}$ and $d_{m-1}$ without having found another arrow from $B(1, m-1)$ into $A(i_1, i_2)$ or from $A(1, l-1)$ into $B(j_1, j_2)$. Then $r(M_1) = 1$. Or we find $i_1, j_1, i_2, j_2$ (we choose $i_1, j_1$ minimal and for these we take the smallest $i_2, j_2$, where arrows go to $j_1, i_1$ respectively) and a $j_3 \in B(j_1, j_2)$ ($j_3 \in A(i_1, i_2)$), we choose again minimal, and an $i_3 \in A(i_1, l-1)$ ($i_3 \in B(j_1, m-1)$) such that there is an arrow from $i_3$ to $j_3$. We do not consider the case in brackets. It is similar and corresponds to the result with the roles of $m$ and $l$, $r_k$ and $s_k$ exchanged. Furthermore $M/A(1, i_1-1) \cup B(1, j_1-1)$ has spectral radius 1.

We prove $j_3 = j_1$. Suppose $j_3 > j_1$. $j_3$ is the end point of an arrow, hence $j_3 = r_k = s_1 + \ldots + s_{P(k)} + 1$ for some $k$ (cf. (1.2) and Lemma 1). Hence at $j_3 - 1 = s_1 + \ldots + s_{P(k)}$ there begins an arrow ending at $i_4 \in A(1, l-1)$ say. $i_4 \geqq i_1$, otherwise we have a contradiction to the minimality of $i_1$. If $i_4 = i_1$ we have a contradiction to the minimality of $j_2$. Hence $i_4 > i_1$. As above there is an arrow from $i_4 - 1$ to $B(1, m-1)$. If this point is not $i_2$, we get a $j_3' \in B(j_1, j_3)$, $j_3' < j_3$, as end point of the arrow and this contradicts the minimality of $j_3$. We have the picture




Set $P = a_0 a_1 \ldots a_{i_1-2}$ and $Q = b_0 b_1 \ldots b_{j_1-2}$. We have no arrow from $A(i_1, i_2-1)$ to $B(1, m-1)$, because this is either a contradiction to the fact that no arrow goes from $A(i_1, i_2)$ to $B(1, j_1-1)$ or to the minimality of $i_2$ or to the minimality of $j_3$. Hence $a_{i_1-1} \ldots a_{i_2-1}$ is the initial segment $u\, b_0 \ldots b_{j_1-2} = u\,Q$ of $\underline{b}^u$ for some $u$ with $1 \leqq u \leqq n-1$ ($u=1$ in case (1.3)) (cf. (1.2)). Using again (1.2) and the fact that no arrow goes to $E$ we get that $\underline{a} = P u Q (v-1) \ldots$ and $\underline{b} = Q v P u Q \ldots v P u Q v P(u+1) \ldots$ for some $v$ with $2 \leqq v \leqq n$ ($v=2$ in case (1.3)). Hence $\sigma^{j_2 - i_1 - j_1 + 1}\, \underline{b} = Q\, v\, P(u+1) \ldots > \underline{b}$, a contradiction to $\underline{b} \in \Sigma_f^+$ (cf. §1).

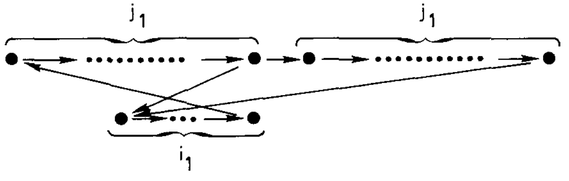
Therefore $j_3 = j_1$. We get the picture




$2j_1 + w$ points

$i_1$ points (2.5)

The characteristic equation for (2.5) is $\lambda^{2j_1+i_1+w}-\lambda^{j_1+w}-1=0$. As (2.5) is a subgraph of $M_1$, $j_1+i_1-1=j_2\leqq m-1$ and $i_1+2j_1+w-1=i_3\leqq l-1$. Hence the largest solution of the above equation is greater than or equal to $\lambda_0$ and hence $r(M_1)\geqq\lambda_0$. Equality holds, iff $i_1+j_1=m$, $i_1+2j_1+w=l$ and there are no more arrows than indicated in (2.5). If $w=0$ we have the picture




If $w>0$ there are arrows going from $A(i_2+1,\, i_3-j_1)$ to $B(1,\infty)$, which must go to $D_2$ in case of equality. These arrows must end at $d_m\in D_2$, because, if one of them ends at a later point, it can end earliest at $d_{l+m}$ (from $d_m$ to $d_{l+m-1}$ there is an initial segment of $\underline{a}$) and hence $A(i_2+1,\, i_3-j_1)$ must contain a block of length $l+m$ (cf. (1.2)), which is impossible ($i_3<l$). Hence $A(i_2+1, i_3-j_1)$ consists of $t$ initial segments of $\underline{b}$, each of length $m$, for some $t$, i.e. $w=tm$.

Set $p=i_1$ and $q=j_1$ (we get $p+q=m$ and $(p+q)(t+1)+q=l$). The properties about $r_k$ and $s_k$ in case of equality are easily deduced.

From Lemmas 2 and 3 we get

**Theorem 1.** *In the case $n=2$, $\Sigma_f^+$ has more than on maximal measure, iff*

$$\underline{a}=P1Q\underbrace{1X1X\ldots1X}_{t\,\text{times}}1Q1X1X1X\ldots \quad \textit{and}$$

$$\underline{b}=Q2P2Y2Y2Y\ldots$$

*or vice versa, where $X=Q2P$ and $Y=P1Q1X1X\ldots1X1Q$ and $P$ and $Q$ are blocks such that $M/A(1,p-1)\cup B(1,q-1)$ has spectral radius* 1 *($p=$length of $2P$, $q=$length of $1Q$). In this case there are exactly two ergodic maximal measures.*

*Proof.* This follows immediately from the lemmas. Choose suitable blocks $P$ and $Q$ and add initial segments of $1\underline{b}$ and $2\underline{a}$ respectively, according to the equations for $r_k$ and $s_k$ in the case of equality in Lemmas 2 and 3 to get $\underline{a}$ and $\underline{b}$. In no other case it can happen that $r(M_1)=r(M_2)$.

In this case of more than one maximal measure one sees that $M_2$ is irreducible (cf. the proof of Lemma 2) and that $M_1$ consists of irreducible parts, all with spectral radius 1, except one, which has spectral radius equal to $r(M_2)=r(M)$ (cf. the proof of Lemma 3). Now apply the theorem in §1. There is at least one maximal measure on $\Sigma_{M_1}$ and on $\Sigma_{M_2}$, because they are shift spaces with finite alphabet, hence expansive.

**Corollary 1.** *If in the case $n=2$ the graph of $f$ is symmetric with respect to $(\frac{1}{2}/\frac{1}{2})$, then $f$ has unique maximal measure.*

*Proof.* In this case we have $a_i=1$, if $b_i=2$ and $a_i=2$, if $b_i=1$. This implies $r_k=s_k$ for all $k$ and hence $l=m$ in the construction of $D_1$ and $D_2$ at the beginning of §2. The result follows from Theorem 1.

**Corollary 2.** *If in the case $n=2$ $f/J_1$ and $f/J_2$ are linear, then $f$ has unique maximal measure.*

*Proof.* If not, we have by Theorem 1 that $\underline{a}=P1Q1X\ldots 1X1Q1X1X\ldots$ and $\underline{b}=Q2P2Y2Y\ldots$ Suppose $f/J_1$ has slope $\lambda$ and $f/J_2$ has slope $\mu$. Applying $f$ to a subinterval of $J_1$ or $J_2$ means to multiply its length by $\lambda$ or $\mu$ respectively. The isomorphism $\varphi$ between $(I,f)$ and $(\Sigma_f^+,\sigma)$ is order preserving, hence intervals in $\Sigma_f^+$ (with respect to the lexicographic ordering) correspond to intervals in $I$. If $[\underline{x},\underline{y}]$ is an interval in $\Sigma_f^+$ and $x_0=y_0$ (i.e. $\varphi^{-1}([\underline{x},\underline{y}])\subset J_1$ or $J_2$), then denoting the length of $\varphi^{-1}(([\underline{x},\underline{y}])\subset I$ by $|[\underline{x},\underline{y}]|$ we have $|\sigma[\underline{x},\underline{y}]|=\lambda|[\underline{x},\underline{y}]|$, if $x_0=y_0=1$ and $=\mu|[\underline{x},\underline{y}]|$, if $x_0=y_0=2$. Set $\alpha=\lambda^u\mu^v$, where $u$ is the number of 1 and $v$ the number of 2 in $P$ and let $\beta$ be the same number for $Q$. We consider the following intervals

$$R=[\sigma^{p+2q-1+t(p+q)}\underline{a},1\underline{b}]=[1X1X\ldots,1X2Y2Y\ldots]$$

and

$$S=[2\underline{a},\sigma^{p+q-1}\underline{b}]=[2Y1X1X\ldots,2Y2Y\ldots],$$

where $p=$ length of $2P$ and $q=$ length of $1Q$. Set $r=|R|$ and $s=|S|$.

$$\sigma^{p+q}R=R\cup S,\qquad \text{i.e. } r(\lambda\mu\alpha\beta)=r+s$$

$$\sigma^{p+2q+t(p+q)}S=R\cup S,\qquad \text{i.e. } s(\lambda\mu\alpha\beta)^{t+1}\lambda\beta=r+s.$$

From this we get because of $r\neq 0$ and $s\neq 0$.

$$(\lambda\mu\alpha\beta)^{t+1}\lambda\beta-(\lambda\mu\alpha\beta)^t\lambda\beta-1=0. \tag{2.6}$$

Now consider the intervals

$$\begin{aligned}F&=[\sigma^{3p+3q-1}\underline{b},\sigma^{p+q-1}\underline{a}]\\&=[\underbrace{1X\ldots 1X}_{t-1}1Q2Y2Y\ldots,\underbrace{1X1X\ldots 1X}_{t}1Q1X1X\ldots]\\G&=[\sigma^{2p+2q-1}\underline{b},\sigma^{p+2q-1+t(p+q)}\underline{a}]=[\underbrace{1X1X\ldots 1X}_{t}1Q2Y2Y\ldots,1X1X\ldots]\\H&=[\sigma^{p+q-1}\underline{a},\sigma^{2p+2q-1}\underline{b}]\\&=[\underbrace{1X1X\ldots 1X}_{t}1Q1X1X\ldots,\underbrace{1X1X\ldots 1X}_{t}1Q2Y2Y\ldots]\end{aligned}$$

and set again $f=|F|$, $g=|G|$ and $h=|H|$.

$$\sigma^{q+t(p+q)}F=G,\qquad \text{i.e. } f(\lambda\mu\alpha\beta)^t\lambda\beta=g$$

$$\sigma^{p+q}G=F\cup H\cup G,\qquad \text{i.e. } g(\lambda\mu\alpha\beta)=f+h+g.$$

From this we get

$$g[(\lambda\mu\alpha\beta)^{t+1}\lambda\beta-(\lambda\mu\alpha\beta)^t-1]=h(\lambda\mu\alpha\beta)^t\lambda\beta$$

and together with (2.6) we have $h=0$, a contradiction, because $\varphi^{-1}(H)$ has distinct end points. The corollary is proved.

**Theorem 2.** *The transformation* $x \mapsto \beta x + \alpha (\bmod 1)$, $\beta > 1$ *and* $0 \leqq \alpha < 1$, *has unique maximal measure.*

*Proof.* The number $n$ of intervals $J_i$ is so that $\alpha + \beta \leqq n < \alpha + \beta + 1$. $n = 2$ is a special case of Corollary 2. For $n \geqq 3$ set $E = \{(A, i, 1) = (B, i, 1)\colon\ 2 \leqq i \leqq n-1\} = \{e_2, \ldots, e_{n-1}\}$. We have transition from every element of $E$ to every element of $E$. Hence for $n \geqq 4$ the irreducible subset of $D$ containing $E$ has spectral radius $\geqq n-2 \geqq 2$ and the remaining part of $D$ has spectral radius $<2$ (if it has spectral radius equal to 2 one computes easily that $n=4$, $\underline{a}$ has to be $13333\ldots$ and $\underline{b}$ has to be $42222\ldots$, i.e. $\beta = 2$, $\alpha = 1$, a case which is not allowed).

The case $n = 3$ remains. If $E = \{e_2\}$ is an irreducible subset of $D$, i.e. the only arrow ending at $E$ is that starting at $E$, then $M/E$ has spectral radius 1 and so it suffices to consider $D \setminus E$. We can apply Lemmas 2 and 3. If there should be more than one irreducible subset of $D \setminus E$ with the same spectral radius the lemmas imply

$$\underline{a} = P u_1 Q u_2 X \ldots u_{t+1} X u_{t+2} Q u_{t+3} X u_{t+4} X \ldots$$

and

$$\underline{b} = Q v_1 P v_2 Y v_3 Y \ldots,$$

where $v_2 - u_1 = 1$, $v_i - u_{t+3} = 1$ for $i \geqq 3$, $u_2 - v_1 = u_{t+3} - v_1 = -1$ and $u_i - v_2 = -1$ for $i \geqq 3$ and $i \neq t+3$. Otherwise there would be transition back to $E$. Set $u_1 = u$ and $v_1 = v$. Then $v_2 = u+1$, $v_i = v$ for $i \geqq 3$, $u_2 = u_{t+3} = v-1$, $u_i = u$ for $i \geqq 3$ and $i \neq t+3$. We have four cases $(u, v) = (1,2)$, $(2,3)$, $(1,3)$, $(2,2)$ ($u = 3$ and $v = 1$ are not possible, because there is the end of an initial segment). In the first two cases one can proceed exactly as in Corollary 2. The other two cases are

$$\underline{a} = P1Q2X1X \ldots 1X1Q2X1X1X \ldots, \qquad \underline{b} = Q3P2Y3Y3Y \ldots$$

and

$$\underline{a} = P2Q1X2X \ldots 2X2Q1X2X2X \ldots, \qquad \underline{b} = Q2P3Y2Y2Y \ldots.$$

Set $R = [\sigma^{(p+q)(t+2)+q-1}\underline{a}, 1\underline{b}]$, $S = [2\underline{a}, \sigma^{p+q-1}\underline{b}]$, $U = [\sigma^{(p+q)(t+1)+q-1}\underline{a}, 2\underline{b}]$, $V = [3\underline{a}, \sigma^{(p+q)(t+2)+q-1}\underline{b}]$ in the first case and $R = [\sigma^{(p+q)(t+2)+q-1}\underline{a}, 2\underline{b}]$, $S = [3\underline{a}, \sigma^{p+q-1}\underline{b}]$, $U = [\sigma^{(p+q)(t+1)+q-1}\underline{a}, 1\underline{b}]$, $V = [2\underline{a}, \sigma^{(p+q)(t+2)+q-1}\underline{b}]$ in the second case. Then

$$\sigma^{p+q} R = R \cup S, \quad \text{i.e. } \beta^{p+q} r = r + s,$$

$$\sigma^{(p+q)(t+1)+q} S = U \cup V, \quad \text{i.e. } \beta^{(p+q)(t+1)+q} s = u + v,$$

$$\sigma^{p+q} U = R \cup S, \quad \text{i.e. } \beta^{p+q} u = r + s,$$

$$\sigma^{(p+q)(t+1)+q} V = U \cup V, \quad \text{i.e. } \beta^{(p+q)(t+1)+q} v = u + v.$$

We get $\beta^{(p+q)(t+1)+q} - \beta^{(p+q)t+q} - 1 = 0$.

Now take $F = [\sigma^{3p+3q}\underline{b}, \sigma^{p+q}\underline{a}]$, $G = [\sigma^{2p+2q}\underline{b}, \sigma^{(p+q)(t+1)+q}\underline{a}]$, $H = [\sigma^{p+q}\underline{a}, \sigma^{2p+2q}\underline{b}]$ in both cases. We have

$$\sigma^{(p+q)t+q} F = G, \quad \text{i.e. } \beta^{(p+q)t+q} f = g$$

$$\sigma^{p+q} G = F \cup H \cup G, \quad \text{i.e. } \beta^{p+q} g = f + h + g.$$

We get $g(\beta^{(p+q)(t+1)+q}-\beta^{(p+q)t+q}-1)=h\,\beta^{(p+q)t+q}$ and hence $h=0$, a contradiction as in Corollary 2.

Therefore there must be an arrow from $D\setminus E$ to $E$, say from $(A,1,l-1)=c_{l-1}$ where we choose $l$ minimal. Then the irreducible subset $D_1$ of $D$ containing $E\cup A(1,l-1)$ has spectral radius greater than or equal to the largest solution of $\lambda^l-\lambda^{l-1}-1=0$. By Lemma 2 the spectral radius of $D\setminus D_1$ is less or equal to this number and equality holds only, if $D_1=E\cup A(1,l-1)$, $r_k=1$, if $r_1+\ldots+r_k\geqq l$ and $s_k=l$ for all $k$. This implies $\sigma^l\underline{a}=222\ldots$ and $\underline{b}=3\,a_0\ldots a_{l-2}(a_{l-1}+1)\,a_0\ldots a_{l-2}\cdot(a_{l-1}+1)\,a_0\ldots$. Because of the minimality of $l$ $a_0\ldots a_{l-1}$ has to be $122\ldots 21$. The last 1 is because of the arrow to $E$, the 2's are because a 1 would imply an earlier arrow to $E$ and a 3 contradicts $\sigma^{l-1}\underline{a}\geqq\underline{a}$. Therefore

$$\underline{a}=1\underbrace{2\ldots 2}_{l-2}1222\ldots \quad\text{and}\quad \underline{b}=31\underbrace{2\ldots 2}_{l-1}1\underbrace{2\ldots 2}_{l-1}1\underbrace{2\ldots 2}_{l-1}1\ldots$$

We proceed again as in Corollary 2. Take

$$R=[\sigma^{l-1}\underline{a},1\underline{b}],\qquad S=[2\underline{a},\sigma^l\underline{b}],\qquad U=[\sigma^l\underline{a},2\underline{b}],\qquad V=[3\underline{a},\underline{b}].$$

We have

$$\sigma R=U\cup V,\quad \sigma^l S=R\cup S,\ \sigma U=U\cup V\ \text{ and }\ \sigma^l V=R\cup S,\qquad \text{i.e.}$$

$$\beta r=u+v,\quad \beta^l s=r+s,\quad \beta u=u+v\ \text{ and }\ \beta^l v=r+s.$$

From this it follows that $\beta^l-\beta^{l-1}-1=0$. Now take $F=[\sigma\underline{b},\sigma^{l-1}\underline{a}]$, $G=[\sigma^l\underline{b},\sigma^l\underline{a}]$. We have $\sigma G=[\sigma\underline{b},\sigma^l\underline{a}]$. $\sigma^{l-1}F=G$, $G\subset\sigma G$ and $\sigma G\setminus G=F\cup R\cup S$, i.e. $\beta^{l-1}f=g$ and $g(\beta-1)=f+r+s$ or $f(\beta^l-\beta^{l-1}-1)=r+s$. Hence $r+s=0$, a contradiction. The theorem is proved.

An example of a function in the case $n=2$ with 2 ergodic maximal measures can be found in part II of [2]. The graph of a function $f$ is constructed there such that $\underline{a}=111212121\ldots$ and $\underline{b}=2211211211\ldots$, the simplest case of Theorem 1. It is not difficult, to find for every $\varepsilon>0$ an $f$ with these $\underline{a}$ and $\underline{b}$ such that there is a subinterval $K$ of $I$, $f/K$ is linear with slope $\lambda$, $f/I\setminus K$ is linear with slope $\mu$ and $|\lambda-\mu|<\varepsilon$. One sees that the transformation $x\mapsto\beta x+\alpha(\mathrm{mod}\,1)$ is not far away from having more than one maximal measure.

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