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On Fatou's Lemma in Several Dimensions

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The existence proof of equilibria for economies with a measure space of economic agents in [7] is based on a lemma which may be considered as Fatou's Lemma in several dimensions. The lemma turned out to be very useful in mathematical economics, e.g., [6], [8] and [10]. The lemma has first been proved by Schmeidler [9]. We give here a different proof which implies an additional result.

Theorem. *Let $(\Omega, \mathscr{F}, \nu)$ be a positive measure space and (f_n) a sequence of integrable functions of Ω into R^l_+. Suppose $\lim_n \int f_n \, d\nu$ exists. Then there is an integrable function f of Ω into R^l_+ such that a.e. in Ω, $f(\omega)$ is adherent in R^l to the sequence $(f_n(\omega))_n$ and*

$$\int f \, d\nu \leqq \lim_n \int f_n \, d\nu.$$

Proof. First we assume that $(\Omega, \mathscr{F}, \nu)$ is an *atomless* and *complete* probability space; the general situation is easily reduced to this case as we shall show at the end of the proof.

Notation. $L^l_p(\Omega, \mathscr{F}, \nu)$ (L^l_p for short) denotes the l-fold product of the linear space $L_p(\Omega, \mathscr{F}, \nu)$, $p=1, \infty$ (Dunford-Schwartz [4, IV.8]), and similarly, $ba^l(\Omega, \mathscr{F}, \nu)$ (ba^l for short) denotes the l-fold product of the linear space $ba(\Omega, \mathscr{F}, \nu)$ (Dunford-Schwartz [4, IV.8, p. 296]).

It is well known that the dual of the Banach space L^l_∞, where every factor space is endowed with the sup-norm topology, is isomorphic to ba^l. The product topology on ba^l, where each factor space ba is endowed with the weak topology $\sigma(ba, L_\infty)$, is denoted by $\sigma^l(ba, L_\infty)$. Finally, the product topology on L^l_1, where each factor space L_1 is endowed with the weak topology $\sigma(L_1, L_\infty)$, is denoted by $\sigma^l(L_1, L_\infty)$.

$$\text{Let} \quad \mu_n(E) = \int_E f_n \, d\nu, \qquad E \in \mathscr{F} \qquad n=1, \ldots.$$

Clearly, the R^l_+-valued measure μ_n belongs to ba^l. Since the sequence $(\mu_n(\Omega))_n$ is bounded, the Theorem of Alaoglu (Dunford-Schwartz [4, V.4, Th. 2, p. 424]) implies that the subset $\{\mu_1, \mu_2, \ldots\}$ in ba^l is relative $\sigma^l(ba, L_\infty)$-compact. Consequently, the sequence (μ_n) has a $\sigma^l(ba, L_\infty)$-adherent point $\mu \in ba^l$. Clearly, $\mu \geqq 0$ and

$$\mu(\Omega) = \lim_n \int f_n \, d\nu. \tag{1}$$

According to a well-known result (Yoshida-Hewitt [11, Th. 1.23, p. 53]) the positive measure $\mu \in ba^l$ can be written in the form

$$\mu = \mu_c + \mu_p, \qquad \mu_c, \mu_p \in ba^l, \qquad \mu_c, \mu_p \geqq 0,$$

where μ_c is countably additive and μ_p purely finitely additive. Clearly, each coordinate of the measure μ_c is dominated by the measure ν. Thus, let $g\colon \Omega \to R^l$ be a Radon-Nikodým derivative of the countably additive part μ_c with respect to ν. We have by (1)

$$\int g\, d\nu = \mu_c(\Omega) \leqq \mu(\Omega) = \lim_n \int f_n\, d\nu .$$

Thus,

$$\int g\, d\nu \leqq \lim_n \int f_n\, d\nu . \tag{2}$$

Since $\mu_p(\Omega)$ may be greater than zero, g is, in general, not $\sigma^l(L_1, L_\infty)$-adherent to the sequence (f_n). However, according to Yoshida-Hewitt [11, Th. 1.22, p. 52], there exists a countable partition $(B_i)_{i\in\mathbf{N}}$ of Ω such that $\mu_p(B_i)=0$ for every $i\in\mathbf{N}$. This clearly implies that in $L_1^l(B_i, \mathscr{F}\cap B_i, \nu|B_i)$ the restriction $g|B_i$ is $\sigma^l(L_1, L_\infty)$-adherent to the restricted sequence $(f_n|B_i)$ for every $i\in\mathbf{N}$.

Now we fix a set B_i and consider for every $n=1, \ldots$ the set

$$A_n = \{f_n|B_i,\ f_{n+1}|\,B_i, \ldots\}.$$

The restriction $g|B_i$ is $\sigma^l(L_1, L_\infty)$-adherent to every A_n and hence to every convex hull co A_n of A_n. But since a strongly closed and convex set is also weakly closed (Dunford-Schwartz [4, V.3.11, Th. 13, p. 422]) there exists a sequence (g_n), where $g_n\in \operatorname{co} A_n$, converging strongly to $g|B_i$. Then there exists a subsequence of (g_n) converging a.e. to $g|B_i$ (Dunford-Schwartz [4, 3.6(i), Th. 3, p. 122 and 6.13(a), Cor. 3, p. 150]). Hence, without loss of generality, we can assume that $\lim_n g_n(\omega) = g(\omega)$ a.e. in B_i. Now we shall prove

$$\text{a.e. in } \Omega, \qquad g(\omega) \in \operatorname{co} \operatorname{adh}(f_n) + R^l_+ . \tag{3}$$

Since $g_n(\omega)\in \operatorname{co}\{f_n(\omega), f_{n+1}(\omega), \ldots\}$ we have according to a result of Caratheordory (e.g., Eggleston [5, p. 34])

$$g_n(\omega) = \sum_{i=0}^{l} \delta_n^i\, y_n^i,$$

where $\delta_n^i \geqq 0$, $\sum_{i=0}^{l} \delta_n^i = 1$ and $\{y_n^0, \ldots, y_n^l\} \subset \{f_n(\omega), f_{n+1}(\omega), \ldots\}$.

In order to prove (3) we can assume without loss of generality that the sequences $(\delta_n^i)_n$ $(i=0, \ldots, l)$ are convergent. Thus, let $\lim_n \delta_n^i = \delta^i$. Clearly, we have $\delta^i \geqq 0$ and $\sum_{i=0}^{l} \delta^i = 1$.

Since $\delta_n^i \geqq 0$, $y_n^i \geqq 0$ and $\lim_n g_n(\omega) = g(\omega)$, the sequences $(\delta_n^i\, y_n^i)_{n=1,\ldots}$ $(i=0, \ldots, l)$ are bounded and hence, we can again assume without loss of generality that $\lim_n \delta_n^i\, y_n^i$ exists $(i=0, \ldots, l)$. Thus, $\delta^i > 0$ implies $\lim_n y_n^i$ exists. Now,

$$g(\omega) = \lim_n g_n(\omega) = \lim_n \sum_{i=0}^{l} \delta_n^i\, y_n^i = \sum_{i=0}^{l} \lim_n \delta_n^i\, y_n^i \geqq \sum_{\substack{i=0\\ \delta_i > 0}}^{l} \delta^i \lim_n y_n^i .$$

According to the definition of y_n^i we have $\lim_n y_n^i \in \operatorname{adh}(f_n(\omega))$. Hence,

$$g(\omega) \geqq \sum_{\substack{i=0 \\ \delta_i > 0}}^{l} \delta^i \lim_n y_n^i \in \operatorname{co} \operatorname{adh}(f_n(\omega)),$$

which proves (3).

The function g which has properties (2) and (3), will now be altered to a function which has the required properties. It is not difficult to show that the set $G = \{(\omega, x) \in \Omega \times R^l | x \in \operatorname{co} \operatorname{adh}(f_n(\omega))\}$ belongs to the product σ-algebra $\mathscr{F} \otimes \mathscr{B}(R^l)$, where $\mathscr{B}(R^l)$ denotes the Borel sets of R^l.

In fact, the following results are known (see Debreu [3, (5.8) and (5.10)]): (i) if H belongs to $\mathscr{F} \otimes \mathscr{B}(R^l)$ and v is a vector in R^l, then the functions S_v of Ω into R, defined by $S_v(\omega) = \sup\{v \cdot x | (\omega, x) \in H\}$, is measurable; (ii) if the subset H in $\Omega \times R^l$ is such that the sections $H(\omega) = \{x \in R^l | (\omega, x) \in H\}$ are compact and convex, then the measurability of the functions S_v for every $v \in R^l$ implies that H belongs to $\mathscr{F} \otimes \mathscr{B}(R^l)$.

Consequently, if H belongs to $\mathscr{F} \otimes \mathscr{B}(R^l)$ and if all $H(\omega)$ are compact, then the set $\{(\omega, x) \in \operatorname{co} H(\omega)\}$ belongs to $\mathscr{F} \otimes \mathscr{B}(R^l)$. Since the set $\operatorname{adh}(f_n(\omega))$ is closed and since the set $\{(\omega, x) \in \Omega \times R^l | x \in \operatorname{adh}(f_n(\omega))\}$ belongs to $\mathscr{F} \otimes \mathscr{B}(R^l)$ it follows the measurability of the set G.

Consider the set $C = G \cap \{(\omega, x) \in \Omega \times R^l | x \leqq g(\omega)\}$. Clearly, the set C belongs to $\mathscr{F} \otimes \mathscr{B}(R^l)$ and (3) implies $\nu(\operatorname{proj}_\Omega C) = 1$. According to a Measurable Choice Theorem (see Aumann [2]), there exists a measurable function f' of Ω into R^l such that a.e. in Ω, $(\omega, f'(\omega)) \in C$, i.e.,

$$\text{a.e. in } \Omega, \quad f'(\omega) \in \operatorname{co} \operatorname{adh}(f_n(\omega)) \quad \text{and} \quad \int f' \, d\nu \leqq \lim_n \int f_n \, d\nu. \tag{4}$$

Finally, since the measure space $(\Omega, \mathscr{F}, \nu)$ is atomless, we have, according to Aumann [1, Th. 3, p. 2],

$$\int \operatorname{adh} f_n \, d\nu = \int \operatorname{co} \operatorname{adh} f_n \, d\nu,$$

where the integral is defined as the set of the integrals of all integrable selections. Hence, there exists an integrable function f of Ω into R_+^l such that

$$\text{a.e. in } \Omega, \quad f(\omega) \in \operatorname{adh}(f_n(\omega)) \quad \text{and} \quad \int f \, d\nu \leqq \lim_n \int f_n \, d\nu.$$

Finally, we prove the theorem for an arbitrary positive measure space $(\Omega, \mathscr{F}, \nu)$. Since the set $\{\omega \in \Omega | \text{there is } n \in N \text{ such that } f_n(\omega) \neq 0\}$ has a σ-finite measure, we can assume that $(\Omega, \mathscr{F}, \nu)$ is σ-finite. Then there exists a countable measurable partition of Ω, $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, where $\nu(\Omega_k) < \infty$ and Ω_k is either an atom or atomless.

Since $\lim_n \int_\Omega f_n \, d\nu$ is assumed to exist and since $f_n \geqq 0$ there is a vector $b \in R^l$ such that for every $k = 1, \ldots$ and every $n = 1, \ldots$ we have

$$\int_{\Omega_k} f_n \, d\nu \leqq b.$$

Without loss of generality we can therefore assume

$$\lim_n \int_{\Omega_k} f_n \, dv \qquad \text{exists for every } k=1, \ldots.$$

For every Ω_k the theorem is either trivial or has been proved above. Therefore, let f^k be a function of Ω_k into R^l such that $f^k(\omega) \in \operatorname{adh}(f_n(\omega))$ a.e. in Ω_k and $\int_{\Omega_k} f^k \, dv \leqq \lim_n \int_{\Omega_k} f_n \, dv$. Define $f = \sum_{k=1}^{\infty} f^k$, when we have set $f^k(\omega)=0$ for $\omega \notin \Omega_k$. Since we have a.e. in $\Omega, f(\omega) \in \operatorname{adh}(f_n(\omega))$, according to the monotone convergence theorem (Levi) we have

$$\int_{\Omega} f \, dv = \sum_{k=1}^{\infty} \int_{\Omega} f^k \, dv$$

and hence,

$$\int_{\Omega} f \, dv \leqq \sum_{k=1}^{\infty} \lim_n \int_{\Omega_k} f_n \, dv.$$

Since

$$\sum_{k=1}^{q} \lim_n \int_{\Omega_k} f_n \, dv = \lim \sum_{k=1}^{q} \int_{\Omega_k} f_n \, dv \leqq \lim_n \int_{\Omega} f_n \, dv$$

we have

$$\int_{\Omega} f \, dv \leqq \lim \int_{\Omega} f_n \, dv.$$

Q.E.D.

In the situation where the theorem has been applied one could show that for every f which has the properties stated in the theorem one has actually $\int f \, dv = \lim_n \int f_n \, dv$. This motivates the following.

Corollary. *Assume one has* $\int f \, dv = \lim_n \int f \, dv$ *whenever* f *is a function with the properties stated in the theorem. Then the sequence* (f_n) *is relative* $\sigma^l(L_1, L_\infty)$-*compact and every* $\sigma^l(L_1, L_\infty)$-*adherent point* g *has the property: a.e. in* Ω, $g(\omega) \in \operatorname{co} \operatorname{adh}(f_n(\omega))$.

Proof. We show that the closure of $\{f_1 \cdot v, f_2 \cdot v, \ldots,\}$ in $(ba^l, \sigma^l(ba, L_\infty))$ consists of σ-additive measures. Let μ be an adherent point of the set $\{f_1 \cdot v, \ldots\}$ and let g, f' and f be as in the proof. Hence, by (1)

$$0 \leqq \mu_p(\Omega) = \mu(\Omega) - \mu_c(\Omega) = \lim_n \int f_n \, dv - \int g \, dv.$$

The additional assumption in the corollary implies

$$\lim_n \int f_n \, dv = \int f \, dv = \int f' \, dv \leqq \int g \, dv \qquad \text{and therefore,}$$

we obtain $\mu_p = 0$. This proves that the sequence (f_n) is relative $\sigma^l(L_1, L_\infty)$-compact.

Finally, since $f' \leqq g$ and $\int f' \, dv = \int g \, dv$ it follows that a.e. in $\Omega, f'(\omega) = g(\omega)$. According to (4), g has the desired properties. Q.E.D.

Remark. If the sequence (f_n) is majorized by an integrable function, then there is a function f such that a.e. in $\Omega, f(\omega) \in \operatorname{adh}(f_n(\omega))$ and $\int f \, dv = \lim_n \int f_n \, dv$. Does this still hold when (f_n) is uniformly integrable?

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