Z. Wahrscheinlichkeitstheorie verw. Geb. 17, 147–150 (1971)

Stopping Rules for S_n/n, and the Class $L \log L$

Burgess Davis

1. Introduction

If $(f_1, f_2, \ldots) = (f_n)$ is a stochastic process a stopping rule for (f_n) will be defined to be a non-defective positive integer valued random variable t such that for every positive integer $k\, \{t=k\} \in \sigma(f_1, \ldots, f_k)$, the σ-field generated by $f_1, \ldots, f_k$. Here we will be concerned with stopping rules for the sequence (S_n/n) where $S_n = X_1 + \cdots + X_n$, $X_1, X_2, \ldots$ independent identically distributed random variables with finite expectation, investigating the possibility of the existence of stopping rules t for (S_n/n) such that $E(S_t/t) = \infty$. It is proved (Theorem 1) that such stopping rules exist if and only if $E(X_1 \log^+ X_1) = \infty$, where $\log^+ a = \log(a)$ if $a > 1$, 0 if $a \leqq 1$. This immediately implies the result of Burkholder in [1] that if $E(X_1 \log^+ X_1) = \infty$ then $E(\sup(S_n/n)) = \infty$. Similar results are proved if (X_n/n) is looked at in place of (S_n/n).

Since $E|S_n/n| \leqq E|X_1|$ for all n, recent results (see, for example, [4]) on stopping a stochastic process (f_n) to get $E|f_t| = \infty$ are not applicable here, since they require that $\sup E|f_n| = \infty$. The work by Chow, Robbins and others on stopping S_n/n to maximize $E(S_t/t)$ has dealt with essentially different (although obviously related) questions than this paper.

From now on "t is a stopping rule" will mean that t is a stopping rule for the particular process under discussion. Note that since $\sigma(S_1, S_2/2, \ldots, S_n/n) = \sigma(X_1, X_2/2, \ldots, X_n/n) = \sigma(X_1, \ldots, X_n)$, the stopping rules for the processes (S_n/n), (X_n/n) and (X_n) are the same.

2. Some Inequalities Concerning S_n/n

Let $f(n) = \sum_{k=1}^{n-1} (n-k)/k$. By approximating $f(n)$ with an integral it is seen that $\lim_{n\to\infty} f(n)/((n+1)\log(n+1)) = 1$. Thus there is a positive number which will be called ε such that

$$f(n) > \varepsilon (n+1) \log(n+1), \qquad n = 2, 3, \ldots.$$

In this section X will be a random variable with finite expectation and $X_1, X_2, \ldots$ will be independent random variables each with the same distribution as X. B will stand for $\max(1, E|X|)$, and if Y is a random variable Y^+ will designate $\max(Y, 0)$.

By a well known inequality, if $\lambda > 0$ then $P(\sup |S_n/n| > \lambda) \leqq E|X_1|/\lambda$. For a proof see the introduction to [3]. Thus

$$P(\sup_n |S_n/n| \geqq 2B) \leqq \tfrac{1}{2}. \tag{1}$$

Let $A_n = ((X_n/n) - 4B)^+ I(\sup_{k<n} |S_k/k| \leqq 2B)$. Then, since $X_n/n \leqq S_n/n + |S_{n-1}/(n-1)|$,

$$A_n \leqq S_n/n \quad \text{on } \{A_n > 0\} \tag{2}$$

Also, since if X_n/n exceeds 4B then at least one of $|S_n/n|$, $|S_{n-1}/(n-1)|$ must exceed $2B$,

$$P(A_i > 0, A_j > 0) = 0 \qquad \text{if } i \neq j. \tag{3}$$

Next a lower bound is found for $\sum EA_k$. Since X_n and $\{\sup_{k<n} |S_k/k| \leqq 2B\}$ are independent, (1) implies $EA_n \geqq E((X_n/n) - 4B)^+)/2$. Let $p_i = P(4Bi \leqq X < 4B(i+1))$. Then

$$\begin{aligned}
\sum_1^\infty EA_n &\geqq (\tfrac{1}{2}) \sum_1^\infty E(((X_n/n) - 4B)^+) \\
&= (\tfrac{1}{2}) \sum_{n=1}^\infty (1/n) E((X - 4Bn)^+) \\
&\geqq (\tfrac{1}{2}) \sum_{n=1}^\infty (4B/n) \sum_{k=n+1}^\infty p_k (k-n) \\
&= 2B \sum_{k=2}^\infty p_k f(k) \geqq 2B\varepsilon \sum_{k=2}^\infty p_k (k+1) \log(k+1) \\
&\geqq 2B\varepsilon [E((X^+/4B) \log^+ (X^+/4B)) - p_1 2\log 2] \\
&\geqq 2B\varepsilon [(1/4B) E(X^+ (\log^+ X^+ - \log 4 - \log B)) - 2\log 2].
\end{aligned}$$

Since $B \leqq 1 + E|X|$, this completes the proof of the following lemma, noting that $\log^+ X^+ = \log^+ X$.

Lemma. *There are positive constants, K, C such that if X is an integrable random variable and $X_1, X_2, \ldots$ are independent random variables each having the distribution of X then $\sum EA_n \geqq CEX \log^+ X - KE|X| \log^+ E|X| - K$.*

3. Construction of Stopping Times

Suppose the conditions of the lemma are satisfied and that $E(X \log^+ X) = \infty$. Then $\sum EA_n = \infty$, and thus if $\{t = n\} = \{A_n > 0\}$, $E((S_t/t) I(t < \infty)) = \infty$, using (2) and (3). Unfortunately t is not a stopping rule since $P(t < \infty) < 1$. Most of the work in proving Theorem 1 below involves changing t slightly to remedy this defect.

Theorem 1. *Let X be an integrable variable and $X_1, X_2, \ldots$ be independent random variables each with the same distribution as X. The following statements are equivalent:*

(i) $E(X \log^+ X) < \infty$.

(ii) $E(S_t/t) < \infty$ *for every stopping rule t.*

(iii) $E(X_t/t) < \infty$ *for every stopping rule t.*

Proof of Theorem 1. (i)$\Rightarrow$(ii), (i)$\Rightarrow$(iii) are immediate consequences of the fact that if $E(X \log^+ X) < \infty$ then $E \sup(S_n/n) < \infty$ and $E \sup(X_n/n) < \infty$. See [1], p. 891.

(ii)$\Rightarrow$(i) Assuming $E(X \log^+ X) = \infty$ a stopping rule t will be constructed so that $E(S_t/t) = \infty$.

Let a be a real number such that $0<P(X>a)<1$. By the lemma, $\sum EA_n=\infty$. Pick integers $1=N_1<N_2<\cdots$ and sets $\phi=D(N_1)$, $D(N_2),\ldots$ to satisfy, if $R_k=\widetilde{D(N_1)}\,\widetilde{D(N_2)}\ldots\widetilde{D(N_k)}$,

(a) $D(N_j)$ is either $\{X_{N_j}>a\}$ or $\{X_{N_j}\leqq a\}$, $j=2,3,\ldots$.

(b) $\sum_{k=N_j+1}^{N_{j+1}} E(A_k I(R_j))\geqq 1$.

(c) $\sum_{k=N_j+1}^{\infty} E(A_k I(R_j))=\infty$.

This is possible, because having chosen $N_1, N_2, \ldots N_u$, $D(N_1),\ldots,D(N_u)$, since (c) holds, N_{u+1} can be picked so that

$$\sum_{k=N_u+1}^{N_{u+1}} E(A_k I(R_u))\geqq 1.$$

Since

$$\sum_{k=N_{u+1}+1}^{\infty} E(A_k I(R_u))=\infty,$$

either

$$\sum_{k=N_{u+1}+1}^{\infty} E(A_k I(R_u,\{X_{N_{u+1}}>a\})=\infty$$

or

$$\sum_{k=N_{u+1}+1}^{\infty} E(A_k I(R_u,\{X_{N_{u+1}}\leqq a\})=\infty.$$

Pick $D(N_{u+1})=\{X_{N_{u+1}}\leqq a\}$ if the first holds and $\{X_{N_{u+1}}>a\}$ if the second holds but the first doesn't.

Let $\tau=\inf\{n\colon A_n>0\}$, $v=\inf\{n\colon n \text{ is some } N_i \text{ and } I(D_n)>0\}$, and let $\eta=\min(\tau, v)$.

$P(\eta<\infty)\geqq P(v<\infty)=1$, since $\sum P(D(N_i))=\infty$ and the sets $D(N_1), D(N_2)\ldots$ are independent. Using (2) and (3),

$$\begin{aligned} E(S_\eta/\eta)^+ &\geqq \sum_{n=1}^{\infty} E(A_n I(\eta=n)) \\ &=\sum_{j=1}^{\infty}\sum_{k=N_j+1}^{N_{j+1}} E(A_k I(\eta=k)) \\ &=\sum_{j=1}^{\infty}\sum_{k=N_j+1}^{N_{j+1}} E(A_k I(R_j))\geqq 1+1+\cdots=\infty. \end{aligned}$$

Now let t be the first time n such that $n\geqq\eta$ and $S_n/n\geqq -2B$. Then t is the required stopping rule, since $S_t/t\geqq S_\eta/\eta$, so $E(S_t/t)^+=\infty$, while $S_t/t\geqq -2B$ so $E(S_t/t)^-<\infty$.

(iii)$\Rightarrow$(i) Again suppose that $E(X\log^+ X)=\infty$ and let the notation be as just above. Since $X_n/n>A_n$ on $\{t=n\}$, the same proof shows that $E(X_\eta/\eta)^+=\infty$. If t^* is the first time k after η such that $X_k>0$, then $X_{t^*}/t^*\geqq X_\eta/\eta$ and $X_{t^*}/t^*\geqq 0$ so $E(X_{t^*}/t^*)\geqq E(X_\eta/\eta)^+=\infty$.

Theorem 2. *There exist positive constants F, G, H such that if X is an integrable random variable, $X_1, X_2, \ldots$ have the same distribution as X, and T is the class of all stopping rules then both $\sup_{t \in T} E|S_t/t|$ and $\sup_{t \in T} E|X_t/t|$ are bounded below by $FE(|X| \log^+ |X|) - GE|X| \log^+ E|X| - G$ and bounded above by $H + HE(|X| \log^+ |X|)$.*

Proof. $\sup |X_n/n| \leqq 2 \sup |S_n/n|$, and since

$$E(\sup |S_n/n|) \leqq e/(e-1) + e/(e-1)\, E(|X| \log^+ |X|),$$

due to Doob (see [1], p. 891), H can be taken as $2e/(e-1)$.

It is no loss of generality to assume $E(X^+ \log^+ X) \geqq (E|X| \log^+ |X|)/2$. Thus, if T_N is the first time that $A_k > 0$ or N, whichever comes first, using (2) and (3)

$$E|S_{T_N}/T_N| \geqq \sum_1^N EA_i, \qquad E|X_{T_N}/T_N| \geqq \sum_1^N EA_i.$$

From an application of the lemma it follows that G may be taken to be K and F to be $C/2$.

References

1. Burkholder, D.L.: Successive conditional expectations of an integrable function. Ann. math. Statistics **33**, 887–893 (1962).
2. Doob, J.L.: Stochastic processes. New York: Wiley 1953.
3. Gundy, R.F.: On the class $L \log L$ martingales, and singular integrals. Studia math. **33**, 109–118 (1969).
4. Stout, William F., Chow, Y.S.: On the expected value of a stopped stochastic sequence. Ann. math. Statistics **40**, 456–461 (1969).

Dr. B. Davis
Statistics Department
Rutgers University
New Brunswick, New Jersey 08903, USA

(Received August 4, 1969)