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Mutual Subordination of Multivariate Stationary Processes Over Any Locally Compact Abelian Group

MILTON ROSENBERG*

Summary. Our purpose is to extend Kolmogorov's theorem [5, Th. 10] on mutual subordination for univariate weakly stationary stochastic processes over the (discrete) group of integers to multivariate processes over any (Hausdorff) locally compact abelian (lca) group. This extension is given in Theorems (1.12) and (3.4) below. We shall lean heavily on the joint paper [10] on the decomposition of matricial measures, to which the present paper may be regarded as a sequel.

In Section 1 of the paper we shall define and prove theorems on the concept of *E-subordination*, where E is a projection-valued measure. In Section 2 we shall examine the structure of stationary processes over an lca group. In Section 3 we shall consider the concept of *subordination* of stationary processes. Finally in Section 4, we shall apply our subordination theorems to deduce that matrix-valued functions in L_2 on the unit circle having no negative frequencies have a constant rank a.e. (Lebesgue) (4.2), a theorem of F. and M. Riesz (4.3), and a theorem on wandering subspaces due to Robertson [9] (4.4).

§ 1. E-Subordination

We make the following assumptions in this entire section. $\mathscr{H}$ is a Hilbert space and $f=(f^i)_{i=1}^q \in \mathscr{H}^q$, $g=(g^j)_{j=1}^p \in \mathscr{H}^p \cdot (f,g)$ is the $q\times p$ Gram matrix $[(f^i, g^j)]$ of scalar products. $\mathscr{B}$ is a σ-algebra of subsets of a set Ω, and E is a spectral measure on $\mathscr{B}$ (for $\mathscr{H}$), i.e., $E(B)$ is an orthogonal projection operator on $\mathscr{H}$ for each $B\in\mathscr{B}$, E is strongly countably additive, and $E(\Omega)=I$, cf. [3, § 6].

(1.1) **Definition.** For each $q\geqq 1$, let $E^q(B)$ be the inflation of $E(B)$ to $\mathscr{H}^q$, i.e. $E^q(B)f=(E(B)f^i)_{i=1}^q$. We define for each $g\in\mathscr{H}^p$ and each $f\in\mathscr{H}^q$, $M_{gf}(B)=(E^p(B)g, E^q(B)f)$.

The next theorem easily follows.

(1.2) **Theorem.** (a) *For each $g\in\mathscr{H}^p$ and each $f\in\mathscr{H}^q$, $M_{gf}(\cdot)$ is a bounded $p\times q$ matrix valued measure on $\mathscr{B}$.*

(b) *For each $f\in\mathscr{H}^q$, $M_{ff}(\cdot)$ is a bounded non-negative hermitian $q\times q$ matrix-valued measure on $\mathscr{B}$.*

(c) *For each $f\in\mathscr{H}^q$, $E^q(\cdot)f$ is an $\mathscr{H}^q$-valued countably-additive orthogonally-scattered (caos) measure, see* [12] *or* [7, *Section* 5].

(1.3) **Definition.** Let $f\in\mathscr{H}^q$ and let $p\geqq 1$. Then (a) $\mathscr{S}_f^p$ is the closed subspace of $\mathscr{H}^p$ spanned by $E^q(B)f\in\mathscr{H}^q(B\in\mathscr{B})$ with $p\times q$ matrix coefficients[1] ($\mathscr{S}_f^p$ is the *p-dimensional E-spectral space of f*).

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1. i.e., spanned by the vectors $AE^q(B)f\in\mathscr{H}^p$, where A is a $p\times q$ matrix.

(b) $(p\times q)\,L_{2,M_{ff}}=\{\Phi\colon \Phi$ is a $p\times q$ matrix-valued $\mathscr{B}$-measurable function on Ω such that $\int_\Omega \Phi(\omega)\cdot M_{ff}(d\omega)\cdot\Phi^*(\omega)$ exists$\}$,

see [12; 3.1 (a) (b)]. We write

$$(\Phi,\Psi)_{M_{ff}}=\int_\Omega \Phi\cdot dM_{ff}\cdot\Psi^*.$$

We quote the following theorem, cf. [12; 3.9, 4.5, 4.6].

(1.4) **Theorem.** *Let $f\in\mathscr{H}^q$ and $p\geqq 1$. Then*

(a) *$(p\times q)\,L_{2,M_{ff}}$ is a Hilbert space under the inner product trace$(\cdot,\cdot)_{M_{ff}}$.*

(b) $\mathscr{S}_f^p=\{x\colon x=\int_\Omega \Phi(\omega)\,E^q(d\omega)\,f,\ \Phi\in(p\times q)\,L_{2,M_{ff}}\}\subseteq\mathscr{H}^p$.

(c) *The correspondence*

$$\Phi\to\int_\Omega \Phi(\omega)\,E^q(d\omega)\,f$$

is unitary on $(p\times q)\,L_{2,M_{ff}}$ onto $\mathscr{S}_f^p$.

We mention the easily proven fact that $\mathscr{S}_f^p=\mathscr{S}_f^1\times\mathscr{S}_f^1\times\cdots\times\mathscr{S}_f^1$ (p factors), see [13; 5.8].

We require the following theorem whose proof we shall omit. Parts (a), (b), and (d) are proved in [11]. Part (c) follows from the results in [12] and basic measure theory.

(1.5) **Theorem.** *Let $\Phi\in(r\times q)\,L_{2,M_{ff}}$, $\Psi\in(s\times p)\,L_{2,M_{gg}}$. Then* (a) *the stochastic integrals*

$$\int_B \Phi\cdot E^q(d\omega)\,f,\ \int_B \Psi\cdot E^p(d\omega)\,g$$

define $\mathscr{H}^r$ and $\mathscr{H}^s$-valued caos measures on $\mathscr{B}$,

(b) $E^r(B)\big(\int_\Omega \Phi\cdot E^q(d\omega)\,f\big)=\int_B \Phi\cdot E^q(d\omega)\,f$ *for each $B\in\mathscr{B}$,*

(c) $\big(\int_B \Phi\cdot E^q(d\omega)\,f,\ \int_C \Psi\cdot E^p(d\omega)\,g\big)=\int_{B\cap C} \Phi\cdot dM_{fg}\cdot\Psi^*$ *for each $B, C\in\mathscr{B}$,*

(d) *if $g=\int_\Omega \chi\cdot E^q(d\omega)\,f$, where $\chi\in(p\times q)\,L_{2,M_{ff}}$, then*

$$\int_B \Psi\cdot E^p(d\omega)\,g=\int_B (\Psi\cdot\chi)\,E^q(d\omega)\,f$$

for each $B\in\mathscr{B}$ (substitution property).

We also need the following lemma.

(1.6) **Lemma.** *Let $M_{ff}(B)=\big(E^q(B)\,f,\ E^q(B)\,f\big)$, and $P(\omega)$ be a $\mathscr{B}$-measurable orthogonal projection matrix-valued function on Ω such that*

$$(*)\qquad \operatorname{range}\big(M'_{ff,\tau}(\omega)\big)\subseteq\operatorname{range}\big(P(\omega)\big)\qquad a.e.\ \ \big(\tau=\operatorname{trace}(M_{ff})\big).^{2}$$

2. $M'_{ff,\tau}$ = the matrix of Radon-Nikodym derivatives of the entry-measures of M_{ff} with respect to τ.

(For a matrix A, range $A=\{y\colon y=xA\}$.*) Then*

$$f=\int_\Omega P(\omega)\,E^q(d\omega)\,f.$$

Proof. By the definition of the stochastic integral we have

$$f=\int_\Omega I\cdot E^q(d\omega)\,f \qquad (I=q\times q \text{ identity matrix}).$$

Let

$$\tilde f=\int_\Omega P(\omega)\cdot E^q(d\omega)\,f.$$

Then by [12; 4.2 (a) (b)] we have

$$f-\tilde f=\int_\Omega (I-P(\omega))\cdot E^q(d\omega)\,f$$

and

$$(f-\tilde f, f-\tilde f)=\int_\Omega (I-P(\omega))\,M'_{ff,\tau}(I-P(\omega))\,d\tau=0, \qquad \text{by } (*).$$

Therefore $\tilde f=f$. //

(1.7) **Definition.** Let $g\in\mathscr{H}^p$, $f\in\mathscr{H}^q$. We say that g *is E-subordinate to* f if and only if $\mathscr{S}_g^1\subseteq\mathscr{S}_f^1\subseteq\mathscr{H}^1$ (equivalently, if and only if $\mathscr{S}_g^n\subseteq\mathscr{S}_f^n\subseteq\mathscr{H}^n$ for any fixed $n\geqq 1$).

The following extended version of [7; 8.7] follows easily from (1.4) (b) (c) and (1.5) (c).

(1.8) **Theorem.** *Let* (i) $g\in\mathscr{H}^p, f\in\mathscr{H}^q$, (ii) $\hat g$ *be the orthogonal projection of* g *onto* $\mathscr{S}_f^p$, (iii) $\Phi_{\hat g}\in(p\times q)\,L_{2,M_{ff}}$ *be the isomorph of* $\hat g$, *i.e.*

$$\hat g=\int_\Omega \Phi_{\hat g}\,E^q(d\omega)\,f.$$

Then

(a) $M_{gf}(B)=M_{\hat g f}(B)=\int_B \Phi_{\hat g}\cdot dM_{ff}, \qquad B\in\mathscr{B}$,

(b) $M_{\hat g\hat g}(B)=\int_B \Phi_{\hat g}\cdot dM_{ff}\cdot\Phi_{\hat g}^*=\int_B \Phi_{\hat g}\,dM_{f\hat g}, \qquad B\in\mathscr{B}$.

Under the assumptions (i) *and* (ii) *of* (1.8), *it therefore follows immediately, see* [10, *Section* 5], *that*

$$\text{(1.9)}\qquad \begin{aligned} &M_{gf}\ll M_{ff},\\ &\hat g=\int_\Omega (dM_{gf}\cdot dM_{ff}^{\#})\,dE^q f.\end{aligned}$$

(Let $M=[M_{ij}]$, $N=[N_{ik}]$ be matrix-valued measures and μ be a σ-finite measure such that $M_{ij}\ll\mu$, $N_{ik}\ll\mu$. *M is strongly a.c. with respect to N* ($M\lll N$) iff range$(M'_\mu(\omega))\subseteq$ range$(N'_\mu(\omega))$ a.e. (μ). In this case $M'_\mu(\omega)=M'_\mu(\omega)\cdot N'_\mu(\omega)^{\#}\,N'_\mu(\omega)$ a.e. (μ), where for a matrix A, $A^{\#}$ denotes its generalized inverse. We call $dM\cdot dN^{\#}=M'_\mu\cdot N'^{\#}_\mu$ a.e. (μ), the R.N. derivative of M with respect to N. Cf. [10].)

The following extended version of Kolmogorov's Theorems [5; Thms. 8, 9], cf. [7; 8.8] now follows easily.

(1.10) **Theorem.** *(E-subordination). Let $g\in\mathscr{H}^p$, $f\in\mathscr{H}^q$. Then the following conditions are equivalent:*

(1) *g is E-subordinate to f, i.e., $\mathscr{S}_g^p\subseteq\mathscr{S}_f^p$,*

(2) *there exists $\Phi\in(p\times q)\,L_{2,M_{ff}}$ such that $g=\int_\Omega \Phi\,dE^q f$,*

(3) *there exists $\Phi\in(p\times q)\,L_{2,M_{ff}}$ such that for any $B\in\mathscr{B}$*

$$M_{gf}(B)=\int_B \Phi\cdot dM_{ff},\qquad M_{gg}(B)=\int_B \Phi\cdot dM_{ff}\cdot\Phi^*.$$

We note that any Φ satisfying (2) also satisfies (3), and conversely. Hence, cf. (1.8), (1.9), [10; Section 5], for $g\in\mathscr{H}^p$ and $f\in\mathscr{H}^q$, the following conditions are equivalent:

(1.11)
(1) g is E-subordinate to f,

(2) $g=\int_\Omega (dM_{gf}\cdot dM_{ff}^{\#})\,dE^q f$,

(3) $M_{gg}(B)=\int_B (dM_{gf}\cdot dM_{ff}^{\#})\cdot dM_{ff}\cdot(dM_{gf}\cdot dM_{ff}^{\#})^*,\qquad B\in\mathscr{B}$.

Finally, we come to our extension of Kolmogorov's theorem on mutual subordination.

(1.12) **Theorem** *(Mutual E-subordination). Let* (i) *$g\in\mathscr{H}^p$, $f\in\mathscr{H}^q$,* (ii) *g be E-subordinate to f,* (iii) *$\Phi_g\in(p\times q)\,L_{2,M_{ff}}$ be the isomorph of g,* (iv) *$\tau=$ trace M_{ff},* (v) *$M'_{ff,\tau}$, $M'_{gg,\tau}$, $M'_{gf,\tau}$, and $M'_{fg,\tau}$ be respectively the matrices of Radon-Nikodym derivatives of the entries of M_{ff}, M_{gg}, M_{gf}, and M_{fg} with respect to τ. Then*

(a) *the following conditions are equivalent*

(1) *f is E-subordinate to g,*

(2) rank $(\Phi_g\cdot M'_{ff,\tau}\,\Phi_g^*)=$ rank $M'_{ff,\tau})$ *a.e.* (τ),

(3) rank $M'_{gg,\tau}=$ rank $M'_{ff,\tau}$ *a.e.* (τ),

(4) rank $(dM_{gf}\cdot dM_{ff}^{\#})=$ rank $M'_{ff,\tau}$ *a.e.* (τ),

(5) rank $M'_{gf,\tau}=$ rank $M'_{ff,\tau}$ *a.e.* (τ),

(6) $M_{ff}\ll M_{gf}$.

(b) *if f and g are mutually E-subordinate, then*

(1) $M_{gf}\ll M_{ff}\ll M_{gf}$, $M_{fg}\ll M_{gg}\ll M_{fg}$, *and*

(2) $dM_{fg}\cdot dM_{gg}^{\#}=(dM_{gf}\cdot dM_{ff}^{\#})^{\#}$ *a.e.* (τ).

Proof. (a) "(2)⇔(4)": By (1.11) (2) and [10; 4.5] it follows that $\Phi_g\equiv dM_{gf}\cdot dM_{ff}^{\#}\,(\text{mod}\,M_{ff})$. Further, by definition, $dM_{gf}\cdot dM_{ff}^{\#}=M'_{gf,\tau}\cdot M'^{\#}_{ff,\tau}$ a.e. (τ). Thus

$$(**)\qquad \Phi_g\,M'_{ff,\tau}\,\Phi_g^*=M'_{gf,\tau}\cdot M'^{\#}_{ff,\tau}\cdot M'^{*}_{gf,\tau}\qquad \text{a.e. } (\tau).$$

But the rank of R. H. S. $(**)$ is easily proven to be the same as the rank of $dM_{gf}\cdot dM_{ff}^{\#}$ a.e. (τ).

"(2)⇔(3)": Since

$$g=\int_{\Omega}\Phi_g\,dE^q f,$$

it follows readily from (1.5) (b) (c) that $M'_{gg,\tau}=\Phi_g\cdot M'_{ff,\tau}\cdot\Phi_g^*$ a.e. (τ).

"(1)⇒(4)": Let

$$\chi(\omega)=(dM_{gf}\cdot dM_{ff}^{\#})(\omega),\qquad \Psi(\omega)=(dM_{fg}\cdot dM_{gg}^{\#})(\omega)\qquad \text{a.e. } (\tau).$$

Then by (1.11) (2) and the substitution property (1.5) (d)

$$f=\int_{\Omega}\Psi\cdot dE^p g=\int_{\Omega}\Psi\,\chi\cdot dE^q f.$$

Hence by (1.1) and (1.5) (b) (c)

$$M_{ff}(B)=\Big(\int_B\Psi\,\chi\,dE^q f,\int_B I\cdot dE^q f\Big)=\int_B\Psi\,\chi\,dM_{ff},\qquad B\in\mathscr{B}.$$

Hence

$$M'_{ff,\tau}=\Psi\,\chi\,M'_{ff,\tau}\qquad \text{a.e. } (\tau).$$

Hence

$$\operatorname{rank} M'_{ff,\tau}\leqq\operatorname{rank}\chi\leqq\operatorname{rank} M'_{ff,\tau}\qquad \text{a.e. } (\tau),$$

where the first inequality is from the preceding equation, and the second inequality is from the definition of χ.

Next, "(4)⇒(1)". Let $\sum=\chi^{\#}$ a.e. (τ). Then $(\sum\chi)(\omega)=$ projection onto the range of $M'_{ff,\tau}(\omega)$ a.e. (τ). Similarly as in [10; 3.3(b)] $\sum\chi\in(q\times q)\,L_{2,M_{ff}}$ implies $\sum\in(q\times p)\,L_{2,M_{gg}}$. Hence by (1.6) and (1.5)(d)

$$f=\int_{\Omega}\sum\chi\,E^q(d\omega)\,f=\int_{\Omega}\sum\cdot E^p(d\omega)\,g.$$

Thus by (1.10)(2) f is E-subordinate to g.

"(4)⇔(5)": Since $M'^{\#}_{ff,\tau}$ is nonnegative hermitian with the same range as $M'_{ff,\tau}$ a.e. (τ), it follows readily that $\operatorname{rank}(dM_{gf}\cdot dM_{ff}^{\#})=\operatorname{rank}(M'_{gf,\tau})$ a.e. (τ).

Finally, "(4)⇔(6)": This is immediate from [10; 5.5].

(b)(1) This is immediate from (1.9) and (1.12)(a)(4).

(b)(2) We have by (1.5)(b)(c) for $B\in\mathscr{B}$

$$M_{ff}(B)=(E^q(B)\,f,E^q(B)\,f)=\Big(\int_B\Psi\,dE^p g,\int_B I\cdot dE^q f\Big)=\int_B\Psi\,dM_{gf}.$$

Hence $M'_{ff,\tau}=\Psi\cdot M'_{gf,\tau}$ a.e. (τ). But range$(\Psi)=(\text{kernel } M'_{gf,\tau})^{\perp}$ a.e. (τ) since M_{gg} and M_{fg} are mutually strongly a.c. Hence

$$dM_{ff}\cdot dM_{gf}^{\#}=\Psi\,M'_{gf,\tau}\cdot M'^{\#}_{gf,\tau}=\Psi=dM_{fg}\cdot dM_{gg}^{\#}\qquad \text{a.e. } (\tau).$$

Thus

$$dM_{fg}\cdot dM_{gg}^{\#}=dM_{ff}\cdot dM_{gf}^{\#}=(dM_{gf}\cdot dM_{ff}^{\#})^{\#}\qquad \text{a.e. } (\tau),$$

where the last equality holds since M_{ff} and M_{gf} are mutually strongly a.c., see [10; 5.5]. //

§ 2. Lca Groups

(2.1) **Definition.** (a) *A q-variate mean-continuous weakly stationary stochastic process (WSSP) over an lca group X* (under the operation $+$) is a function $(f_x)_{x\in X}$, such that (i) $f_x \in \mathscr{H}^q$ for each $x \in X$ ($\mathscr{H}$ is a fixed Hilbert space), (ii) the $q\times q$ Gram matrix $(f_x, f_y) = \Gamma_{x-y}$ depends only on $x-y$, (iii) $(f_x - f_y, f_x - f_y) \to 0$ as $x-y\to 0$ (mean-continuity).

(b) The *p-dimensional temporal domain* $\mathscr{M}_f^p$ of a q-variate WSSP $(f_x)_{x\in X}$ is the closed subspace of $\mathscr{H}^p$ spanned by the $f_x \in \mathscr{H}^q$ with $p\times q$ matrix coefficients.

(c) Let $(f_x)_{x\in X}$ and $(g_x)_{x\in X}$ be q and p-variate WSSP's respectively over the same X. We say that $(f_x)_{x\in X}$ and $(g_x)_{x\in X}$ are *stationarily cross-correlated* if (f_x, g_y) depends only on $x-y$.

In many papers, e.g. [14], $(f_x)_{x\in X}$ is referred to as a *homogeneous random field.*

(2.2) **Lemma.** *Let $(f_x)_{x\in X}$, $(g_x)_{x\in X}$, $(h_x)_{x\in X}$, ... be $q, p, r, \ldots$-variate WSSP's, which are pairwise stationarily correlated. Then there exists a strongly continuous group of unitary operators $(U_x)_{x\in X}$ on $\mathscr{H}$ such that for each $x\in X$, we have*

$$\text{(A)} \qquad f_x = U_x^{(q)} f_0, \qquad g_x = U_x^{(p)} g_0, \qquad h_x = U_x^{(r)} h_0, \ldots,$$

where $U_x^{(q)}$ is the inflation of U_x to $\mathscr{H}^q$ $(U_x^{(q)} f_0 = (U_x f_0^i)_{i=1}^q)$, etc.

Proof. The proof of the existence of a not necessarily unique unitary group with property (A) follows as in Kolmogorov's paper [5] or as in the reproof of this result in [2; 1.1, 1.2]. The strong continuity of the group $(U_x)_{x\in X}$ follows from the assumed mean continuity of each WSSP. //

Let X be an lca group. A continuous complex-valued function $\omega(x)$ on X (denoted $[x,\omega]$) is called a character on X if $[x+y,\omega] = [x,\omega]\cdot[y,\omega]$ and $|[x,\omega]| = 1$. The set Ω of characters is again an lca group under multiplication with respect to the dual topology, see [8; p. 408 II]. By the Borel subsets of an lca group we shall mean the smallest σ-algebra generated by its open subsets (rather than the σ-ring generated by compact subsets, cf. [3; 111], [4; 118]). We now recall the generalization of Stone's theorem [1; Th. 1] [8; 419] and of Bochner's theorem [1; I, II] [8; 410].

(2.3) **Stone's Theorem.** *Let $(U_x)_{x\in X}$ be a weakly continuous family of unitary operators on a Hilbert space $\mathscr{H}$ over an lca group X. Then there exists a unique spectral measure $E(\cdot)$ defined on the Borel subsets of the dual group Ω such that*

$$U_x = \int_\Omega \overline{[x,\omega]}\, E(d\omega)$$

(the conjugate of $[x,\omega]$ is used here to conform with the usual conventions for stochastic processes).

(2.4) **Bochner's Theorem.** (a) *f is a continuous positive definite complex-valued function on the lca group $X \Leftrightarrow$ there exists a bounded non-negative measure m on the Borel subsets $\mathscr{B}$ of the dual group Ω such that for all $x\in X$*

$$f(x) = \int_\Omega \overline{[x,\omega]}\, m(d\omega).$$

(b) *If for all* $x \in X$

$$\int_{\Omega} \overline{[x, \omega]}\, m(d\omega) = \int_{\Omega} \overline{[x, \omega]}\, \mu(d\omega)$$

where m *and* μ *are bounded complex measures on* $\mathscr{B}$, *then* $m = \mu$.

(2.5) **Lemma** *(equality of temporal and spectral subspaces). Let* $(U_x)_{x \in X}$ *be the shift group of the q-variate WSSP* $(f_x)_{x \in X}$ *so that* $U_x^{(q)} f_y = f_{x+y}$, *and let E be the spectral measure for this group. Then for each* $p \geqq 1$, $\mathscr{M}_f{}^p = \mathscr{S}_{f_0}^p$.

Proof. It is easy to prove that

$$f_x = \int_{\Omega} \overline{[x, \omega]} \cdot I \cdot E^q(d\omega)\, f_0$$

(stochastic integral), $x \in X$. It readily follows that for each $p \times q$ matrix A,

$$A f_x = \int_{\Omega} \left(\overline{[x, \omega]}\, A\right) E^q(d\omega)\, f_0 \in \mathscr{S}_{f_0}^p.$$

It also readily follows that $\mathscr{M}_f^p \subseteq \mathscr{S}_{f_0}^p$.

Since $U^q(\mathscr{M}_f^q) = \mathscr{M}_f^q$ (i.e. $\mathscr{M}_f^1$ reduces U) it may be shown that for $B \in \mathscr{B}$, $E^q(B)\, f_0 \in \mathscr{M}_f^q$ and is unique. Hence for each $p \times q$ matrix A, $A E^q(B)\, f_0 \in \mathscr{M}_f^p$, and thus $\mathscr{S}_{f_0}^p \subseteq \mathscr{M}_f^p$. //

§ 3. Subordination of Weakly Stationary Processes

(3.1) **Definition.** Let $(f_x)_{x \in X}$ and $(g_x)_{x \in X}$ be q and p-variate WSSP's over X, having the same shift group $(U_x)_{x \in X}$, with spectral measure E. Then the matrix-valued measures $M_{f_0 f_0}$, $M_{g_0 f_0}$ defined in (1.1) are called the *(auto-) spectral measure* of $(f_x)_{x \in X}$ and the *(cross-) spectral measure* of $(g_x)_{x \in X}$ with respect to $(f_x)_{x \in X}$.

(3.2) **Definition.** Let $(g_x)_{x \in X}$, $(f_x)_{x \in X}$ be p and q-variate WSSP's which are stationarily cross-correlated. We say that $(g_x)_{x \in X}$ *is subordinate to* $(f_x)_{x \in X}$ if and only if $\mathscr{M}_g^1 \subseteq \mathscr{M}_g^1$ (equivalently, if $\mathscr{M}_g^n \subseteq \mathscr{M}_g^n$ for some integer $n \geqq 1$).

From (2.5) we obtain the following basic lemma.

(3.3) **Lemma.** *Let* $(g_x)_{x \in X}$ *and* $(f_x)_{x \in X}$ *be p and q-variate WSSP's having a common shift group* $(U_x)_{x \in X}$ *with spectral measure E. Then* $(g_x)_{x \in X}$ *is subordinate to* $(f_x)_{x \in X} \Leftrightarrow g_0$ *is E-subordinate to* f_0.

Let $(g_x)_{x \in X}$ and $(f_x)_{x \in X}$ be p and q-variate WSSP's with common shift group $(U_x)_{x \in X}$ with spectral measure E. On applying theorems 1.10 and 1.12 to $f = f_0$ and $g = g_0$ we at once have:

(3.4) **Theorem.** *Let* $(g_x)_{x \in X}$, $(f_x)_{x \in X}$ *be p and q-variate WSSP's with common shift group* $(U_x)_{x \in X}$ *having spectral measure E, and let* $g = g_0$, $f = f_0$. *Then*

(a) *any one of the conditions* (1)–(3) *of* (1.10) *is necessary and sufficient for the subordination of* $(g_x)_{x \in X}$ *to* $(f_x)_{x \in X}$,

(b) *if* $(g_x)_{x\in X}$ *is subordinate to* $(f_x)_{x\in X}$, *any one of the conditions* (1)–(6) *of* (1.12) *is necessary and sufficient for the mutual subordination of* $(g_x)_{x\in X}$ *and* $(f_x)_{x\in X}$.

Theorem (3.4) generalizes Kolmogorov's original theorems [5, Thms. 8–10] concerning simple discrete parameter processes, i.e., with $q=p=1$ and $X=$ {integers}.

§ 4. Applications

We shall now assume that the reader is familiar with the fundamentals of the prediction theory of multivariate WSSP's over the group of integers, $-\infty<n<\infty$; cf., e.g., [7] or [13; Section 6].

For the following application it is convenient first to quote a slightly generalized version of the condition for pure non-determinism [13; 6.13].

(4.1) **Theorem.** *Let* $(f_n)_{-\infty}^{\infty}$ *be a q-variate WSSP. Then each of the following conditions is equivalent to the pure non-determinism of* $(f_n)_{-\infty}^{\infty}$.

(a) $(f_n)_{-\infty}^{\infty}$ *is a one-sided moving average:*

$$f_n=\sum_{k=0}^{\infty} A_k\,\phi_{n-k},$$

A_k *are* $q\times p$ *matrices*, $\phi_n\in\mathscr{H}^p$, $(\phi_m,\phi_n)=\delta_{mn}K$,

$$\sum_{k=0}^{\infty}|A_k\sqrt{K}|_E^2<\infty,$$

(b) $\mathscr{M}_{-\infty}^{f}=\{0\}$.

In the following theorem we prove that matrix-valued L_2^{0+} functions have constant rank a.e. (Lebesgue), without recourse to analytic extension, see [6; 2.3, 2.5].

(4.2) **Theorem.** *If* $\Phi\in(p\times q)\,L_2^{0+}$, *then* $\operatorname{rank}(\Phi(e^{i\theta}))=\text{const}$ *a.e.* ($L=$ *Lebesgue measure*).

Proof. By assumption,

$$\Phi=\text{l.i.m.}\sum_{0}^{\infty}A_k\,e^{ik\theta}$$

in $L_2(\Omega=(0,2\pi],\ \mathscr{B}=\text{Borel subsets, Lebesgue measure})$, where the A_k's are $p\times q$ matricial Fourier coefficients of Φ and

$$\sum_{0}^{\infty}|A_k|_E^2<\infty.$$

Let $(g_n)_{-\infty}^{\infty}$ be an arbitrary q-variate orthonormal sequence, i.e., $(g_m,g_n)=\delta_{mn}\cdot I$, with shift operator U, having spectral measure E. Since

$$\int_{\Omega}e^{-im\theta},\,dM_{g_0g_0}=(g_m,g_0)=\delta_{m0}\cdot I,$$

it follows from (2.4) (b) that

$$M_{g_0 g_0} = \frac{1}{2\pi} I \cdot L.$$

Let

$$f_n = \sum_{k=0}^{\infty} A_k g_{n-k}.$$

It readily follows that

$$f_0 = \int_{\Omega} \left(\sum_0^{\infty} A_k e^{ik\theta} \right) E^q(d\theta)\, g_0.$$

Hence by (1.5) (c),

$$M_{f_0 f_0}(B) = (E^p(B) f_0, E^p(B) f_0) = \frac{1}{2\pi} \int_B \left(\sum_0^{\infty} A_k e^{ik\theta} \right) \cdot \left(\sum_0^{\infty} A_k e^{ik\theta} \right)^* dL.$$

Further by (3.1) $(f_n)_{-\infty}^{\infty}$ is purely nondeterministic and is therefore mutually subordinate with its orthogonal p-variate process $(\psi_n)_{-\infty}^{\infty}$ of "innovation vectors". But

$$(\psi_m, \psi_n) = \delta_{mn} G, \quad \text{so} \quad M_{\psi_0 \psi_0} = \frac{1}{2\pi} G \cdot L.$$

From (3.4) (b) (3) it follows that rank

$$\left(\sum_0^{\infty} A_k e^{ik\theta} \right) \cdot \left(\sum_0^{\infty} A_k e^{ik\theta} \right)^* = \text{rank}\, G \qquad \text{a.e. } (L). \qquad //$$

The following proof of the F. and M. Riesz Theorem is in essence due to D. Sarason. Its present very brief format, emphasizing subordination theorems, is due to Masani.

(4.3) **Theorem.** *Let μ be a complex measure on the σ-algebra of Borel subsets of $\Omega = (0, 2\pi]$. If the negative Fourier coefficients of μ vanish, i.e.,*

$$\int_{\Omega} e^{-in\theta}\, d\mu = 0, \qquad n < 0,$$

then μ is a.c. with respect to Lebesgue measure.

Proof. All processes shall be discrete univariate. Let $(x_n)_{-\infty}^{\infty}$ be a WSSP with spectral measure $|\mu|$ (total variation measure). Let

$$y_n = \int_{\Omega} e^{-in\theta} (d\mu / d|\mu|)\, E(d\theta)\, x_0.$$

Then it is readily verified that $(y_n)_{-\infty}^{\infty}$ and $(x_n)_{-\infty}^{\infty}$ are mutually subordinate (Show stationarily correlated; next compute $M_{y_0 y_0} = |\mu|$). Thus $\mathscr{M}_{\infty}^{x} = \mathscr{M}_{\infty}^{y}$, see [13, section 6] or [7, 2.10]. But

$$(x_k, y_n) = \left(\int_\Omega e^{-ki\theta} E(d\theta)\, x_0, \int_\Omega e^{-in\theta}(d\mu/d|\mu|)\, E(d\theta)\, x_0\right)$$
$$= \int_\Omega e^{(n-k)i\theta}\, d\mu = 0 \qquad \text{if } n>k.$$

Thus the remote past $\mathcal{M}^x_{-\infty} \perp \mathcal{M}^y_\infty = \mathcal{M}^x_\infty$. But this implies $\mathcal{M}^x_{-\infty} = \{0\}$. Hence $(x_n)^\infty_{-\infty}$ is purely non-deterministic. Hence as seen in the proof of (4.2), $|\mu| = M_{x_0 x_0} \ll L$. Hence $\mu \ll |\mu| \ll L$.

We conclude the paper with a subordination proof of a theorem on wandering subspaces due to Robertson [9; Th. 1]. Let V be a unitary operator on the complex Hilbert space $\mathcal{H}$. X is said to be a *wandering subspace* for V if it is a subspace of $\mathcal{H}$ such that $V^m(X) \perp V^n(X)$ for all $m \neq n$.

(4.4) **Theorem.** *Let X and Y be wandering subspaces for a unitary operator V such that*

(a) $\sum_{k=-\infty}^{\infty} V^k(X) \subseteq \sum_{k=-\infty}^{\infty} V^k(Y)$

(b) $\dim(X) = \dim(Y) < \infty$.

Then

$$\sum_{k=-\infty}^{\infty} V^k(X) = \sum_{k=-\infty}^{\infty} V^k(Y).$$

Proof. Let $x^1, \ldots, x^q$ and $y^1, \ldots, y^q$ be orthonormal bases for X and Y, respectively. Let $x_0 = (x^i)^q_{i=1} \in \mathcal{H}^q$, $y_0 = (y^i)^q_{i=1} \in \mathcal{H}^q$, $x_n = (V^{(q)})^n x_0$, $y_n = (V^{(q)})^n y_0$. Then $(x_n)^\infty_{-\infty}$ and $(y_n)^\infty_{-\infty}$ are WSSP's which are stationarily correlated such that

$$\mathcal{M}^1_x = \sum_{k=-\infty}^{\infty} V^k(X), \qquad \mathcal{M}^1_y = \sum_{k=1}^{\infty} V^k(Y)$$

and $(x_n)^\infty_{-\infty}$ is subordinate to $(y_n)^\infty_{-\infty}$. But since

$$\int_\Omega e^{-im\theta}\, dM_{x_0 x_0} = (x_m, x_0) = \delta_m \cdot I = (y_m, y_0) = \int_\Omega e^{-im\theta}\, dM_{y_0 y_0},$$

it follows from (2.4) (a) (b) that we must have

$$M_{x_0 x_0}(B) = M_{y_0 y_0}(B) = \frac{1}{2\pi} \int_B I \cdot dL$$

for each Borel subset B of $\Omega = (0, 2\pi]$ ($I = q \times q$ identity, L = Lebesgue measure). Hence by (3.4) (b) (3), $(y_n)^\infty_{-\infty}$ is subordinate to $(x_n)^\infty_{-\infty}$. //

Note added in proof. We can strengthen Theorem (4.3) on noting in the proof that $|\mu|$ is the spectral measure of a univariate purely non-deterministic process and thus by (4.1) and (4.2) the rank of the one-by-one matrix-valued function $d|\mu|/dL(\theta)$ is a constant a.e. (L). Thus either μ is identically zero or μ and L are mutually a.c. Further, in (4.2) it is not true that range $\Phi(e^{i\theta})$ = const a.e. (L). For example, let

$$\Phi(e^{i\theta}) = \begin{bmatrix} 1 & e^{i\theta} \\ e^{i\theta} & e^{2i\theta} \end{bmatrix}.$$

Then range $\Phi(e^{i\theta_1}) \neq$ range $\Phi(e^{i\theta_2})$ if $\theta_1 \neq \theta_2$.

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Professor Milton Rosenberg
Department of Mathematics
University of Kansas
Lawrence, Kansas 66044, USA

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