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Sample Path Properties of Processes with Stable Components

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Summary. In this paper, processes in R^d of the form $X(t)=(X_1(t), X_2(t), \ldots, X_N(t))$, where $X_i(t)$ is a stable process of index α_i in Euclidean space of dimension d_i and $d=d_1+\cdots+d_N$, are considered. The asymptotic behaviour of the first passage time out of a sphere, and of the sojourn time in a sphere is established. Properties of the space-time process $(X_1(t), t)$ in R^{d+1} are obtained when $X_1(t)$ is a stable process in R^d. For each of these processes, a Hausdorff measure function $\varphi(h)$ is found such that the range set $R(s)$ of the sample path on $[0, s]$ has Hausdorff φ-measure cs for a suitable finite positive c.

1. Introduction

The object of the present paper is to investigate a class of Markov processes in R^d with stationary independent increments. Suppose $X_i(t)$ is a stable process of index α_i in Euclidean space of dimension d_i for $i=1, 2, \ldots, N$. Then, if the $X_i(t)$ are independent, the process

$$X(t)=(X_1(t), X_2(t), \ldots, X_N(t))$$

in R^d, where $d=d_1+d_2+\cdots+d_N$, and the d_i-dimensional subspaces in which the $X_i(t)$ take their values are orthogonal, is called a process with stable components. We may assume, without loss of generality, that the indices α_i are all different, and that

$$\alpha_N<\alpha_{N-1}<\cdots<\alpha_2<\alpha_1. \tag{1.1}$$

Processes of this type provide an interesting class of examples illustrating the general theory of Markov processes. The case $d_1=d_2=1$, $d=2$ arose naturally in [5] as a tool for obtaining information about the collision set of two independent stable processes. It turns out that most of the possible kinds of behaviour for general N and d are already obtainable when $N=2=d$ (and the general proofs are not much harder than this case) so we can illustrate the general results obtained by describing them in the planar case.

We consider the first passage time process

$$P(a)=P(a, \omega)=\inf\{t\colon\ |X(t, \omega)|\geqq a\}, \qquad 0\leqq a<\infty; \tag{1.2}$$

and the sojourn time process,

$$T(a)=T(a, \omega)=\int_0^\infty I_a(X(t, \omega))\,dt, \qquad 0\leqq a<\infty, \tag{1.3}$$

where I_a is the indicator function of the closed sphere of radius a, centre the origin. Now, if $1<\alpha_2<\alpha_1\leqq 2$, and $X(t)=(X_1(t), X_2(t))$ is the planar process with

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1-dimensional components $X_1(t), X_2(t)$, these components are recurrent while $X(t)$ is transient. We can state the asymptotic laws for $P(a)$ and $T(a)$ as follows.

Theorem 1.1. *Let* $1<\alpha_2<\alpha_1\leqq 2$, $d_1=d_2=1$, $N=2$,

$$\rho_1=1+\alpha_2(1-1/\alpha_1), \qquad \rho_2=1+\alpha_1(1-1/\alpha_2) \tag{1.4}$$

and

$$\varphi_\alpha(a)=a^\alpha \log|\log a|. \tag{1.5}$$

Then there are positive constants c_1, c_2, c_3, c_4 *such that with probability one*

$$\limsup_{a\to 0} \frac{P(a)}{\varphi_{\alpha_1}(a)}=c_1; \qquad \limsup_{a\to\infty} \frac{P(a)}{\varphi_{\alpha_2}(a)}=c_2;$$

$$\limsup_{a\to 0} \frac{T(a)}{\varphi_{\rho_1}(a)}=c_3; \qquad \limsup_{a\to\infty} \frac{T(a)}{\varphi_{\rho_2}(a)}=c_4.$$

The relationship between these parameters in this case is

$$\alpha_2<\rho_2<\rho_1<\alpha_1$$

and we have in Theorem 1.1 statements of four different rates of growth. But when $\alpha_1=\alpha_2<2$, the planar process is stable, and all these rates become the same. The asymptotic behaviour of $P(a)$ and $T(a)$ for general processes with stable components is given in Theorems 4.1, 5.1, 7.1, and 7.2 (except for some critical cases discussed in Section 9). These include Theorem 1.1 as a special case.

A related result is concerned with the correct Hausdorff measure function for the range of the sample path $X(t)$ on $[0, s]$. Let the set of points in R^d visited by the process up to time s be

$$R(s)=R(s,\omega)=\{X(t,\omega):\ 0\leqq t\leqq s\}.$$

Theorem 1.2. *Let* $1<\alpha_2<\alpha_1\leqq 2$, $d_1=d_2=1$, *and* ρ_1, $\varphi_{\rho_1}(a)$ *be as in* (1.4) *and* (1.5). *Then there is a positive constant* c_5 *such that*

$$\varphi_{\rho_1}-m[R(s)]=c_5 s$$

for all $s\geqq 0$ *almost surely.*

In particular, this implies that the Hausdorff dimension of the range of the sample paths is almost surely equal to ρ_1. It is worth pointing out that in the present case the parameters defined by Blumenthal and Getoor [1] are

$$\beta=\alpha_1, \qquad \beta'=\rho_1.$$

These authors showed that the dimension of the range satisfies the inequality

$$\dim R(s)\geqq\beta'$$

almost surely; Theorem 1.2 shows that there is actually equality here. This adds some weight to the conjecture that $\dim R(s)=\beta'$ for arbitrary processes with stationary, independent increments; this has recently been proved by Horowitz (personal communication) for subordinators.*

The correct Hausdorff measure function for the range of $X(t)$ in the general case is given in Theorem 6.1 (with the critical cases again in Section 9). When $\alpha_1<1$, it is a consequence of Theorem 6.1 that the dimension of the range is α_1. This would also follow from [1] since when $\alpha_1<1$, the indices become

$$\beta=\beta'=\alpha_1,$$

and for the case $\beta<1$, Blumenthal and Getoor proved that

$$\beta'\leqq \dim R(s)\leqq \beta$$

almost surely. Results corresponding to Theorems 1.1, 1.2 were obtained in [15] for the transient stable processes.

Now, if we replace $X_2(t)$ by the degenerate process $X_2(t)\equiv t$, we can obtain information about the space-time process $(X_1(t), t)$ whose range can be thought of as the graph of the process $X_1(t)$. The dimensional number of the graph was obtained by Taylor [13] for Brownian motion, and by Blumenthal, Getoor [2] for a symmetric stable process, while Jain and Pruitt [6] obtained the correct measure function for the transient case $\alpha_1<d_1$. The techniques of the present paper allow us to obtain the correct Hausdorff measure function for the graph of a point recurrent stable process in R^1. Define, in R^2, the set of points on the graph up to time s,

$$G(s)=G(s,\omega)=\{(X(t,\omega),t)\colon\ 0\leqq t\leqq s\},$$

and let

$$\varphi_{\alpha\beta}(a)=a^{\alpha}(\log|\log a|)^{\beta}.$$

Theorem 1.3. *Let $X(t)$ be a stable process of index α, $1<\alpha\leqq 2$, in R^1. Then there is a positive constant c_6 such that*

$$\varphi_{2-1/\alpha,\,1/\alpha}-m[G(s)]=c_6\, s,$$

for all $s\geqq 0$ almost surely.

The proof of Theorem 1.3 is in Section 8. The correct measure function for the graph of a general process with stable components is also obtained.

Throughout this paper we rely extensively on the scaling property of stable processes, so we have to completely exclude the case where one of the components does not satisfy this property. The scaling property can only fail if $\alpha_i=1$ for some i, so if one of the components is Cauchy, we assume it to be such that it satisfies the scaling property. However, the form of the results is often different in the critical cases where one of the components is interval recurrent, but not point recurrent, that is, $\alpha_i=d_i=1$ or 2 for some i. Most of the results in Sections 5 to 7 exclude this critical case, which we take up in a final Section 9.

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* *Added in proof.* Horowitz has shown that the dimension for subordinators is σ and not β' (see [1] for the definition). Furthermore, there are subordinators with $\sigma\neq\beta'$ so that the dimension cannot be β' in general. We now have an index which gives the dimension for all processes with stationary independent increments and this index *is* equal to β' at least for all *symmetric* processes with $\beta'\leqq d$.

2. Preliminaries

The d-dimensional characteristic function of a stable process $X(t)$ of index α has the form $\exp[t\psi(z)]$ where

$$\psi(z)=i(b,z)-c_7|z|^\alpha \int_{S_d} w_\alpha(z,\theta)\mu(d\theta),$$

with $b\in R^d$, $c_7>0$,

$$w_\alpha(z,\theta)=[1-i\,\mathrm{sgn}(z,\theta)\tan\pi\alpha/2]\,|(z/|z|,\theta)|^\alpha, \qquad \alpha\neq 1,$$

$$w_1(z,\theta)=|(z/|z|,\theta)|+(2i/\pi)(z/|z|,\theta)\log|(z,\theta)|,$$

and μ is a probability measure on the surface of the unit sphere S_d in R^d [7]. We shall assume that μ is not supported by a proper subspace of R^d, and that $b=0$, $c_7=1$. The process is called symmetric when μ is uniform. It is assumed that all the processes considered have been defined so as to have sample functions $X(t)$ which are right continuous and have left limits everywhere. The processes will also have the strong Markov property, which we will use extensively without specific mention.

The density function $p(t,x)$ of $X(t)$ is continuous and bounded in x for fixed t. It also satisfies the scaling property (except for some nonsymmetric processes of index 1)

$$p(t,x)=p(rt,r^{1/\alpha}x)\,r^{d/\alpha} \tag{2.1}$$

for all $r>0$, or in terms of the process itself, $X(rt)$ and $r^{1/\alpha}X(t)$ have the same distribution. This scaling property will be used extensively and so the processes of index 1 for which it fails will be excluded throughout the paper. The stable processes have been classified in [15] as being of type A if $p(1,0)>0$ and of type B otherwise. When $\alpha\geqq 1$, only processes of type A can occur.

The first passage time and sojourn time processes were defined in (1.2) and (1.3). We shall also need to consider the sojourn time of the process in a sphere of radius a up to time s,

$$T(a,s)=T(a,s,\omega)=\int_0^s I_a(X(t,\omega))\,dt. \tag{2.2}$$

In the case $d=1$ and $1<\alpha\leqq 2$, there is a local time for the process which we denote by $L(x,s)=L(x,s,\omega)$ which satisfies

$$T(a,s,\omega)=\int_{|x|\leqq a} L(x,s,\omega)\,dx.$$

The existence and some basic properties of the local time are proved in [3].

Finally, we quote some lemmas which will be used. The first combines a result due to Dynkin [4] on the range of a stable subordinator and one due to Stone [12] relating the zero set of a stable process to the closure of the range of a stable subordinator. The other two are in [15].

Lemma 2.1. *Let $X(t)$ be a stable process in R^1 of index $\alpha>1$. There is a positive constant c_8 such that the probability that an interval $[u,v)$ contains no zero of $X(t)$ is equal to*

$$c_8\int_{v/u}^\infty s^{-1}(s-1)^{-1+1/\alpha}\,ds.$$

Lemma 2.2. *Let $X(t)$ be a stable process of type A in R^d of index $\alpha<d$. There are positive constants c_9, c_{10}, and λ_0 such that for all $\lambda\geqq\lambda_0$*

$$e^{-c_9\lambda}\leqq P[P(a)\geqq\lambda a^\alpha]\leqq P[T(a)\geqq\lambda a^\alpha]\leqq e^{-c_{10}\lambda}.$$

Lemma 2.3. *Let $X(t)$ be a stable process of type B in R^d of index $\alpha<1$. There are positive constants c_{11}, c_{12}, and λ_0 such that for all $\lambda\geqq\lambda_0$, if $\beta=1/(1-\alpha)$,*

$$\exp(-c_{11}\lambda^\beta)\leqq P[P(a)\geqq\lambda a^\alpha]\leqq P[T(a)\geqq\lambda a^\alpha]\leqq\exp(-c_{12}\lambda^\beta).$$

3. The Sojourn Time $T(a,s)$ for $\alpha>d=1$

In the case of a one-dimensional stable process $X(t)$ which is point recurrent, $T(a)$ is almost surely infinite. Since one or more of the components of our process may be of this type we require information about the tail of the distribution of $T(a,s)$ at least for the case where $s\geqq a^\alpha$. (The other case when s is much smaller than a^α is of less interest for then $T(a,s)$ is likely to be s). This is the objective of the present section. We start with a lemma which is of some independent interest. It turns out to be just as easy to prove in a more general context, and shows that, under very weak assumptions, the tail of the distribution of $T(a,s)$ is not greater than negative exponential. We state the result in R^d, though it obviously holds in any normed linear space. For any set $B\subset R^d$, define the difference set $D=D(B)$ by

$$D=\{z\in R^d\colon\ z=x-y,\, x, y\in B\}.$$

Lemma 3.1. *Let $X(t)$ be a process in R^d with stationary independent increments, and let $T_B(s)$, $T_D(s)$ denote the sojourn times of $X(t)$ in a Borel set B and its difference set D, up to time s $(0<s\leqq+\infty)$. Then, for any $0<\delta<1$, there is a $\lambda_0=\lambda_0(\delta)$ such that if $x\in B$ and $0<ET_D(s)<\infty$,*

$$P^x[T_B(s)\geqq\lambda ET_D(s)]\leqq e^{-\delta\lambda}\qquad \text{for } \lambda\geqq\lambda_0.$$

Remark. If B is a closed set, and the sample paths are right continuous, it is clear that the inequality holds for all starting points $x\in R^d$, since we can restart the process at σ, the hitting time of B.

Proof. The first step is to estimate the moments of $T_B(s)$.

$$\begin{aligned}
E^x\{T_B(s)\}^n &= \int_0^s\cdots\int_0^s P^x[X(t_i)\in B, i=1,\dots,n]\,dt_1\dots dt_n\\
&= n!\int\cdots\int_{0\leqq t_1\leqq\cdots\leqq t_n\leqq s} P^x[X(t_i)\in B, i=1,\dots,n]\,dt_1\dots dt_n\\
&\leqq n!\int\cdots\int_{0\leqq t_1\leqq\cdots\leqq t_n\leqq s} P^x[X(t_i)\in B, i=1,\dots,n-1]\\
&\qquad\qquad\cdot P^0[X(t_n-t_{n-1})\in D]\,dt_1\dots dt_n\\
&\leqq n!\,E^0T_D(s)\int\cdots\int_{0\leqq t_1\leqq\cdots\leqq t_{n-1}\leqq s} P^x[X(t_i)\in B, i=1,\dots,n-1]\,dt_1\dots dt_{n-1}\\
&= n\,ET_D(s)\cdot E^x\{T_B(s)\}^{n-1}.
\end{aligned}$$

By induction, we obtain $E^x\{T_B(s)\}^n \leqq n!\{E T_D(s)\}^n$ for $x\in B$. Now let $k=\frac{1}{2}(1+\delta^{-1})$, $u=[k\, E\, T_D(s)]^{-1}$, and we have

$$E^x \exp(u\, T_B(s)) = E^x \sum_{n=0}^{\infty} \frac{u^n}{n!} \{T_B(s)\}^n \leqq \sum_{n=0}^{\infty} k^{-n} = c_{13}.$$

Finally, by the basic inequality ([8], p. 157), applied to the function $g(a)=\exp(u\,a)$,

$$P^x[T_B(s) \geqq \lambda E T_D(s)] \leqq e^{-\lambda/k} E^x \exp(u\, T_B(s)) \leqq c_{13}\, e^{-\lambda/k} \leqq e^{-\delta\lambda}, \qquad \text{for } \lambda \geqq \lambda_0(\delta).$$

Corollary. *For any process in R^d with stationary independent increments, if $T(a, s)$ is the sojourn time in a closed sphere of radius a up to time s, then there is a λ_0 such that, if $0<ET(2a, s)<\infty$,*

$$P^x[T(a, s) \geqq \lambda E T(2a, s)] \leqq e^{-\frac{1}{2}\lambda} \qquad \textit{for } \lambda \geqq \lambda_0.$$

Proof. This follows from the lemma, with $\delta=\frac{1}{2}$, by observing that when B is a closed sphere radius a, then $D(B)$ is a closed sphere radius $2a$ centred at the origin.

Lemma 3.2. *Let $X(t)$ be a stable process in R^1 of index $\alpha>1$. Then there are positive constants c_{14}, c_{15} such that for $s \geqq 2a^\alpha$,*

$$c_{14}\, a\, s^{1-1/\alpha} \leqq E\{T(a, s)\} \leqq c_{15}\, a\, s^{1-1/\alpha}.$$

The restriction on s and a is only needed in the lower bound.

Proof. By the scaling property (2.1) and the boundedness of $p(1, x)$,

$$E\{T(a, s)\} = \int_0^s P[|X(t)| \leqq a]\, dt = \int_0^s P[|X(1)| \leqq a\, t^{-1/\alpha}]\, dt$$
$$\leqq \int_0^s c_{16}\, a\, t^{-1/\alpha}\, dt = c_{15}\, a\, s^{1-1/\alpha}.$$

For the lower bound, we use the fact that $p(1, x)$ is bounded below for $x\in[-1, 1]$ since $X(t)$ is necessarily of type A. Thus

$$E\{T(a, s)\} = \int_0^s P[|X(1)| \leqq a\, t^{-1/\alpha}]\, dt \geqq \int_{a^\alpha}^s c_{17}\, a\, t^{-1/\alpha}\, dt$$
$$\geqq c_{14}\, a\, s^{1-1/\alpha}, \qquad \text{since } s \geqq 2a^\alpha.$$

Remark. For the processes included in Lemma 3.2, $T(a, s)\to\infty$ a.s. as $s\to\infty$ since $X(t)$ is recurrent. The result shows how $ET(a, s)$ grows as s increases. We could immediately apply Lemma 3.1 and its Corollary to Lemma 3.2 to obtain a negative exponential upper bound for the tail of the $T(a, s)$ distribution. However, this would give a bound of the wrong order of magnitude. We can do better by using the scaling property, and splitting the interval $(0, s)$ into a suitable number of smaller pieces.

Lemma 3.3. *Let $X(t)$ be a stable process in R^1 of index $\alpha>1$. Then there are positive constants c_{18}, c_{19}, c_{20} and λ_0 such that for all λ with $\lambda_0 \leqq \lambda \leqq c_{18}\, a^{-1} s^{1/\alpha}$,*

$$\exp(-c_{19}\lambda^\alpha) \leqq P[T(a,s) \geqq \lambda\, a\, s^{1-1/\alpha}] \leqq \exp(-c_{20}\lambda^\alpha).$$

Proof. We first obtain the lower bound. By Lemma 2.1,

$$P[X(t)=0 \text{ for some } t\in[u,2u)] = c_{21} > 0, \tag{3.1}$$

for all $u>0$. The real number u will be chosen later so as to be relatively large compared to a^α, but small compared to s. The next step is to see that $T(a,u)$ is not too small. Choose c_{22} so that it is a continuity point of the distribution (it seems certain that the distribution of $L(0,1)$ is continuous, but we do not stop to prove this) of the local time $L(0,1)$ for $X(t)$ and so that

$$P[L(0,1) \geqq c_{22}] \geqq 1 - c_{21}/3.$$

This is possible by Theorem 1 of [12]. The scaling property (2.1) implies that $T(a,u)$ and $u\,T(a\,u^{-1/\alpha},1)$ have the same distribution. Therefore

$$P[T(a,u) \geqq 2c_{22}\, a\, u^{1-1/\alpha}] = P[(2a)^{-1} u^{1/\alpha}\, T(a\,u^{-1/\alpha},1) \geqq c_{22}].$$

The random variable on the left of the inequality in the last expression converges almost surely to $L(0,1)$ as $a\,u^{-1/\alpha} \to 0$ since the local time is continuous. Since this implies convergence in distribution, there is an $\varepsilon>0$ such that

$$P[T(a,u) \geqq 2\,c_{22}\, a\, u^{1-1/\alpha}] \geqq 1 - 2c_{21}/3,$$

whenever $a\,u^{-1/\alpha} \leqq \varepsilon$. Recalling (3.1), we have

$$P[T(a,u) \geqq 2c_{22}\, a\, u^{1-1/\alpha};\ X(t)=0 \text{ for some } t\in[u,2u)] \geqq c_{21}/3 \tag{3.2}$$

for all a, u such that $a\,u^{-1/\alpha} \leqq \varepsilon$. Now let $\tau_0=0$, and

$$\tau_k = \inf\{t \geqq \tau_{k-1}+u\colon X(t)=0\},$$

$$T_k = \int_{\tau_{k-1}}^{\tau_{k-1}+u} I_a(X(t))\, dt.$$

Re-starting the process at each τ_k, we see from (3.2) that

$$\begin{aligned}(\tfrac{1}{3}c_{21})^n &\leqq P[T_k \geqq 2c_{22}\, a\, u^{1-1/\alpha},\ \tau_k - \tau_{k-1} < 2u,\ k=1,\dots,n]\\ &\leqq P[T(a,2nu) \geqq 2n\, c_{22}\, a\, u^{1-1/\alpha}],\end{aligned}$$

so long as $a u^{-1/\alpha} \leqq \varepsilon$. Letting $n=[\lambda^\alpha/c_{22}^\alpha]$, $u=s/2n$, $c_{18}=\varepsilon\, c_{22}/2^{1/\alpha}$, and $c_{19}=\log(3/c_{21})/c_{22}^\alpha$;

$$\begin{aligned}\exp(-c_{19}\lambda^\alpha) \leqq \left(\frac{c_{21}}{3}\right)^n &\leqq P[T(a,2nu) \geqq (2n)^{1/\alpha}\, c_{22}\, a(2nu)^{1-1/\alpha}]\\ &\leqq P[T(a,s) \geqq \lambda\, a\, s^{1-1/\alpha}]\end{aligned}$$

for $c_{22} \leqq \lambda \leqq c_{18}\, a^{-1}\, s^{1/\alpha}$.

For the upper bound, we first use the estimate of Lemma 3.2 in the Corollary to Lemma 3.1 to obtain

$$P[T(2a, w) \geqq 4 c_{15} a \lambda w^{1-1/\alpha}] \leqq e^{-\lambda/2},$$

so that $\exp[v a^{-1} w^{-1+1/\alpha} T(2a, w)]$ has an expectation which is bounded for all a and w whenever v is fixed and sufficiently small. Fix such a value of v and let $e^{c_{23}}$ be a bound on the expectation.

Let $\sigma_0 = 0$, and define inductively:

$$\sigma_k = \inf\{t \geqq \sigma_{k-1} + w \colon |X(t)| \leqq a\},$$

$$S_k = \int_{\sigma_{k-1}}^{\sigma_{k-1}+w} I_{2a}(X(t) - X(\sigma_{k-1}))\, dt.$$

The random variables $S_1, S_2, \ldots$ are independent, identically distributed with the same distribution as $T(2a, w)$. Since $|X(\sigma_k)| \leqq a$, the sphere of radius $2a$, centre at $X(\sigma_k)$, contains the sphere of radius a centered at the origin. Hence $T(a, s)$ is no larger than $S_1 + S_2 + \cdots + S_n$ where $w = s/n$. Putting $n = [(v \lambda / 2 c_{23})^\alpha]$ gives

$$\begin{aligned} P[T(a, s) \geqq \lambda a s^{1-1/\alpha}] &\leqq P[S_1 + S_2 + \cdots + S_n \geqq \lambda a (n w)^{1-1/\alpha}] \\ &\leqq \exp(-v \lambda n^{1-1/\alpha}) [E\{\exp v a^{-1} w^{-1+1/\alpha} T(2a, w)\}]^n, \end{aligned}$$

on using the basic inequality. If $\lambda \geqq 2 c_{23}/v$, and we take $c_{20} = c_{23}^{1-\alpha} v^\alpha 2^{-\alpha-1}$ this gives

$$\begin{aligned} P[T(a, s) \geqq \lambda a s^{1-1/\alpha}] &\leqq \exp\{-n(v \lambda n^{-1/\alpha} - c_{23})\} \\ &\leqq e^{-c_{23} n} \leqq e^{-c_{20} \lambda^\alpha}. \end{aligned}$$

Remark. In the tail estimate of Lemma 3.3, the upper bound does not require λ to be bounded above. However our proof of the lower estimate did not allow λ to be arbitrarily large. This is not surprising since, if $\lambda a s^{-1/\alpha} > 1$, the probability must be zero.

4. First Passage Times

We first show that the exponential estimates given by Lemma 2.2 for the tail of the distribution of $P(a)$ are still valid when $\alpha \geqq d$. Note, however, $T(a)$ will be infinite in this case since the process is neighbourhood-recurrent.

Lemma 4.1. *Let $X(t)$ be a stable process in R^d with index $\alpha \geqq d$. There are positive constants c_{24}, c_{25}, and λ_0 such that for $\lambda \geqq \lambda_0$,*

$$e^{-c_{24} \lambda} \leqq P[P(a) \geqq \lambda a^\alpha] \leqq e^{-c_{25} \lambda}.$$

Proof. The proof of the lower bound given in [15], pp. 1238/39, uses only the fact that the process is type A which must be the case here since $\alpha \geqq d$. Thus the same proof may be used now. As for the upper bound, if $n = [\lambda] - 1$, then

$$\begin{aligned} P[P(a) \geqq \lambda a^\alpha] &= P[P(1) \geqq \lambda] \\ &\leqq P[|X(i) - X(i-1)| \leqq 2, i = 1, \ldots, n] \\ &\leqq c_{26}^n, \end{aligned}$$

and this gives the desired result if c_{25} is chosen less than $\log(1/c_{26})$.

The asymptotic behaviour of the first passage times for a general process with stable components can now be easily obtained from the corresponding information for the stable processes. The process under consideration will be

$$X(t)=\big(X_1(t), X_2(t), \ldots, X_N(t)\big),$$

where the $X_i(t)$ are mutually independent stable processes with $X_i(t)$ being in dimension d_i and having index α_i. We recall the convention (1.1) that

$$\alpha_N<\alpha_{N-1}<\cdots<\alpha_2<\alpha_1.$$

The notation $P_i(a)$, $T_i(a)$ will be used for the first passage time and sojourn time processes derived from the process $X_i(t)$. Since the form of the tail of the distribution of $P(a)$ for a stable process depends on whether the process is of type A or type B it is clear that the types of the components will be relevant now. It is also clear that the component $X_1(t)$ with the largest index will be dominant since, for small t, $|X_1(t)|$ is likely to be much larger than the other components.

Lemma 4.2. *Let $X(t)$ be a process with stable components satisfying* (1.1). *If $X_1(t)$ is of type A and i is the smallest index such that $X_i(t)$ is of type B, then there are positive constants c_{27}, c_{28}, and λ_0 such that for $a\leqq 1$ and $\lambda_0\leqq\lambda\leqq a^{1-\alpha_1/\alpha_i}$,*

$$e^{-c_{27}\lambda}\leqq P[P(a)\geqq\lambda\, a^{\alpha_1}]\leqq e^{-c_{28}\lambda}.$$

The upper restriction on λ is needed only for the lower bound and is not needed even then if all of the $X_i(t)$ are of type A. If $X_1(t)$ is of type B, then there are positive constants c_{29} and c_{30} such that for $a\leqq 1$ and all λ sufficiently large

$$\exp(-c_{29}\lambda^{\beta})\leqq P[P(a)\geqq\lambda\, a^{\alpha_1}]\leqq\exp(-c_{30}\lambda^{\beta}),$$

where $\beta=1/(1-\alpha_1)$.

Proof. These results follow by simply applying the bounds given in Lemmas 2.2, 2.3, and 4.1 to the inclusion relations

$$\bigcap_{i=1}^{N}\left[P_i\left(\frac{a}{N}\right)\geqq\lambda\, a^{\alpha_1}\right]\subset[P(a)\geqq\lambda\, a^{\alpha_1}]\subset[P_1(a)\geqq\lambda\, a^{\alpha_1}].$$

In dealing with the first passage time process, it is often convenient to utilize the supremum process

$$M(s)=\sup_{0\leqq t\leqq s}|X(t)|.$$

This process is related in a trivial way to the first passage time process, and the estimates of Lemma 4.2 lead directly to estimates on

$$P[M(s)\leqq\lambda\, s^{1/\alpha_1}]$$

for $s\leqq 1$ and λ sufficiently small. But we shall also need estimates on the other tail of the distribution of $M(s)$.

Lemma 4.3. *Let $X(t)$ be a stable process in R^d of index α. Then there is a positive constant λ_0 such that for all s and $\lambda\geqq\lambda_0$,*

$$P[|X(s)|>\lambda\, s^{1/\alpha}]\leqq P[M(s)>\lambda\, s^{1/\alpha}]\leqq 2d P[|X(s)|>\lambda\, s^{1/\alpha}/2d].$$

Proof. All three probabilities are independent of s by the scaling property, and the left hand inequality is trivial. Letting $Y_i(t)$ denote the projection of $X(t)$ on the i^{th} coordinate axis and $S_i(t)$ the corresponding supremum process, we have

$$[M(1)>\lambda]\subset\bigcup_{i=1}^{d}[S_i(1)>\lambda/d]; \qquad [|Y_i(1)|>\lambda/2d]\subset[|X(1)|>\lambda/2d]$$

and these inclusions reduce the d-dimensional case to the one-dimensional one. We shall therefore assume $X(t)$ is in R^1. By the scaling property, if $m(t)$ denotes the median of $X(t)$

$$\tfrac{1}{2}=P[X(t)\geqq m(t)]=P[X(1)\geqq t^{-1/\alpha}m(t)]$$

so that $m(t)=t^{1/\alpha}m(1)$ and $|m(t)|\leqq|m(1)|$ for all $t\leqq 1$. Now let $t_{kn}=k2^{-n}$ for $k=1,2,\dots,2^n$. For $\lambda\geqq 2|m(1)|$, the Lévy inequality ([8], p. 247) yields

$$P\left[\max_{k\leqq 2^n}|X(t_{kn})+m(1-t_{kn})|>\frac{\lambda}{2}\right]\leqq 2P[|X(1)|>\lambda/2]$$

or

$$P[\max_{k\leqq 2^n}|X(t_{kn})|>\lambda]\leqq 2P[|X(1)|>\lambda/2]. \tag{4.1}$$

Finally, since $X(t)$ has right continuous paths,

$$P[M(1)>\lambda]=\lim_{n\to\infty}P[\max_{k\leqq 2^n}|X(t_{kn})|>\lambda], \tag{4.2}$$

and (4.1), (4.2) combine to give the right hand inequality of the lemma for $d=1$.

Lemma 4.4. *Let $X(t)$ be a process with stable components satisfying* (1.1). *Then there is a positive constant c_{31} such that for all $s\leqq 1$ and all λ*

$$P[M(s)>\lambda s^{1/\alpha_1}]\leqq c_{31}\lambda^{-\frac{1}{2}\alpha_N}.$$

Proof. For $s\leqq 1$, and all λ sufficiently large,

$$\begin{aligned}P[M(s)>\lambda s^{1/\alpha_1}]&\leqq\sum_{i=1}^{N}P[M_i(s)>\lambda s^{1/\alpha_1}/N]\\&\leqq\sum_{i=1}^{N}2d_iP[|X_i(1)|>\lambda s^{1/\alpha_1-1/\alpha_i}/2Nd_i]\\&\leqq\sum_{i=1}^{N}2d_iP[|X_i(1)|>\lambda/2Nd_i].\end{aligned}$$

The fact that $\lambda^{\alpha_i-\varepsilon}P[|X_i(1)|>\lambda]$ is bounded for any $\varepsilon>0$ is a consequence of Theorem 4.2 of [11] (this is presumably true even with $\varepsilon=0$, as it is in R^1, in which case the bound in the lemma could be improved to be $c_{31}\lambda^{-\alpha_N}$). This completes the proof of the lemma since it is trivially true for small λ.

The tail estimates of Lemma 4.2 and 4.4 would now allow us to prove that if $\varphi(a)=a^{\alpha_1}(\log|\log a|)^{1/\beta}$, where $\beta=1$ or $1/(1-\alpha_1)$ according as $X_1(t)$ is of type A or type B, then for a suitable small c_{32} and all $\delta>0$,

$$P[\sup_{0<a\leqq\delta}P(a)/\varphi(a)<c_{32}]=0$$

by using standard Borel-Cantelli arguments. This would imply that when γ is small $P[\sup_{\gamma \leqq a \leqq \delta} P(a)/\varphi(a) < c_{32}]$ is small. At a later stage in the argument we will require quantitative estimates of these probabilities so we estimate them now. We make no attempt to obtain best possible results here.

Lemma 4.5. *Let $X(t)$ be a process with stable components satisfying* (1.1). *Then there are positive constants c_{32}, c_{33}, and γ_0 such that*

$$P[\sup_{\gamma \leqq a \leqq \delta} P(a)/\varphi(a) < c_{32}] < \exp[-c_{33}(\log 1/\gamma)^{\frac{1}{8}}]$$

provided $0 < \gamma \leqq \gamma_0$ and $\delta \geqq \gamma^{\frac{1}{6}}$ where

$$\varphi(a) = \begin{cases} a^{\alpha_1} \log|\log a|, & \text{if } X_1(t) \text{ is type } A, \\ a^{\alpha_1}(\log|\log a|)^{1-\alpha_1}, & \text{if } X_1(t) \text{ is type } B. \end{cases}$$

Proof. With Lemmas 4.2 and 4.4 at our disposal, we have all the necessary estimates so that we can follow the proof of Lemma 8 of [15] (pp. 1242/43). (Lemma 4.4 is required since the event H_k defined on page 1243 should read $\{M(t_{k+1}) > c_{26}\,\psi(t_k)\}$.) The only change necessary in the proof is that the upper estimate for q_k becomes $e^{-c_{34}k}$, where $c_{34} = \alpha_N/2\alpha_1$.

Theorem 4.1. *Let $X(t)$ be a process with stable components satisfying* (1.1). *Then there is a positive constant c_{35} such that with probability one*

$$\limsup_{a \to 0} \frac{P(a)}{\varphi(a)} = c_{35},$$

where $\varphi(a)$ is as in Lemma 4.5.

Proof. Letting $\gamma \to 0$ in Lemma 4.5 shows that

$$\limsup_{a \to 0} P(a)/\varphi(a) \geqq c_{32}$$

with probability one. By Lemma 4.2,

$$P[P(a) \geqq c_{36}\,\varphi(a)] \leqq \exp(-2\log|\log a|) = |\log a|^{-2}$$

if $c_{36} \geqq \max[2c_{28}^{-1}, (2c_{30}^{-1})^{1-\alpha_1}]$. Letting $a_k = e^{-k}$ and applying Borel-Cantelli shows that

$$\limsup_{k \to \infty} P(a_k)/\varphi(a_k) \leqq c_{36}$$

almost surely. But for $a_{k+1} \leqq a < a_k$,

$$\frac{P(a)}{\varphi(a)} \leqq \frac{P(a_k)}{\varphi(a_k)} \cdot \frac{\varphi(a_k)}{\varphi(a_{k+1})}$$

and the latter ratio is bounded. Thus we have the desired lim sup positive and finite with probability one. The fact that it is constant follows from the Blumenthal zero-one law.

5. Sojourn Times

The results of the last two sections can now be combined to give estimates on the tail of the sojourn time distributions. But first we need to estimate the first moment of $T(a)$. We use ρ to denote the constant ρ_1 of Section 1, since ρ_2 is not relevant for the present.

Lemma 5.1. *Let $X(t)$ be a process with stable components satisfying* (1.1), $N \geqq 2$. *Then there are positive constants c_{37} and c_{38} such that for all $s \geqq a^{\alpha_2}$, $a \leqq 1$,*

$$c_{37}\, a^{\alpha_1} \leqq E\{T(a,s)\} \leqq E\{T(a)\} \leqq c_{38}\, a^{\alpha_1}, \qquad \text{if } \alpha_1 < d_1,$$

$$c_{37}\, a^{\rho} \leqq E\{T(a,s)\} \leqq E\{T(a)\} \leqq c_{38}\, a^{\rho}, \qquad \text{if } \alpha_1 > d_1,$$

where $\rho = 1 + \alpha_2 - \alpha_2/\alpha_1$.

Proof. If $\alpha_1 < d_1$,

$$E\{T(a)\} = \int_0^\infty P[|X(t)| \leqq a]\, dt \leqq \int_0^\infty P[|X_1(1)| \leqq a\, t^{-1/\alpha_1}]\, dt$$

$$\leqq \int_0^{a^{\alpha_1}} 1\, dt + \int_{a^{\alpha_1}}^\infty c_{39} (a\, t^{-1/\alpha_1})^{d_1}\, dt = c_{38}\, a^{\alpha_1}.$$

When $\alpha_1 > d_1$,

$$E\{T(a)\} \leqq \int_0^\infty P[|X_1(1)| \leqq a\, t^{-1/\alpha_1}]\, P[|X_2(1)| \leqq a\, t^{-1/\alpha_2}]\, dt$$

$$\leqq \int_0^{a^{\alpha_2}} c_{40}\, a\, t^{-1/\alpha_1}\, dt + \int_{a^{\alpha_2}}^\infty c_{41}\, a^2\, t^{-1/\alpha_1 - 1/\alpha_2}\, dt$$

$$= c_{38}\, a^{\rho}.$$

For the lower bound, when $\alpha_1 < d_1$,

$$E\{T(a,s)\} \geqq \int_0^s \prod_{i=1}^N P[|X_i(t)| \leqq a/N]\, dt$$

$$\geqq \int_0^{a^{\alpha_1}} \prod_{i=1}^N P[|X_i(1)| \leqq a\, t^{-1/\alpha_i}/N]\, dt$$

$$\geqq c_{37}\, a^{\alpha_1},$$

since the probability of being in the sphere of radius $1/N$, center the origin, is positive for any stable process. If $\alpha_1 > d_1$,

$$E\{T(a,s)\} \geqq \int_{\frac{1}{2}a^{\alpha_1}}^{a^{\alpha_2}} \prod_{i=1}^N P[|X_i(1)| \leqq a\, t^{-1/\alpha_i}/N]\, dt$$

$$\geqq \int_{\frac{1}{2}a^{\alpha_1}}^{a^{\alpha_2}} c_{42}\, a\, t^{-1/\alpha_1}\, dt = c_{43} (a^{\rho} - 2^{1/\alpha_1 - 1}\, a^{\alpha_1})$$

$$\geqq c_{37}\, a^{\rho}$$

for $a \leqq 1$. In this part, we have also used the fact that $X_1(t)$ must be of type A and so the density of $X_1(1)$ is bounded below on the interval $[-2, 2]$.

We need to estimate the tail of the distribution of $T(a, s)$ for the process $X(t) = (X_1(t), X_2(t), \ldots, X_N(t))$. The form of the tail is determined by the first component when this is transient, but when $\alpha_1 > d_1$, the type of the second component is important. The four cases resulting are tabulated in the statement of Lemma 5.2. For convenience we introduce a notation

$$X_0(t) = (X_2(t), \ldots, X_N(t))$$

for the process with the first component deleted, and $P_0(a)$ for the first passage time process obtained from $X_0(t)$.

Lemma 5.2. *Let $X(t)$ be a process with stable components satisfying* (1.1), $N \geqq 2$. *Then there are positive constants c_{44}, c_{45}, c_{46}, c_{47}, and λ_0 such that for $a \leqq c_{44}$ and $\lambda_0 \leqq \lambda \leqq a^{-c_{45}}$,*

$$\exp(-c_{46}\lambda^{\beta}) \leqq P[T(a, \lambda^{\xi} a^{\delta}) \geqq \lambda a^{\gamma}]$$
$$\leqq P[T(a) \geqq \lambda a^{\gamma}] \leqq \exp(-c_{47}\lambda^{\beta}),$$

where β, γ, δ, and ξ are given by

Case	β	γ	δ	ξ
$\alpha_1 < d_1$, $X_1(t)$ *is type A*	1	α_1	α_1	1
$\alpha_1 < d_1$, $X_1(t)$ *is type B*	$1/(1-\alpha_1)$	α_1	α_1	1
$\alpha_1 > d_1$, $X_2(t)$ *is type A*	1	ρ	α_2	1
$\alpha_1 > d_1$, $X_2(t)$ *is type B*	$1/(2-\rho)$	ρ	α_2	$(1-\alpha_2)/(2-\rho)$

and $\rho = 1 + \alpha_2 - \alpha_2/\alpha_1$. The upper restrictions on λ and a are only needed for the lower bounds.

Proof. The first two cases ($\alpha_1 < d_1$) follow on applying the estimates of Lemmas 2.2, 2.3, and 4.2 to the inclusion relations

$$[P(a) \geqq \lambda a^{\alpha_1}] \subset [T(a, \lambda a^{\alpha_1}) \geqq \lambda a^{\alpha_1}] \subset [T(a) \geqq \lambda a^{\alpha_1}] \subset [T_1(a) \geqq \lambda a^{\alpha_1}].$$

When $\alpha_1 > d_1$, the lower bound follows on using Lemmas 3.3 and 4.2 and the independence of P_0 and T_1, with

$$[P_0(a/2) \geqq \lambda^{\xi} a^{\alpha_2}] \cap [T_1(a/2, \lambda^{\xi} a^{\alpha_2}) \geqq \lambda^{\beta/\alpha_1} a(\lambda^{\xi} a^{\alpha_2})^{(1-1/\alpha_1)}] \subset [T(a, \lambda^{\xi} a^{\alpha_2}) \geqq \lambda a^{\rho}].$$

For the upper bound when $\alpha_1 > d_1$ and $X_2(t)$ is type A, it suffices to use the estimate on $E\{T(a)\}$ given in Lemma 5.1 in the Corollary to Lemma 3.1. To obtain the upper bound in the final case, we need the fact ([15], p. 1233) that since $X_2(t)$ is of type B, there is a line L through the origin such that if $X_L(t)$ is the projection of $X_2(t)$ on L, then $X_L(t)$ is a stable subordinator of index α_2. Now

$$P[T(a) \geqq \lambda a^{\rho}] \leqq P[T_1(a, \lambda^{\xi} a^{\alpha_2}) \geqq \lambda a^{\rho}] + P[P_L(2a) \geqq \lambda^{\xi} a^{\alpha_2}],$$

where $P_L(a)$ is the first passage process derived from $X_L(t)$. This is valid since once $|X_L(s)| > a$, it cannot decrease and thus $|X(t)| > a$ for all $t \geqq s$. Applying Lemmas 3.3 and 4.2 completes the proof.

Lemma 5.3. *Let $X(t)$ be a process with stable components satisfying* (1.1), $N \geqq 2$. *Then for every $s>0$ there are positive constants c_{48}, c_{49}, and γ_0 such that*

$$P\Big[\sup_{\gamma \leqq a \leqq \delta} T(a,s)/\varphi(a) < c_{48}\Big] < \exp\big[-c_{49}(\log 1/\gamma)^{\frac{1}{8}}\big]$$

provided $0<\gamma \leqq \gamma_0$ and $\delta \geqq \gamma^{\frac{1}{6}}$ where

$$\varphi(a)=\begin{cases} a^{\alpha_1}\log|\log a| & \text{if } \alpha_1<d_1 \text{ and } X_1(t) \text{ is type } A,\\ a^{\alpha_1}(\log|\log a|)^{1-\alpha_1} & \text{if } \alpha_1<d_1 \text{ and } X_1(t) \text{ is type } B,\\ a^{\rho}\log|\log a| & \text{if } \alpha_1>d_1 \text{ and } X_2(t) \text{ is type } A,\\ a^{\rho}(\log|\log a|)^{2-\rho} & \text{if } \alpha_1>d_1 \text{ and } X_2(t) \text{ is type } B. \end{cases}$$

Proof. When $\alpha_1<d_1$, this follows from Lemma 4.5 and the inclusion

$$\left[\sup_{\gamma \leqq a \leqq \delta} \frac{T(a,s)}{\varphi(a)} < c_{48}\right] \subset \left[\sup_{\gamma \leqq a \leqq \gamma^{\frac{1}{6}}} \frac{P(a)}{\varphi(a)} < c_{48}\right],$$

which is valid if γ is small enough to ensure that $c_{48}\,\varphi(\gamma^{\frac{1}{6}}) \leqq s$.

However, if $\alpha_1>d_1$, $T(a)$ has a different order of magnitude, and we have to work harder to overcome the independence difficulties. For positive integers k, let

$$a_k=\exp(-k^2), \qquad c_{48}=2^{-\rho}(3c_{46})^{-1/\beta},$$
$$s_k=c_{48}^{\xi}\, 2^{\xi\rho-\alpha_2}\, a_k^{\alpha_2}(\log|\log a_k|)^{\xi/\beta},$$

where ξ and β are as defined in the last two cases of Lemma 5.2. For a fixed integer m, define stopping times τ_k, $k=2m, 2m-1, \ldots, m$ by

$$\tau_{2m}=0,$$
$$\tau_k=\inf\{t \geqq \tau_{k+1}+s_{k+1}:\ X_1(t)=0\}.$$

These are all finite with probability 1 since X_1 is point recurrent. Put

$$T_k=\int_{\tau_k}^{\tau_k+s_k} I_{\frac{1}{2}a_k}(X(t)-X(\tau_k))\,dt;$$

clearly the random variables T_k are independent. Using the last two cases of Lemma 5.2 we have

$$P[T_k \geqq c_{48}\,\varphi(a_k)] \geqq \exp[-c_{46}(c_{48}\,2^{\rho}\{\log|\log a_k|\}^{1/\beta})^{\beta}]$$
$$=|\log a_k|^{-\frac{1}{3}}=k^{-\frac{2}{3}}.$$

If Q_m is the event $[T_k<c_{48}\,\varphi(a_k);\ k=2m, 2m-1, \ldots, m+1]$,

$$P(Q_m) \leqq \prod_{k=m+1}^{2m}(1-k^{-\frac{2}{3}}) \leqq \exp\Big(-\sum_{k=m+1}^{2m} k^{-\frac{2}{3}}\Big).$$

Thus

$$P(Q_m) \leqq \exp(-c_{50}\, m^{\frac{1}{3}}). \tag{5.1}$$

If we put

$$R_m=\bigcup_{k=m+1}^{2m}\big[|X(\tau_k)|=|X_0(\tau_k)|>\tfrac{1}{2}a_k\big],$$

we can estimate $P(R_m)$ by

$$\begin{aligned} R_m &\subset \bigcup_{k=m+1}^{2m-1} [|X_0(\tau_k)-X_0(\tau_{k+1})|>\tfrac{1}{2}(a_k-a_{k+1})] \\ &\subset \bigcup_{k=m+1}^{2m-1} (A_k \cup B_k)=S_m, \end{aligned}$$

where

$$\begin{aligned} A_k &= [|\tau_k-\tau_{k+1}|>e^{\alpha_2 k} s_{k+1}], \\ B_k &= [\sup_{\tau_{k+1}\leqq t\leqq \tau_{k+1}+e^{\alpha_2 k} s_{k+1}} |X_0(t)-X_0(\tau_{k+1})|>\tfrac{1}{2}(a_k-a_{k+1})]. \end{aligned}$$

Lemma 2.1 gives an estimate for

$$P(A_k)=P[X_1(t) \text{ has no zero in } [s_{k+1}, e^{\alpha_2 k} s_{k+1}),$$

while Lemma 4.4 provides an estimate for

$$P(B_k)=P[M_0(e^{\alpha_2 k} s_{k+1})>\tfrac{1}{2}(a_k-a_{k+1})],$$

yielding

$$P(A_k \cup B_k)<e^{-c_{51} k}, \qquad \text{for large } k;$$

and then

$$P(R_m)\leqq P(S_m)<m\, e^{-c_{51} m}, \qquad \text{for large } m. \tag{5.2}$$

Now if the path is not in R_m, the sphere of radius $\frac{1}{2}a_k$, centre at $X(\tau_k)$, will be contained in the sphere of radius a_k, centred at the origin, so that

$$T(a_k, \tau_k+s_k)\geqq T_k, \qquad k=m+1, m+2, \ldots, 2m.$$

Furthermore, since

$$\sum_{k=1}^{\infty} e^{\alpha_2 k} s_{k+1}<\infty,$$

if the path is not in S_m, then

$$\tau_k+s_k\leqq \tau_{m+1}+s_{m+1}\leqq s_{m+1}+\sum_{k=m+1}^{2m-1} e^{\alpha_2 k} s_{k+1}\leqq s, \qquad k=m+1,\ m+2, \ldots, 2m,$$

for m sufficiently large. It follows that

$$\begin{aligned} P[\sup_{m\leqq k\leqq 2m} T(a_k, s)/\varphi(a_k)<c_{48}] &\leqq P(Q_m)+P(S_m) \\ &\leqq \exp(-c_{50}\, m^{\frac{1}{3}})+m\exp(-c_{51}\, m) \\ &< \exp(-m^{\frac{1}{4}}), \end{aligned}$$

for sufficiently large m, using (5.1) and (5.2). Now if γ is sufficiently small and $\delta\geqq\gamma^{\frac{1}{6}}$, and if we set $m=[(-\log\gamma^{\frac{1}{5}})^{\frac{1}{2}}]$, then

$$\gamma<a_k<\delta \qquad \text{for } m\leqq k\leqq 2m.$$

It follows that

$$P[\sup_{\gamma\leqq a\leqq\delta} T(a, s)/\varphi(a)<c_{48}]<\exp(-m^{\frac{1}{4}})<\exp[-c_{49}(\log 1/\gamma)^{\frac{1}{8}}].$$

Theorem 5.1. *Let $X(t)$ be a process with stable components satisfying* (1.1) *and $N \geqq 2$. Then there is a positive constant c_{52} such that with probability one*

$$\limsup_{a \to 0} \frac{T(a,s)}{\varphi(a)} = c_{52},$$

for all $0 < s \leqq \infty$ where $\varphi(a)$ is defined in Lemma 5.3.

Proof. This is proved using Lemmas 5.2 and 5.3 exactly as Theorem 4.1 was.

6. The Exact Hausdorff Measure of the Sample Paths

The only estimate that must still be obtained in order to follow the usual line of proof (see [9, 14, 15], and [17]) for a correct measure function is one for the expected number of cubes in a covering collection that the process $X(t)$ will hit. Once more it is just as easy to obtain the necessary lemma for general processes with stationary, independent increments. A similar method was used in [6], but it was simpler in that case since the number of cubes could be related to first passage times instead of sojourn times.

Instead of taking a particular family of cubes, we note that for a collection $\Lambda(a)$ of cubes of side a in R^d, the property that no point of R^d is contained in more that c_{53} cubes of $\Lambda(a)$ is equivalent to the property that no sphere of radius a in R^d intersects more than c_{54} cubes of $\Lambda(a)$. For any fixed positive constant c_{54} we call a collection satisfying the latter property *c_{54}-nested.*

Lemma 6.1. *Let $X(t)$ be a process in R^d with stationary independent increments and suppose $\Lambda(a)$ is a fixed c_{54}-nested collection of cubes of side $a (a \leqq 1)$, in R^d. If $M(a,s)$ is the number of these cubes hit by the path $X(t)$ at some time $t \in [0,s]$, then*

$$E\,M(a,s) \leqq 2s\,c_{54}\,[E\,T(a/3,s)]^{-1}.$$

Proof. Let $\tau_0 = 0$, and define stopping times

$$\tau_k = \inf\{t \geqq \tau_{k-1} \colon\ |X(t) - X(\tau_j)| > a \ \text{ for } j = 0, 1, \ldots, k-1\},$$

for $k = 1, 2, \ldots$. Then $|X(\tau_k) - X(\tau_j)| \geqq a$ for $j \neq k$ so that if S_k is the sphere centre $X(\tau_k)$ and radius $a/3$, the S_k are disjoint. Let T_k be the sojourn time in S_k after τ_k and before $\tau_k + s$; put

$$\eta = \min\{k \colon\ \tau_k > s\},$$

and let I_k be the indicator of the event

$$[\eta - 1 \geqq k] = [\tau_k \leqq s].$$

Since I_k is determined by the path up to τ_k, and T_k by the increments after τ_k,

$$E(I_k\,T_k) = E(I_k)\,E\{T(a/3,s)\}.$$

But $\sum_{k=0}^{\infty} I_k\,T_k \leqq 2s$, so that, since $\eta = \sum_{k=0}^{\infty} I_k$,

$$E(\eta)\,E\{T(a/3,s)\} = E\Big(\sum_{k=0}^{\infty} I_k\,T_k\Big) \leqq 2s.$$

Finally, note that the path $X(t)$ for $0 \leqq t \leqq s$ is covered by spheres centre $X(\tau_k)$, radius a, for $k=0, 1, \ldots, \eta-1$; and that each of these spheres can intersect at most c_{54} cubes of $\Lambda(a)$, so that

$$M(a, s) \leqq c_{54} \eta.$$

Corollary. *Let $X(t)$ be a process with stable components satisfying* (1.1), *and suppose $\Lambda(a)$, $M(a, s)$ are as in Lemma* 6.1. *Then there is a positive constant c_{55} such that, for all $a \leqq s^{1/\alpha_2}$,*

$$E\,M(a, s) \leqq c_{55}\, s\, a^{-\alpha_1}, \quad \text{if } \alpha_1 < d_1,$$

$$E\,M(a, s) \leqq c_{55}\, s\, a^{-\rho}, \quad \text{if } \alpha_1 > d_1,$$

where $\rho = 1 + \alpha_2 - \alpha_2/\alpha_1$.

Proof. This is a combination of Lemma 6.1 with the estimates of Lemma 5.1.

Remark. As in [15] we could use the particular collections $\Lambda(a_n)$ of half-open cubes of side $a_n = 2^{-n}$ centered at the dyadic points $(x_1/2^n, x_2/2^n, \ldots, x_d/2^n)$ where $x_1, \ldots, x_d$ are integers. It is clear that this collection is c_{54}-nested with $c_{54} = 3^d$.

We are now ready to state our main theorem giving the precise Hausdorff measure function for which the sample paths have positive finite measure. If $\alpha_1 < d_1$, only the first component matters while, for $\alpha_1 > d_1$ both the index and the type of the second component affect the result.

Theorem 6.1. *Let $X(t)$ be a process with stable components satisfying* (1.1), *$N \geqq 2$, and*

$$\varphi(h) = \begin{cases} h^{\alpha_1} \log|\log h| & \text{if } \alpha_1 < d_1 \text{ and } X_1(t) \text{ is type } A, \\ h^{\alpha_1} (\log|\log h|)^{1-\alpha_1} & \text{if } \alpha_1 < d_1 \text{ and } X_1(t) \text{ is type } B, \\ h^{\rho} \log|\log h| & \text{if } \alpha_1 > d_1 \text{ and } X_2(t) \text{ is type } A, \\ h^{\rho} (\log|\log h|)^{2-\rho} & \text{if } \alpha_1 > d_1 \text{ and } X_2(t) \text{ is type } B. \end{cases}$$

Then there is a positive, finite constant c_{56} such that $\varphi - m[R(s)] = c_{56}\, s$ for all $s \geqq 0$ with probability one, where $R(s)$ denotes the range of $X(t)$ up to time s.

Proof. The proof is now exactly that of Theorem 6 of [15] using Theorem 5.1 and the density theorem of [10] for the lower bound; and Lemma 5.3 and the Corollary to Lemma 6.1 in the usual way to estimate the number of "bad" cubes in the collection which the path hits but visits for only a short time, for the upper bound.

7. Asymptotic Behaviour of $P(a)$, $T(a)$ for Large a

Many authors have observed that, for a stable process, the asymptotic laws for $P(a)$, $T(a)$ as $a \to \infty$ have a form very similar to those as $a \to 0$. This is no longer true for the class of processes we are considering since, while a^{α_1} is the smallest function for small a, a^{α_N} will be for large a. We shall make no use of the results of the present section as they are not relevant to the local structure of the sample paths — or to the right measure function which depends only on the local structure.

Theorem 7.1. *Let $X(t)$ be a process with stable components satisfying* (1.1). *Then there is a positive constant c_{57} such that with probability one*

$$\limsup_{a\to\infty} \frac{P(a)}{\varphi(a)} = c_{57},$$

where

$$\varphi(a) = \begin{cases} a^{\alpha_N} \log\log a, & \text{if } X_N(t) \text{ is type } A, \\ a^{\alpha_N} (\log\log a)^{1-\alpha_N}, & \text{if } X_N(t) \text{ is type } B. \end{cases}$$

Theorem 7.2. *Let $X(t)$ be a process with stable components satisfying* (1.1). *Then there is a positive constant c_{58} such that with probability one*

$$\limsup_{a\to\infty} \frac{T(a)}{\varphi(a)} = c_{58},$$

with

$$\varphi(a) = \begin{cases} a^{\alpha_N} \log\log a, & \text{if } \alpha_N < d_N,\ X_N(t) \text{ is type } A, \\ a^{\alpha_N} (\log\log a)^{1-\alpha_N}, & \text{if } \alpha_N < d_N,\ X_N(t) \text{ is type } B, \\ a^{\rho_N} \log\log a, & \text{if } \alpha_N > d_N, \end{cases}$$

where $\rho_N = 1 + \alpha_{N-1} - \alpha_{N-1}/\alpha_N$.

Remark. There is no longer a dichotomy based on the type of $X_{N-1}(t)$ when $\alpha_N > d_N$ since then $\alpha_{N-1} > 1$ and $X_{N-1}(t)$ must be type A.

8. Graphs of Processes with Stable Components

In [6] the graphs of transient stable processes were considered. The correct measure function for the graph of a point recurrent stable process has been given in Theorem 1.3, which we shall prove in this section. A combination of the methods of these two papers will suffice to obtain the correct measure function for the graph of a general process with stable components. The basic idea is that if $\alpha_1 \leqq 1$, the time component is dominant, if $\alpha_1 > d_1 \geqq \alpha_2$, the graph of $X_1(t)$ is dominant, while if $d_1 \geqq \alpha_1 > 1$ or $\alpha_2 > 1$, the range of $X(t)$ is the relevant part.

Theorem 8.1. *Let $X(t)$ be a process with stable components satisfying* (1.1), *and $G(s)$ be the set of points in R^{d+1} on the graph up to time s,*

$$G(s) = G(s, \omega) = \{(X(t, \omega), t):\ 0 \leqq t \leqq s\}.$$

If

$$\varphi(a) = \begin{cases} a & \text{if } \alpha_1 \leqq 1, \\ a^{2-1/\alpha_1} (\log|\log a|)^{1/\alpha_1} & \text{if } \alpha_1 > d_1 \geqq \alpha_2, \\ \text{as in Theorems 6.1 and 9.1} & \text{if } d_1 \geqq \alpha_1 > 1 \text{ or } \alpha_2 > 1, \end{cases}$$

then there is a positive constant c_{59} such that

$$\varphi - m[G(s)] = c_{59}\, s$$

for all $s \geqq 0$ almost surely.

The first case can be handled most easily by the methods used in [6], while the last case can be treated in much the same way as the range has been treated. The middle case is essentially the same as the graph of a recurrent stable process, which we now consider in detail.

Proof of Theorem 1.3. Let $X(t)$ be a stable process in R^1 of index $\alpha>1$ and $Y(t)=(X(t), t)$ the corresponding space-time process. Then if $T(a,a)$ denotes the time spent by the $X(t)$ process in the sphere of radius a up to time a, we can think of it also as the total sojourn time of $Y(t)$ in the rectangle $\{|X(t)|\leqq a, 0\leqq t\leqq a\}$. (We could, if we wished, obtain information about the sojourn time of $Y(t)$ in a sphere of radius a, but the results for rectangles of this type are sufficient.) Now Lemma 3.3 yields the estimates

$$\exp(-c_{19}\lambda^{\alpha})\leqq P[T(a,a)\geqq \lambda a^{2-1/\alpha}]\leqq \exp(-c_{20}\lambda^{\alpha})$$

which plays the role of Lemma 5.2 for this case. Given an integer m, the sequence of stopping times τ_k, for $k=2m, 2m-1, \ldots, m$ is defined by $\tau_{2m}=0$, and for $k<2m$

$$\tau_k=\inf\{t\geqq \tau_{k+1}+\tfrac{1}{2}a_{k+1}\colon X(t)=0\};$$

also let

$$T_k=\int_{\tau_k}^{\tau_k+\frac{1}{2}a_k} I_{\frac{1}{2}a_k}(X(t))\,dt,$$

where

$$a_k=\exp(-k^2).$$

Then, as in the proof of Lemma 5.3, if $c_{60}=2^{-2+1/\alpha}(3c_{19})^{-1/\alpha}$,

$$P\left[\sup_{m+1\leqq k\leqq 2m}\frac{T_k}{\varphi(a_k)}<c_{60}\right]\leqq \exp(-c_{50}m^{\frac{1}{3}}), \tag{8.1}$$

where $\varphi(a)=a^{2-1/\alpha}(\log|\log a|)^{1/\alpha}$. In this case, we let

$$R_m=\bigcup_{k=m+1}^{2m}[\tau_k>\tfrac{1}{2}a_k]\subset \bigcup_{k=m+1}^{2m}[\tau_k-\tau_{k+1}>\tfrac{1}{2}(a_k-a_{k+1})]$$

and Lemma 2.1 provides an estimate of the form

$$P(R_m)<m\,e^{-c_{61}m} \tag{8.2}$$

as before. Then, if the path is not in R_m, we have

$$(T(a_k, a_k))\geqq T_k, \qquad k=m+1,\ m+2, \ldots, 2m,$$

and (8.1), (8.2) may be combined to give an estimate which takes the place of Lemma 5.3. Finally, $EM(a,s)$ for this situation can be estimated from Lemmas 3.2 and 6.1. This is enough to complete the proof of Theorem 1.3.

9. Some of the Critical Cases

As pointed out in the introduction we cannot deal at all with the case where one of the $\alpha_i=1$, unless the corresponding component satisfies the scaling property (2.1). This will be true if and only if the defining stable distribution is given by a

μ on S_d whose centre of mass

$$\int_{S_d} \theta\, \mu(d\theta)=0;$$

so, in particular, we will have the scaling property if μ is symmetric about the origin. If the scaling property is satisfied, then all the arguments of Sections 4, 5, and 6 are valid provided we do not have

$$\alpha_1=d_1=1 \quad \text{or} \quad \alpha_1=d_1=2. \tag{9.1}$$

The case $\alpha_i=d_i=2$, $i\geqq 2$ cannot arise, and $\alpha_i=d_i=1$, $i\geqq 2$ will make no difference to any of the arguments since, for components other than the first, it is the first passage time distribution that matters and this has the same form for $\alpha=1$. In Section 7, the critical case will be $\alpha_N=d_N=1$, and in Section 8, the only critical case is $\alpha_1=d_1=2$. We do not give the detailed computations in the present section, but rather just summarise the results and indicate any differences in the methods required to obtain them.

In order to obtain the results of Sections 5 and 6 when $X_1(t)$ is interval recurrent, we simply need the analogs of Lemmas 5.1, 5.2, and 5.3. Arguments similar to those used in the proof of Lemma 5.1 yield:

Lemma 9.1. *If $X(t)$ is a process with stable components satisfying* (1.1) *and $N\geqq 2$, $\alpha_1=d_1$, then there are positive constants c_{62}, c_{63} such that for all $s\geqq a^{\alpha_2}$ and $a\leqq\frac{1}{2}$,*

$$c_{62}\, a^{\alpha_1}|\log a|\leqq E\{T(a,s)\}\leqq E\{T(a)\}\leqq c_{63}\, a^{\alpha_1}|\log a|.$$

When $N=1$, $T(a)=\infty$, but the upper and lower bounds are still valid for $E\{T(a,s)\}$ provided $a^{\gamma}\leqq s\leqq 1$ for some $\gamma<\alpha_1$.

In order to replace Lemma 5.2, we first need a result to take the place of Lemma 3.3 for the interval recurrent processes. To prove this result we shall make use of an inequality which holds for both planar Brownian motion and the symmetric Cauchy process on the line. For $\gamma<\alpha$, $a<1$, and $|x|\leqq\frac{1}{2}a$,

$$P^x\Big[\inf_{a^{\alpha}\leqq t\leqq a^{\gamma}}|X(t)|\geqq\tfrac{1}{2}a\Big]\leqq\frac{c_{64}}{|\log a|}. \tag{9.2}$$

For Brownian motion this can be proved by combining the logarithmic potential theory estimates for the probability of exiting from the sphere of radius $a^{(\gamma+\alpha)/2\alpha}$ prior to hitting the sphere of radius $\frac{1}{2}a$ starting from $X(a^{\alpha})$ with the estimates for the distribution of $P(a^{(\gamma+\alpha)/2\alpha})$ given in Lemma 4.1. In the Cauchy case, it can be obtained by utilizing the connection between the Cauchy process and planar Brownian motion as in the proof of Lemma 12 of [16].

Lemma 9.2. *Let $X(t)$ be a stable process in R^d of index $\alpha=d=1$ or 2. Then if $\gamma<\alpha$, there are positive constants c_{65}, c_{66}, c_{67}, and λ_0 such that for all $a^{\gamma}\leqq s\leqq 1$, $\lambda_0\leqq\lambda\leqq a^{\frac{1}{2}(\gamma-\alpha)}$, $a\leqq c_{67}$,*

$$e^{-c_{65}\lambda}\leqq P[T(a,s)\geqq\lambda\, a^{\alpha}|\log a|]\leqq e^{-c_{66}\lambda}.$$

Proof. The upper bound is obtained in the usual way from Lemma 9.1 and the Corollary to Lemma 3.1. For the lower bound, we use a combination of the methods of Ray [9] with those of the present paper. The methods of [9] need to be modified since the Cauchy process is not continuous and they can be somewhat simplified at the same time. Define a sequence of stopping times by

$$\tau_0=0, \quad \tau_k=\inf\{t\geqq\tau_{k-1}+a^\alpha\colon\ |X(t)|\leqq\tfrac{1}{2}a\}.$$

Since the process is interval recurrent, these times are finite a.s. and $|X(\tau_k)|\leqq\frac{1}{2}a$. Let T_k be the time spent in the sphere of centre $X(\tau_{k-1})$, radius $\frac{1}{2}a$, during the interval $(\tau_{k-1}, \tau_{k-1}+a^\alpha)$. Then $T_1, T_2, \ldots, T_n$ are independent, identically distributed and

$$T(a,\tau_n)\geqq\sum_{i=1}^{n} T_i.$$

It follows that

$$P[T(a,s)<\lambda\, a^\alpha\,|\log a|]\leqq P[\tau_n\geqq a^\gamma]+P\left[\sum_{i=1}^{n} T_i<\lambda\, a^\alpha\,|\log a|\right]. \tag{9.3}$$

To estimate the first term, we note that for $n\leqq a^{2(\gamma-\alpha)/3}$,

$$P[\tau_n<a^\gamma]\geqq P[\tau_k-\tau_{k-1}<a^\gamma/n,\ k=1,2,\ldots,n]\geqq\left(1-\frac{c_{64}}{|\log a|}\right)^n, \tag{9.4}$$

where the final inequality is a consequence of (9.2). For the other term we can use the basic inequality since there is a positive constant c_{68} such that

$$E\{\exp(-a^{-\alpha}\,T_1)\}=e^{-c_{68}},$$

and then

$$P\left[\sum_{i=1}^{n} T_i<\lambda\, a^\alpha\,|\log a|\right]\leqq e^{\lambda|\log a|-c_{68}\,n}. \tag{9.5}$$

Letting $n=[2c_{68}^{-1}\,\lambda\,|\log a|]$ and combining (9.3), (9.4), and (9.5) completes the proof.

Arguments similar to those in Section 5 will now suffice to prove the next two lemmas which take the place of Lemmas 5.2 and 5.3. One difference that occurs in the proof of Lemma 9.4 is that the sequence a_k must be defined by

$$a_k=\exp(-c_{69}^k)$$

where $1<c_{69}<\alpha_1/\gamma$. Lemma 9.4 is a modification of Lemma 3 of [14].

Lemma 9.3. *Let $X(t)$ be a process with stable components satisfying* (1.1) *and $N\geqq 2$, $\alpha_1=d_1$. Then, if $\alpha_2<\gamma<\alpha_1$, there are positive constants c_{70}, c_{71}, c_{72}, and λ_0 such that for $a\leqq c_{70}$ and $\lambda_0\leqq\lambda\leqq a^{\frac{1}{2}(\gamma-\alpha_1)}$,*

$$\begin{aligned} e^{-c_{71}\lambda}&\leqq P[T(a,a^\gamma)\geqq\lambda\, a^{\alpha_1}\,|\log a|]\\ &\leqq P[T(a)\geqq\lambda\, a^{\alpha_1}\,|\log a|]\leqq e^{-c_{72}\lambda}.\end{aligned}$$

Lemma 9.4. *Let $X(t)$ be a process with stable components satisfying* (1.1) *and* $\alpha_1 = d_1$. *Then for every $s > 0$ there are positive constants c_{73}, c_{74}, and γ_0 such that*

$$P[\sup_{\gamma \leqq a \leqq \delta} T(a, s)/\varphi(a) < c_{73}] < \exp[-c_{74}(\log|\log \gamma|)^{\frac{1}{2}}]$$

provided $0 < \gamma \leqq \gamma_0$ *and* $\delta \geqq \exp\{-|\log \gamma|^{\frac{1}{3}}\}$, *where*

$$\varphi(a) = a^{\alpha_1} |\log a| \log\log|\log a|.$$

As in Section 5, Lemma 9.4 shows that

$$\limsup_{a \to 0} \frac{T(a, s)}{\varphi(a)} \geqq c_{73}$$

almost surely. The upper bound is a consequence of the corresponding limit law for T_1, since $T(a, s) \leqq T_1(a, s)$. (The result for $T_1(a, s)$ is in [9] for planar Brownian motion, and similar techniques will prove it for the symmetric Cauchy process on the line.) The asymptotic law

$$\limsup_{a \to 0} \frac{T(a, s)}{\varphi(a)} = c_{75} \tag{9.6}$$

then follows; (9.6) also holds for $s = \infty$ when $N \geqq 2$. Now the arguments outlined in Section 6 lead to

Theorem 9.1. *Let $X(t)$ be a process with stable components satisfying* (1.1) *with* $\alpha_1 = d_1 = 1$ *or* 2, *and*

$$\varphi(h) = h^{\alpha_1} |\log h| \log\log|\log h|.$$

Then there is a finite positive constant c_{76} such that $\varphi - m[R(s)] = c_{76}\, s$ for all $s \geqq 0$ with probability one, where $R(s)$ denotes the range of $X(t)$ up to time s.

Finally, we remark that in Section 7 the result of Theorem 7.2 is still valid when $X_N(t)$ is a symmetric Cauchy process on the line provided one takes $\varphi(a) = a \log a \log\log\log a$. The proof of this is similar to that of (9.6).

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