Z. Wahrscheinlichkeitstheorie verw. Gebiete 31, 57–67 (1974)

Point Processes and Completely Monotone Set Functions

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1. Introduction

Inherent in the proof of Renyi's Theorem [9] (as observed by Kallenberg [4] and Mönch [8]) is the fact that the distribution of an orderly point process (a process without multiple points) is determined by the zero probability function $\phi(B) \equiv P\{N(B)=0\}$ ($N(B)$=number of points in B). It is natural to ask what set functions ϕ can arise as zero probability functions of point processes. In this paper we characterize these set functions in terms of a property that can naturally be called *complete monotonicity.*

In his classical paper on capacities [1], Choquet studies a class of set functions which he calls alternating capacities of order infinity. Based on the integral representation of these functions in terms of extremal functions, Choquet observes that a class of these functions can be interpreted as giving the probability that sets intersect a random compact set. The extremal functions are of the form

$$f_T(B) = \begin{cases} 0 & \text{if } T \cap B = \phi \\ 1 & \text{if } T \cap B \neq \phi. \end{cases}$$

An alternating capacity of order infinity $\psi(B)$ can be represented by

$$\psi(B) = \int f_T(B)\, d\mu(T).$$

If μ is in an appropriate sense a probability measure on the collection of compact sets, then Choquet's interpretation is clear. Of course $1-\psi(B)$ is what we call a zero probability function.

Based on Choquet's work Mathéron [6] characterizes the zero probability functions of processes whose realizations are convex closed sets.

The proof of our characterization is based on a simple probabilistic construction of the point process corresponding to a zero probability function and is much closer to the work of Kendall [5] than to that of Choquet. Kendall uses the notion of complete monotonicity in his study of very general set valued processes. His avoidance function corresponds to our zero probability function.

We will assume that the points of our processes lie in a complete, separable, locally compact, metric space X that is σ-compact (i.e. is a countable union of compact sets). We will denote the metric by $\rho(x, y)$ and the σ-algebras of Borel sets by $\mathscr{B}$. All sets considered will be Borel sets. $\bar{B}$ will denote the closure of B and B^0 the interior.

* Research supported in part by the National Science Foundation.

Let $S(X)$ denote the set of all countable collections of points in X, possibly with multiplicities. For $\mathbf{x} \in S(X)$ and $B \in \mathscr{B}$, $N(B, \mathbf{x})$ will denote the cardinality of $\{x \in \mathbf{x}: x \in B\}$, counting multiplicities. Let $\mathscr{S}$ denote the σ-algebra generated by sets of the form $\{\mathbf{x}: N(B, \mathbf{x}) = k\}$. We will think of a point process as a measurable mapping ξ of a probability space $(\Omega, \mathscr{F}, P)$ into $(S(X), \mathscr{S})$. Ordinarily $N(B, \xi)$ will be denoted by $N(B)$, perhaps with appropriate subscripts, and $\phi(B) = P\{N(B) = 0\}$. The process is orderly if $P\{\xi$ has multiple points$\} = 0$.

2. Characterization of Zero Probability Functions

Let ϕ be a real valued function defined on $\mathscr{B}$. For $A \in \mathscr{B}$ define

$$\Delta_A \phi(B) = \phi(A \cup B) - \phi(B).$$

If ϕ is the zero probability function of a point process then

$$-\Delta_A \phi(B) = P\{N(A) > 0, N(B) = 0\}$$

and in general if $A_1 A_2 \ldots A_n \in \mathscr{B}$ then

$$(2.1) \qquad (-1)^n \Delta_{A_1} \Delta_{A_2} \cdots \Delta_{A_n} \phi(B) = P\{N(A_1) > 0 \ldots N(A_n) > 0, \mathscr{N}(B) = 0\}.$$

Classically a completely monotone function is a non-negative function f defined on $(0, \infty)$ and satisfying $(-1)^n \Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_n} f(x) \geqq 0$ for all $x, h_1, h_2, \ldots, h_n > 0$ where $\Delta_h f(x) = f(x+h) - f(x)$. (All such functions have the representation $f(x) = \int_0^\infty e^{-xt} \mu(dt)$ for some measure μ. See, for example, Meyer [7, p. 237].) Analogously, we call a non-negative function ϕ defined on $\mathscr{B}$ completely monotone if $(-1)^n \Delta_{A_1} \Delta_{A_2} \ldots \Delta_{A_n} \phi(B) \geqq 0$ for all $B, A_1, A_2, \ldots, A_n \in \mathscr{B}$. The identity (2.1) insures that all zero probability functions are completely monotone.

In order to avoid subscripts with subscripted subscripts in what follows we introduce the notation

$$\Delta(A_1, A_2, \ldots, A_n) \equiv \Delta_{A_1} \Delta_{A_2} \ldots \Delta_{A_n}.$$

If $A \subset B$ then complete monotonicity implies $\phi(A) \geqq \phi(B)$ (in particular $\phi(\emptyset) \geqq \phi(B)$) and

$$\phi(A) - \phi(B) \leqq \phi(\emptyset) - \phi(B - A).$$

Complete monotonicity also implies

$$\psi(A) \equiv \phi(\emptyset) - \phi(A)$$

is subadditive, i.e.

$$\psi(A \cup B) \leqq \psi(A) + \psi(B),$$

in fact

$$\psi(A \cup B) + \psi(A \cap B) \leqq \psi(A) + \psi(B).$$

For any completely monotone function with $\phi(\emptyset) = 1$ (in fact for any function such that $1 - \phi$ is subadditive) we can define a measure by

$$\lambda_\phi(A) = \sup \sum (1 - \phi(A_i))$$

where the supremum is taken over all collections of disjoint sets with $A=\bigcup A_i$. If ϕ is a zero probability function, then λ_ϕ is the intensity measure for the point process. (See Daley and Vere-Jones [1, p. 349].)

We now prove three lemmas that will be needed for the proof of the main characterization theorem, Theorem (2.13) below.

(2.2) **Lemma.** *Let ϕ be any real valued function defined on $\mathscr{B}$. Suppose $A_1, A_2, \ldots, A_n$ are disjoint subsets on $\mathscr{B}$ and let $A=\bigcup_{i=1}^n A_i$. Then*

$$-\Delta_A \phi(B)=\sum (-1)^k \Delta(A_{i_1} \ldots A_{i_k})\, \phi\left(B \cup \left(A-\bigcup_{l=1}^{k} A_{i_l}\right)\right) \tag{2.3}$$

and

$$\sum_{i=1}^{n} 1-\phi(A_i)=\sum k(-1)^k \Delta(A_{i_1} \ldots A_{i_k})\, \phi\left(A-\bigcup_{l=1}^{k} A_{i_l}\right) \tag{2.4}$$

where the sums on the right range over all non-empty subcollections of $\{A_1, A_2, \ldots, A_n\}$.

Proof. We observe that $A=A_1 \cup A_2$ implies

$$\begin{aligned} -\Delta_A \phi(B)&=\phi(B)-\phi(B \cup A)\\ &=\phi(B \cup A_1)-\phi(B \cup A_1 \cup A_2)+\phi(B \cup A_2)-\phi(B \cup A_1 \cup A_2)\\ &\quad+\phi(B \cup A_1 \cup A_2)-\phi(B \cup A_1)-\phi(B \cup A_2)+\phi(B)\\ &=-\Delta(A_2)\, \phi(B \cup A_1)-\Delta(A_1)\, \phi(B \cup A_2)+\Delta(A_1, A_2)\, \phi(B). \end{aligned} \tag{2.5}$$

We emphasize that both (2.3) and (2.4) are purely algebraic and could be verified by an induction argument starting with (2.5). However, if ϕ is the zero probability function of a point process, then the probabilistic interpretation of the identities makes their verification immediate. The identity (2.3) becomes

$$P\{N(A)>0, N(B)=0\}=\sum P\left\{N(A_{i_1})>0 \ldots N(A_{i_k})>0, N\left(B \cup \left(A-\bigcup_{l=1}^{k} A_{i_l}\right)\right)=0\right\}.$$

Observing that $N(A)>0$ if and only if some $N(A_i)>0$, see that the events on the right are disjoint and their union is the event on the left.

To verify (2.4), let Y be the cardinality of $\{A_i \colon N(A_i)>0\}$ then

$$\begin{aligned} E(Y)&=\sum_{i=1}^{n} P\{N(A_i)>0\}=\sum_{i=1}^{n} 1-\phi(A_i)\\ &=\sum kP\left\{N(A_{i_1})>0 \ldots N(A_{i_k})>0, N\left(A-\bigcup_{l=1}^{k} A_{i_l}\right)=0\right\}\\ &=\sum k(-1)^k \Delta(A_{i_1} \ldots A_{i_k})\, \phi\left(A-\bigcup_{l=1}^{k} A_{i_l}\right). \end{aligned}$$

(2.6) **Lemma.** *Let ϕ be completely monotone with $\phi(\emptyset)=1$. Suppose $A_1, A_2, \ldots$ are disjoint and let $A=\bigcup_{i=1}^\infty A_i$. If*

$$1-\phi(A)=\sum (-1)^k \Delta(A_{i_1} \ldots A_{i_k})\, \phi\left(A-\bigcup_{l=1}^{k} A_{i_l}\right) \tag{2.7}$$

where the sum ranges over all finite non-empty subcollections of $\{A_1, A_2, \ldots\}$ *then*

$$1-\phi\left(\bigcup_{i=m}^{\infty} A_i\right)=\sum(-1)^k \Delta(A_{i_1} \ldots A_{i_k})\,\phi\left(A-\bigcup_{l=1}^{k} A_{i_l}\right) \tag{2.8}$$

where the sum ranges over all finite non empty subcollections of $\{A_m, A_{m+1}, \ldots\}$.

Remark. For a finite collection of sets $\{A_1 A_2 \ldots A_n\}$, (2.7) is just a special case of (2.3) ($B=\emptyset$). A limiting argument implies that the left hand side of (2.7) is always greater than or equal to the right hand side. If ϕ is a zero probability function $1-\phi(A)=P\{N(A)>0\}$ and the right hand side is $P\{N(A)\}>0$ and only finitely many $N(A_i)>0\}$.

Proof. Identity (2.3) implies

$$\phi\left(\bigcup_{i=m}^{\infty} A_i\right)-\phi(A)=\sum(-1)^k \Delta(A_{i_1} \ldots A_{i_k})\,\phi\left(A-\bigcup_{l=1}^{k} A_{i_l}\right) \tag{2.9}$$

where the sum ranges over non empty subcollections of $\{A_1 A_2 \ldots A_{m-1}\}$. Identity (2.8) is the difference between (2.7) and (2.9).

(2.10) **Lemma.** *Let* ϕ *be the zero probability function of a point process. Then for* $B\in\mathscr{B}$,

$$\phi(B)=\inf_{K\subset B}\phi(K)=\sup_{U\supset B}\phi(U), \tag{2.11}$$

where the K *are compact and the* U *open.*

Proof. First assume $B\subset\Gamma$ where Γ is compact. Define

$$\gamma_k(A)=E\big(N(A)\,\chi_{\{N(\Gamma)\leqq k\}}\big).$$

Then $\gamma_k(A)$ is a regular measure on the Borel subsets of Γ. (See Rudin [10, *p.* 47].) For K compact, $K\subset B$ we have

$$\begin{aligned}&P\{N(K)=0,\ N(\Gamma)\leqq k\}-P\{N(B)=0,\ N(\Gamma)\leqq k\}\\&\quad=P\{N(B-K)>0,\ N(K)=0,\ N(\Gamma)\leqq k\}\\&\quad\leqq\gamma_k(B-K).\end{aligned} \tag{2.12}$$

For $\varepsilon>0$, the regularity of γ_k implies the existence of a compact set $K_k\subset B$ for which the right hand side of (2.12) is less than ε. Let $A=\bigcup_{k=1}^{\infty} K_k$. Replacing K by A in (2.12) and letting k go to infinity we have $\phi(A)-\phi(B)\leqq\varepsilon$. But

$$\phi(A)=\lim_{n\to\infty}\phi\left(\bigcup_{k=1}^{n} K_k\right)$$

and, since $\bigcup_{k=1}^{n} K_k$ is compact, $\inf_{K\subset B}\phi(K)\leqq\phi(B)+\varepsilon$. Hence we have the first part of (2.11) for B relatively compact. Since X is σ-compact, we can represent B as a countable union of relatively compact sets and obtain the first part of (2.11) in general. The second equality follows in a similar fashion.

(2.13) **Theorem.** *A non-negative function* ϕ *defined on* $\mathscr{B}$ *is the zero probability function of a point process if and only if the following hold:*

(2.14) *ϕ is completely monotone;*

(2.15) $\phi(\emptyset)=1$;

(2.16) *let $K_1 \subset K_2 \subset K_3 \ldots$ be a sequence of compact sets with $X=\bigcup_{n=1}^{\infty} K_n$, then*

$$\phi(B)=\lim_{n\to\infty}\phi(B\cap K_n) \quad \text{for every } B\in\mathscr{B};$$

(2.17) *for every compact set K and every $\varepsilon>0$ there exists a k_ε such that if $K=\bigcup_{i=1}^{\infty} B_i$ and the B_i are disjoint then*

$$\phi(K)+\sum(-1)^k \Delta(B_{i_1}, B_{i_2}, \ldots, B_{i_k})\,\phi\left(K-\bigcup_{l=1}^{k} B_{i_l}\right)\geqq 1-\varepsilon$$

where the sum ranges over all non-empty, finite subcollections $\{B_{i_1}\ldots B_{i_k}\}\subset\{B_1, B_2, \ldots\}$ with $k\leqq k_\varepsilon$.

Remark. We need condition (2.16) to exclude the pathological completely monotone function: $\phi(B)=1$ if B is relatively compact, $\phi(B)=0$ otherwise. If ϕ satisfies conditions (2.14), (2.15) and (2.17) then $\hat{\phi}(B)=\lim_{n\to\infty}\phi(B\cap K_n)$ satisfies all four conditions and $\hat{\phi}(B)=\phi(B)$ for all relatively compact B. Condition (2.17) insures that compact sets contain only a finite number of points. A sufficient condition for (2.17) is that $\lambda_\phi(K)<\infty$ for every K. Note that (2.17) implies that the left hand side of the inequality is one if the sum is over *all* finite subcollections.

Proof. The necessity of (2.14), (2.15) and (2.16) is immediate. The necessity of (2.17) follows from the observation that if ϕ is a zero probability function then the left hand side of the inequality is greater than $P\{N(K)\leqq k_\varepsilon\}$.

The proof of sufficiency divides into two parts. First we construct a point process using ϕ and then we verify that ϕ is the zero probability function of that process. The proof, in fact, represents an alternative approach to the existence of point processes.

Let Π_n be an increasing sequence of countable partitions of X, that is, $\Pi_n=\{A_i^n\}$ where the A_i^n's are disjoint and $X=\bigcup_i A_i^n$; and each set in Π_n is a union of sets in Π_{n+1}. Assume also that $\lim_{n\to\infty}\sup_i \text{diameter}(A_i^n)=0$, where $\text{diameter}(B)=\sup_{x,y\in B}\rho(x,y)$, that the A_i^n are relatively compact, and that only a finite number of A_i^n's intersect a compact set for any fixed n.

Let $\Pi=\bigcup_{n=1}^{\infty}\Pi_n$. Using the Kolmogorov Extension Theorem we construct a family of random variables $\{X_B\colon B\in\Pi\}$ such that X_B is 0 or 1 and

$$\begin{aligned}(2.18)\qquad & P\{X_{B_1}=1\ldots X_{B_k}=1,\ X_{C_1}=0\ldots X_{C_l}=0\}\\ &\quad=(-1)^k\Delta(B_1, B_2, \ldots, B_k)\,\phi\left(\bigcup_{i=1}^{l} C_i\right).\end{aligned}$$

It is easy to check that these distributions are consistent, and that, except for an event with probability zero, $B, C\in\Pi$, $B\subset C$ and $X_B=1$ imply $X_C=1$. It also follows that

$$X_{A_i^n}=\max_{A_j^{n+1}\subset A_i^n} X_{A_j^{n+1}}.$$

If $X_B=1$ then X_B is part of one or more sequences $\{X_{B_n}\}$ with $B_n\in\Pi_n$ and $B_n\supset B_{n+1}$ such that $X_{B_n}=1$ for all n. For each such sequence $\bigcap_{n=1}^{\infty}\bar{B}_n$ consists of a single point. Our point process is

$$\xi=\left\{\bigcap_{n=1}^{\infty}\bar{B}_n\colon B_n\in\Pi_n B_n\supset B_{n+1},\ X_{B_n}=1\right\}. \tag{2.19}$$

It remains to be shown that $N(K,\xi)$ is finite with probability one for every compact set.

Let K be compact and let $\delta>0$ be such that $K_\delta=\{x\colon \inf_{y\in K}\rho(x,y)\leqq\delta\}$ is also compact. If $\{B_n\}$ is one of the sequences contributing to ξ in (2.19), then $\bigcap_{n=1}^{\infty}\bar{B}_n\in K$ implies $B_n\subset K_\delta$ for all sufficiently large n. Let Y_n be the cardinality of $\{A_i^n\colon A_i^n\subset K_\delta$ and $X_{A_i^n}=1\}$. Then $Y_n\leqq Y_{n+1}$ and $N(K,\xi)\leqq\lim_{n\to\infty}Y_n$. For $\varepsilon>0$, let k_ε be the integer associated with K_δ by condition (2.17). We claim $P\{Y_n>k_\varepsilon\}\leqq\varepsilon$. To see this let

$$B=K_\delta-\bigcup_{A_i^n\subset K_\delta}A_i^n.$$

Then by (2.17)

$$\begin{aligned}&\sum(-1)^{k+1}\Delta(B,A_{i_1}^n\dots A_{i_k}^n)\,\phi\Big(K_\delta-B\cup\bigcup_{l=1}^{k}A_{i_l}^n\Big)\\&\quad+\sum(-1)^k\Delta(A_{i_1}^n\dots A_{i_k}^n)\,\phi\Big(K_\delta-\bigcup_{l=1}^{k}A_{i_l}^n\Big)\leqq\varepsilon\end{aligned} \tag{2.20}$$

where the sums are over subcollections of $\{A_i^n\colon A_i^n\subset K_\delta\}$ with $k>k_\varepsilon$.

Using the definition of Δ_B the left hand side of (2.20) becomes

$$\begin{aligned}&\sum(-1)^k\Delta(A_{i_1}^n\dots A_{i_k}^n)\,\phi\Big((K_\delta-B)-\bigcup_{l=1}^{n}A_{i_l}^n\Big)\\&\quad=\sum P\{X_{A_{i_1}^n}=1,\dots,X_{A_{i_k}^n}=1,\ X_{A_i^n}=0\ \text{for all other}\ A_i^n\subset K_\delta\}\\&\quad=P\{Y_n>k_\varepsilon\}.\end{aligned}$$

Let $\hat{\phi}(B)=P\{N(B,\xi)=0\}$. We must verify that $\phi(B)=\hat{\phi}(B)$. Let $\mathscr{A}$ be the algebra containing all finite unions of sets in $\bigcup_{n=1}^{\infty}\Pi_n$. For $A\in\mathscr{A}$, $\phi(A)=P\{X_{A_i^n}=0$ all $A_i^n\subset A\}$. It follows that $\phi(A)\leqq\hat{\phi}(A^0)$ and $\hat{\phi}(\bar{A})\leqq\phi(A)$. Any open set U can be represented as a union of an increasing sequence $A_n\in\mathscr{A}$ in such a way that $U=\bigcup A_n=\bigcup A_n^0$. We then have

$$\phi(U)\leqq\lim_{n\to\infty}\phi(A_n)\leqq\lim_{n\to\infty}\hat{\phi}(A_n^0)=\hat{\phi}(U). \tag{2.21}$$

The last equality follows from the fact that $\hat{\phi}$ is a zero probability function. Similarly any compact set K can be represented as an intersection of a decreasing sequence of sets $A_n\in\mathscr{A}$ in such a way that $K=\bigcap A_n=\bigcap\bar{A}_n$. As before

$$\phi(K)\geqq\lim_{n\to\infty}\phi(A_n)\geqq\lim_{n\to\infty}\hat{\phi}(\bar{A}_n)=\hat{\phi}(K). \tag{2.22}$$

Combining (2.21) and (2.22) we have

$$\hat{\phi}(K)\leqq\phi(K)\leqq\phi(K^0)\leqq\hat{\phi}(K^0). \tag{2.23}$$

Consequently, if $\hat{\phi}(K)=\hat{\phi}(K^0)$ we have equality. To see that there is no shortage of K for which this is true observe that for $B_r(x)=\{y\colon \rho(x,y)<r\}$

$$\begin{aligned}1 &\geqq \sum_r P\{N(\overline{B}_r(x))>0,\ N(B_r(x))=0\}\\ &=\sum_r \hat{\phi}(B_r(x))-\hat{\phi}(\overline{B}_r(x)).\end{aligned}$$

Consequently, $\hat{\phi}(B_r(x))=\hat{\phi}(\overline{B}_r(x))$ for all but countably many values of r. We also observe that if $\hat{\phi}(K_k)=\hat{\phi}(K_k^0)$ $k=1,2,\dots$, then $\hat{\phi}(\bigcup_{k=1}^n K_k)=\hat{\phi}(\bigcup_{k=1}^n K_k^0)$. To complete the proof we need the following lemma.

(2.24) **Lemma.** *Let $B_1\supset B_2\supset B_3\dots$ and suppose B_1 is relatively compact. Then*

$$\lim_{n\to\infty}\phi(B_n)=\phi\left(\bigcap_{n=1}^{\infty}B_n\right).$$

b) *Let $B_1\subset B_2\subset\cdots$ and suppose $B=\bigcup_{n=1}^{\infty}B_n$ is relatively compact. Then*

$$\lim_{n\to\infty}\phi(B_n)=\phi(B).$$

Proof. In part a) we may as well assume B_1 is compact and $\bigcap B_n=\emptyset$ (since $\phi(\bigcap_{k=1}^{\infty}B_k)-\phi(B_n)\leqq 1-\phi(B_n-\bigcap_{k=1}^{\infty}B_k)$). Let $A_n=B_n-B_{n+1}$. Condition (2.17) implies Condition (2.7) of Lemma (2.6) with $A=B_1$, and hence

$$\begin{aligned}1-\phi(B_m)&=1-\phi\left(\bigcup_{i=m}^{\infty}A_i\right)\\ &=\sum(-1)^k\Delta(A_{i_1}\dots A_{i_k})\,\phi\left(A-\bigcup_{l=1}^{k}A_{i_l}\right)\end{aligned}$$

where the sum ranges over finite subcollections of $\{A_m, A_{m+1},\dots\}$. Consequently as m goes to infinity the sum goes to zero. Part b) follows from part a) and the inequality $\phi(B_n)-\phi(B)\leqq 1-\phi(B-B_n)$.

Every compact K can be represented as an intersection of compact sets K_n with $\hat{\phi}(K_n)=\hat{\phi}(K_n^0)$ and every open, relatively compact U can be represented as a union of such sets. Consequently Lemma (2.24) implies $\phi(K)=\hat{\phi}(K)$ for all compact K and $\phi(U)=\hat{\phi}(U)$ for all open, relatively compact U. The fact that $\phi(B)=\hat{\phi}(B)$ for all relatively compact B now follows by Lemma (2.10). Condition (2.16) then gives the equality for all B.

Remark. Given *any* completely monotone set function with $\phi(\emptyset)=1$, the construction used produces a process ξ whose realizations are closed sets.

These considerations are related to what Kendall [5] calls avoidance functions. We should also note that Kendall has shown that a function $f(x)=P\{N([0,x])=0\}$ for some stationary point process on the line if and only if $f(x)=\int_0^{\infty}\left(1-\frac{x}{t}\right)^+\mu(dt)$ for some probability measure μ, i.e. f must be nonnegative, convex and decreasing and $f(0)=1$.

It is natural to ask if there is any relationship between classical completely monotone functions and completely monotone set functions. The following result in that direction is easy to verify.

(2.25) **Proposition.** *Let γ be an infinite non-atomic measure on $\mathscr{B}$. Then $\phi(A) \equiv f(\gamma(A))$ is a completely monotone set function if and only if f is completely monotone as a function on $(0, \infty)$ and $\lim_{x \to 0} f(x) = f(0) < \infty$. If $f(0) = 1$ and γ is finite on compact sets, then $\phi(A)$ is the zero probability function of a mixture of Poisson processes (i.e. $\phi(A) = \int_0^\infty e^{-t\gamma(A)} \mu(dt)$ for some probability measure μ).*

Remark. This is essentially Theorem 5.1 of Kallenberg [4].

If ϕ is a function on $\mathscr{B}$ that satisfies condition (2.16), then to show that ϕ is completely monotone it is sufficient to verify

$$(-1)^k \Delta(A_1 \dots A_k)\, \phi(B) \geqq 0 \tag{2.26}$$

for relatively compact sets. We also note that $(-1)^k \Delta(A_1 \dots A_k)\, \phi(B)$ can be written as a sum of similar expressions in which each term involves only disjoint sets. (This can be verified most readily by using the probabilistic interpretation in the special case that ϕ is a zero probability function.) Consequently, we only need to verify (2.26) for $B, A_1, A_2 \dots A_k$ disjoint.

Using Lemma (2.2) we observe that we can take $A_1 A_2 \dots A_k$ to have diameter less than ε for any $\varepsilon > 0$. This suggests an "infinitesimal" approach to complete monotonicity.

For each k let μ_k be a measure on $(X^k, \mathscr{B}^{(k)})$, where X^k is the k-fold Cartesian product of X and $\mathscr{B}^{(k)}$ is the product σ-algebra. Let Π^n be as in the proof of Theorem (2.13). Define

$$\phi(x_1, x_2 \dots x_k, B) = \lim_{n \to \infty} \frac{(-1)^k \Delta(C_1, \dots, C_k)\, \phi\left(B - \bigcup_{i=1}^k C_i\right)}{\mu_k(C_1 \times C_2 \times \dots \times C_k)} \tag{2.27}$$

where for each n, $C_1 \dots C_k \in \Pi_n$ and $x_i \in C_i$, whenever the limit exists. Observe that $x_i = x_j$ some $i \neq j$ implies $\phi(x_1, x_2 \dots x_k, B) = 0$.

(2.28) **Theorem.** *Let ϕ be a function satisfying $\phi(B) = \lim_{n \to \infty} \phi(K_n \cap B)$ for an increasing sequence of compact sets with $X = \bigcup_{n=1}^\infty K_n$, and $\phi(\bigcap_{n=1}^\infty B_n) = \lim_{n \to \infty} \phi(B_n)$ for every decreasing sequence of relatively compact sets. Let $\mathscr{A}$ be the algebra of finite unions of sets on $\Pi \equiv \bigcup_{n=1}^\infty \Pi_n$.*

Let μ_k be a measure in $(X^k, \mathscr{B}^{(k)})$ such that $\sum \frac{1}{k!} \mu_k(K \times K \times \dots \times K) < \infty$ for every compact K, and suppose that

$$|\Delta(C_1 \dots C_k)\, \phi(B)| \leqq C(K)\, \mu_k(C_1 \dots C_k) \tag{2.29}$$

where B, $C_1 \dots C_k$ are disjoint subsets of K compact with $B \in \mathscr{A}$, $C_1 C_2 \dots C_k \in \Pi$ and $C(K)$ is a constant depending only on K.

If for every k, $\phi(x_1, x_2 \dots x_k, B)$ exists and is nonnegative a.e. μ_k for every $B \in \mathscr{A}$, then ϕ is completely monotone.

Proof. By the continuity assumptions made on ϕ, it is sufficient to verify the monotonicity inequality (2.26) for $A_1 A_2 \dots A_k$ and B disjoint subsets in $\mathscr{A}$. For sufficiently large n, $A_1 A_2 \dots A_k$ are finite unions of sets in Π_n and hence

$$(-1)^k \Delta(A_1 \dots A_k)\, \phi(B) = \sum (-1)^l \Delta(C_1 \dots C_l)\, \phi\left(B \cup A_1 \cup \dots \cup A_k - \bigcup_{i=1}^l C_i\right) \tag{2.30}$$

where the sum is over all finite nonempty subsets $\{C_1 \dots C_l\}$ of Π_n with

$$\bigcup_{i=1}^{l} C_i \subset \bigcup_{j=1}^{k} A_j \quad \text{and} \quad A_j \cap \bigcup_{i=1}^{l} C_i \neq \emptyset$$

for each j. This identity follows from Lemma (2.2) and the fact that $A_1, A_2 \dots A_k$ and B are disjoint. Let $\chi_C(x)$ denote the indicator function of C, and note that

$$(-1)^l \Delta(C_1 \dots C_l)\, \phi\left(B \cup A_1 \cup \dots \cup A_k - \bigcup_{i=1}^{l} C_i\right)$$
$$= \int \frac{(-1)^l \Delta(C_1 \dots C_l)\, \phi\left(B \cup A_1 \cup \dots \cup A_k - \bigcup_{i=1}^{l} C_i\right)}{\mu_l(C_1 \times C_2 \dots C_l)} \prod_{i=1}^{l} \chi_{C_i}(x_i)\, d\mu_l.$$

Splitting the sum into a series of sums with l fixed, the right hand side of (2.30) can be written as

$$(2.31)\quad \sum \int_D \sum \frac{(-1)^l \Delta(C_1 \dots C_l)\, \phi\left(B \cup A_1 \cup A_2 \cup \dots \cup A_k - \bigcup_{i=1}^{l} C_i\right)}{\mu_l(C_1 \dots C_l)} \prod_{i=1}^{l} \chi_{C_i}(x_i) \frac{1}{l!}\, d\mu_l$$

where the inner sum is over *all ordered* collections $\{C_1\, C_2 \dots C_l\}$ with $C_i \in \Pi_n$ (the fact that the collections in this sum are ordered introduces the factor $1/l!$) and $D_l = \{(x_1, x_2 \dots x_l) \in X^l\colon x_i \in \bigcup_{j=1}^{k} A_j$ for all i and some $x_i \in A_j$ for all $j = 1, 2 \dots k\}$. Note that $D_l \subset (\bigcup_{i=1}^{k} A_i)^l$ and $\bigcup_{i=1}^{k} A_i$ is relatively compact so that $\sum \frac{1}{l!} \mu_l(D_l) < \infty$. Condition (2.29) insures that the integrand in (2.31) is bounded and hence the Dominated Convergence Theorem implies

$$\Delta(A_1 \dots A_k)\, \phi(B) = \sum \int_{D_l} \phi(x_1, x_2 \dots x_k, B \cup A_1 \cup \dots \cup A_k) \frac{1}{l!}\, d\mu_l \geqq 0.$$

(2.32) **Corollary.** *Let $X = \mathbb{R}$ and let $f_l(x_1 \dots x_l)$ be nonnegative, continuous functions on $\mathbb{R}^l$ such that for every compact K there is a $\rho > 0$ with $f_l(x_1, x_2 \dots x_l) \leqq \rho^l$ for all $x_1, x_2 \dots x_l \in K$. Take $f_0 \equiv 1$, and assume f_l is invariant under permutation of the arguments. If*

$$(2.33)\qquad \phi(B) \equiv \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_B \dots \int_B f_l(y_1, y_2 \dots y_l)\, dy_1\, dy_2 \dots dy_l \geqq 0$$

and

$$(2.34)\qquad \phi(x_1, x_2 \dots x_k, B) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_B \dots \int_B f_{k+l}(x_1, x_2 \dots x_k, y_1, y_2 \dots y_l)\, dy_1 \dots dy_l \geqq 0$$

for all k, $x_1, x_2 \dots x_k$, and B then ϕ is completely monotone. Furthermore, ϕ is a zero probability function, the functions f_l are the product density functions for the point process (see Fisher [3, p. 473]), and

$$(2.35)\quad E(N(B)(N(B)-1) \dots (N(B)-l+1)) = \int_B \dots \int_B f_l(y_1, y_2 \dots y_l)\, dy_1 \dots dy_l.$$

Remark. The above Theorem also holds for more general X and measures other than Lebesgue measure.

Proof. Let K be compact and let ρ satisfy $f_l(x_1\, x_2 \dots x_l) \leqq \rho^l$ for $x_1, x_2 \dots x_k \in K$. Then letting μ_k be $\rho^k m_k$, where m_k is Lebesgue measure on $\mathbb{R}^k$, Theorem (2.28) implies $\phi(B)$ is completely monotone on the Borel subsets of K, but this being true for every compact K implies $\phi(B)$ is completely monotone on $\mathscr{B}$.

There are many examples of completely monotone functions with the form (2.33). For example taking $f_l(x_1\, x_2 \dots x_l) = \prod_{i=1}^l g(x_i)$, $\phi(B)$ is the zero probability function of a Poisson process. More generally, if $\lambda(x)$ is a nonnegative stochastic process bounded by a constant on each compact set, then the zero probability function of the doubly stochastic Poisson process with parameter function λ is

$$\phi(B) = E(\exp\{-\int_B \lambda(x)\, dx\}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left(\int_B \dots \int_B E(\lambda(x_1)\lambda(x_2) \dots \lambda(x_l))\right) dx_1 \dots dx_l.$$

Unfortunately, examples of completely monotone set functions which are not already known to be zero probability functions are harder to find. The following is a not very satisfying example:

Let $f_l(x_1\, x_2 \dots x_l)$ satisfy

$$\int_{\mathbb{R}} f_1(y)\, dy \leqq 1 \tag{2.36}$$

and

$$\int_{\mathbb{R}} f_{l+1}(x_1\, x_2 \dots x_l, y)\, dy \leqq f_l(x_1, x_2 \dots x_l). \tag{2.37}$$

Then it is easy to verify that the terms on the sums in (2.33) and (2.34) alternate in sign and decrease in absolute value. Consequently, the sums are nonnegative and ϕ is a zero probability function. However, by (2.36)

$$E(N(\mathbb{R})) = \int_{\mathbb{R}} f_1(y)\, dy \leqq 1.$$

If (2.36) and (2.37) hold with integration over a set K rather than $\mathbb{R}$, then ϕ will be completely monotone on the Borel subsets of K.

Acknowledgements. This paper was written while the author was enjoying the hospitality of the Statistics Department in the Institute of Advanced Studies of The Australian National University. Particular thanks are due to Dr. D.J. Daley who first aroused my interest in point processes and then patiently responded to a steady stream of questions relating to this work. His insight into point processes greatly improved the final results.

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(Received June 20, 1973; in revised form October 15, 1974)

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