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Diffusion Branching Processes with Several Types of Particles

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Let us consider a sequence of one-type-particle branching processes with basic first moments $M_n = 1 + \frac{\alpha}{n}$ and let us transform the time parameter and the states of the n^{th} process in such a way that one new time unit corresponds to n time units of the original process and one unit in the new state space corresponds to n particles of the original process. Feller discovered (see [1]) that the transition probabilities of the transformed processes converge with $n \to \infty$ to transition probabilities of a diffusion process and that the characteristic functions of the limit transition probabilities satisfy a partial differential equation the explicit solution of which is known. A complete proof of this assertion is contained in [2]; more precisely, it is proved in [2] that the logarithms of the Laplace transforms of the transformed transition probabilities converge to a function $\psi(t, s)$ defined by an explicit formula (see [2] (8)). This function satisfies a partial differential equation analoguous to the partial differential equation mentioned above, but this fact is not used in the proof of [2].

It is natural to expect that branching processes with r types of particles will behave in a similar way and that the logarithms $\psi_{ni}(t, s)$ of the Laplace transforms of the transformed basic transition probabilities will converge to functions $\psi_i(t, s)$ satisfying analoguous differential equations (see (5) and (6) of this paper). It seems that there is no explicit solution of these equations if $r > 1$ and the method of [2] is therefore not applicable. An attempt to prove directly (using estimates similar to those of [2]) that the sequences $\psi_{ni}(t, s)$ are convergent, failed too. For these reasons the method of the present paper is different from that used in [2]. It follows the ideas of the proof of the continuity theorem for characteristic functions, i.e. it consists of the following three steps: (a) the existence of a convergent subsequence of $\psi_{ni}(t, s)$ is established, (b) it is proved that the limits $\psi_i(t, s)$ satisfy the system (5) of differential equations and (c) the uniqueness of the solution of (5) is then used in proving that the whole sequence $\psi_{ni}(t, s)$ is convergent.

In the paper we shall deal with r-dimensional vectors and $r \times r$ matrices. For a vector $a = (a_1, \ldots, a_r)$, $|a|$ will denote the vector $(|a_1|, \ldots, |a_r|)$ and $\|a\| = \max\{|a_1|, \ldots, |a_r|\}$; similarly, for a matrix $A = (A_{ij})_{i,j}$, $|A|$ will denote the matrix $(|A_{ij}|)_{i,j}$. If a, b are vectors then $a b$ will denote their inner product. We shall not distinguish explicitly between row and column vectors. Thus, in the product $A a$ of a matrix A and a vector a, a will be automatically considered a column vector. Inequalities between vectors or matrices will mean that the indicated inequalities hold between all corresponding components of the vectors or matrices. The i^{th} unit vector $(0, \ldots, 1, \ldots, 0)$ will be denoted by $e^{(i)}$ and, consequently, $e_j^{(i)} = 0$ if

$i \neq j$, $e_i^{(i)} = 1$; e will denote the vector $(1, 1, \ldots, 1)$, 0 the zero vector or zero matrix, E the diagonal unit matrix and I the matrix all elements of which are equal to 1. If $A(t)$ is a matrix $(A_{ij}(t))_{i,j}$ of functions of a variable t, then $\int A(t)\,dt$ will denote the matrix $(\int A_{ij}(t)\,dt)_{i,j}$.

Let $\mathscr{P}_n$ be a sequence of homogeneous Markov branching processes with discrete time parameter $t \in T = \{0, 1, \ldots\}$ and with r types of particles, i.e. with state space N^r, where $N = \{0, 1, \ldots\}$. The probability of transition from the state $a = (a_1, \ldots, a_r) \in N^r$ to the state $b \in N^r$ in $\mathscr{P}_n$ will be denoted by $P_n(t, a, b)$. The probability distribution induced on N^r by $P_n(t, a, b)$ will be denoted by $P_n(t, a, \cdot)$. To each $\mathscr{P}_n$ we shall assign a new Markov process $\mathscr{Q}_n$ with discrete time parameter $t \in T_n = \{0, 1/n, 2/n, \ldots\}$ and with state space N_n^r, where $N_n = \{0, 1/n, 2/n, \ldots\}$. The transition probabilities $Q_n(t, a, b)$ of $\mathscr{Q}_n$ will be defined by

$$Q_n(t, a, b) = P_n(n\,t, n\,a, n\,b), \qquad t \in T_n, \qquad a, b \in N_n^r. \tag{1}$$

Let R_+^r be the non-negative cone of the r-space. For $s = (s_1, \ldots, s_r) \in R_+^r$, put

$$\Phi_n(t, a, s) = \sum_{b \in N^r} e^{-sb} P_n(t, a, b)$$

and

$$\phi_n(t, a, s) = -\log \Phi_n(t, a, s)$$

(i.e. ϕ_n is the negative logarithm of the Laplace transform of $P_n(t, a, \cdot)$). Similarly, put

$$\psi(t, a, s) = -\log \sum_{b \in N_n^r} e^{-sb} Q_n(t, a, b)$$

$(t \in T_n,\ a \in N_n^r)$.

It is well known that the process $\mathscr{P}_n$ is uniquely determined by the basic transition probabilities $P_n(t, e^{(i)}, b)$. For simplicity reasons we shall write $P_{n,i}(t, b)$ instead of $P_n(t, e^{(i)}, b)$ and the same rule will apply to all other functions; thus, $\phi_{ni}(t, s) = \phi_n(t, e^{(i)}, s)$ e.g. $\phi_n(t, s)$ will denote the vector $(\phi_{n1}(t, s), \ldots, \phi_{nr}(t, s))$, $\psi_n(t, s)$ the vector $(\psi_{n1}(t, s), \ldots, \psi_{nr}(t, s))$ etc. In $\mathscr{P}_n$ processes, the value $t = 1$ will be also ommitted, i.e. we shall write $P_{ni}(b)$ instead of $P_{ni}(1, b)$, $\phi_n(s)$ instead of $\phi_n(1, s)$ etc. To simplify symbols for derivatives, we shall write $\phi_{nij}(t, s)$ instead of $\dfrac{\partial}{\partial s_j} \phi_{ni}(t, s)$ and $\phi_{nijk}(t, s)$ instead of $\dfrac{\partial^2}{\partial s_j\, \partial s_l} \phi_{ni}(t, s)$. The same rule will apply to $\psi_{ni}(t, s)$ and other functions.

We shall assume that the first and second moments of $\mathscr{P}_n$ are finite and we shall denote the first moments of $P_{ni}(t, \cdot)$ by

$$M_{nij}(t) = \phi_{nij}(t, 0).$$

$M_n(t)$ will denote the matrix $(M_{nij}(t))_{i,j}$. It is well known that

$$M_n(t) = M_n^t \qquad (\text{with } M_n = M_n(1)).$$

The covariances of $P_{ni}(\cdot)$ will be denoted by

$$D_{nijk} = -\phi_{nijk}(0).$$

For each $t \geqq 0$ let $[t]_n$ denote the largest $\tau \in T_n$ less or equal to t, i.e. $[t]_n = [t_n]/n$, where $[y] = [y]_1$ denotes the integral part of y. Similarly, for a vector $a = (a_1, \ldots, a_r)$, we shall write $[a]_n = ([a_1]_n, \ldots, [a_r]_n)$.

Theorem. *Let us assume that finite limits*

$$\lim_{n\to\infty} n(M_n - E) = A = (\alpha_{ij})_{i,j} \tag{2}$$

and

$$\lim_{\substack{n\to\infty\\ s\to 0}} \phi_{nijk}(s) = -2\beta_{ijk} \tag{3}$$

exist. Then there exist, for each $t \geqq 0$ *and* $a \in R_+^r$, *probability measures* $Q(t, a, \cdot)$ *on* R_+^r *such that*

$$Q_n([t]_n, [a]_n, \cdot) \xrightarrow[n\to\infty]{} Q(t, a, \cdot) \qquad (weakly) \tag{4}$$

and the functions

$$\psi_i(t, s) = -\log \int e^{-sb} Q(t, e^{(i)}, db)$$

satisfy the following system of ordinary differential equations

$$\frac{\partial}{\partial t} \psi_i(t, s) = \sum_j \alpha_{ij} \psi_j(t, s) - \sum_{j,k} \beta_{ijk} \psi_j(t, s) \psi_k(t, s) \tag{5}$$

with initial conditions $\psi_i(0, s) = s_i$ $(i = 1, \ldots, r)$. *Each* $\psi_i(t, s)$ *satisfies also the partial differential equation*

$$\frac{\partial}{\partial t} \psi_i(t, s) = \sum_j C_j(s) \frac{\partial}{\partial s_j} \psi_i(t, s) \tag{6}$$

where

$$C_i(s) = \sum_j \alpha_{ij} s_j - \sum_{j,k} \beta_{ijk} s_j s_k. \tag{7}$$

Remarks. (a) It follows from (3) that

$$\tfrac{1}{2} D_{nijk} \to \beta_{ijk} \tag{8}$$

but the stronger condition (3) is needed in the proof. It can be proved that (3) follows from (8), if the third cummulants $\phi_{nijkl}(0)$ are uniformly bounded with respect to n.

(b) The notation introduced in the theorem might be confusing, because $\psi_n(t, s)$ denotes by the previous agreement the vector of functions $(\psi_{n1}, \ldots, \psi_{nr})$, while $\psi_i(t, s)$ defined in the theorem means another (single) function. All possible confusion will be avoided, if we make an agreement that the letter n will be always used as the first index only and its presence will express the fact that the corresponding function (or vector etc.) belongs to the n^{th} process $\mathscr{P}_n$ or $\mathscr{Q}_n$, while n missing will mean that the function belongs to the limit process.

Proof of the Theorem. In order to prove (4) it is sufficient to show that for each $t \geqq 0$ and $a \in R_+^r$ the functions $\psi([t]_n, [a]_n, s)$ converge to a function $\psi(t, a, s)$ and that $\psi(t, a, s)$ is the negative logarithm of the Laplace transform of a probability measure $Q(t, a, \cdot)$. By (1)

$$\psi_n(t, a, s) = \phi_n\left(n t, n a, \frac{1}{n} s\right) \qquad (t \in T_n,\ a \in N_n^r) \tag{9}$$

and from the basic identity for branching processes

$$\phi_n(t, a, s) = \sum_i a_i \phi_{ni}(t, s) \qquad (t \in T,\ a \in N^r) \tag{10}$$

we have

$$\psi_{ni}([t]_n, s) = n\,\phi_{ni}\left([t\,n], \frac{1}{n}s\right) \quad \text{for all } t \geqq 0 \tag{11}$$

and

$$\psi_n([t]_n, [a]_n, s) = \sum_i [a_i]_n \psi_{ni}([t]_n, s)$$

for all $t \geqq 0$ and all $a \in R^r_+$. Since $[a_i]_n \to a_i$, it is sufficient to prove the convergence of $\psi_{ni}([t]_n, s)$. We have defined the functions $\psi_{ni}(t, s)$ for $t \in T_n$ only, but it will simplify the notation and the proof, if we define them for all $t \geqq 0$, i.e., if we write

$$\psi_{ni}(t, s) = n\,\phi_{ni}\left([t\,n], \frac{1}{n}s\right) \quad \text{for all } t \geqq 0.$$

Put $A_n = n(M_n - E)$ and let α_{nij} be the elements of A_n. By (2), $\alpha_{nij} \xrightarrow[n\to\infty]{} \alpha_{ij}$ for all i, j. The well known formula

$$\phi_n(t_1 + t_2, s) = \phi_n(t_1, \phi_n(t_2, s)) \tag{12}$$

and also the obvious relations

$$\phi_n(0, s) = s, \qquad \psi_n(0, s) = s$$

will be used frequently. The proof of the theorem will consist of several lemmas.

(i) *To each $t_0 \geqq 0$ there exists $k > 0$ such that*

$$M_n^t \leqq k\,I \quad \textit{for all } n \textit{ and all } t = 0, 1, \ldots, [t_0\,n].$$

Proof. α_{nij} are convergent (with $n \to \infty$) and therefore bounded by a constant $c > 0$. Then

$$M_n^t = \left(E + \frac{1}{n}A_n\right)^t \leqq \left(E + \frac{c}{n}I\right)^t \leqq \left(E + \frac{c}{n}I\right)^{[t_0 n]} \xrightarrow[n\to\infty]{} \exp(c\,t_0\,I).$$

(ii) *To each $t_0 \geqq 0$ there exists $k > 0$ such that*

$$0 \leqq \phi_{ni}(t, s) \leqq k\,\|s\|$$

for all n, all $t = 0, 1, \ldots, [t_0\,n]$, all i and all $s \in R^r_+$.

Proof.

$$\phi_{ni}(t, s) = \sum_j M_{nij}(t)\,s_j + \tfrac{1}{2}\sum_{j,k} \phi_{nijk}(t, \sigma)\,s_j\,s_k \tag{13}$$

where $0 \leqq \sigma \leqq s$. Since $(-\phi_{nijk}(t, \sigma))_{j,k}$ is the covariance matrix of the probability distribution $e^{-\sigma b} P_{ni}(t, b)/\phi_{ni}(t, \sigma)$ (on N^r),

$$\sum_{jk} \phi_{nijk}(t, \sigma)\,s_j\,s_k \leqq 0.$$

Hence $\phi_{ni}(t, \sigma) \leqq \sum_j M_{nij}(t)\,s_j$ and the assertion follows from (i).

(iii) *To each* $t_0 \geqq 0$ *there exists* $x>0$ *such that*

$$0 \leqq \phi_{nij}(t, s) \leqq 2M_{nij}(t)$$

for all n, *all* $t=0, 1, \ldots, [t_0 n]_1$ *all* i, j *and all* $s \in [0x]^r$.

Proof.

$$\phi_{nij}(t, s) = -\frac{\Phi_{nij}(t, s)}{\Phi_{ni}(t, s)}$$

and $0 \leqq -\Phi_{nij}(t, s) \leqq M_{nij}(t)$ for all s. Hence,

$$\Phi_{ni}(t, s) = 1 + \sum_j \Phi_{nij}(t, \sigma)\, s_j \geqq 1 - \sum_j M_{nij}(t)\, s_j.$$

By (i), $M_{nij}(t)$ are uniformly bounded for all n and all $t=0, 1, \ldots, [t_0 n]$ and, consequently, $\Phi_{ni}(t, s) > \frac{1}{2}$ for sufficiently small s.

(iv) *To each* $x>0$ *there exists* $k>0$ *such that*

$$0 \leqq \phi_{nij}\left(\frac{1}{n} s\right) \leqq e_j^{(i)} + \frac{k}{n}$$

for all n, *all* i, j *and all* $s \in [0x]^r$.

Proof. For $i \neq j$, the assertion follows from (iii) and the fact that $e_j^{(i)}=0$, $M_{nij} = \frac{1}{n}\alpha_{nij}$ (at least for sufficiently large n; but then it must hold for all n). For $i=j$ we have

$$\phi_{nii}\left(\frac{1}{n} s\right) - 1 = -\frac{\Phi_{ni}\left(\frac{1}{n} s\right) + \Phi_{nii}\left(\frac{1}{n} s\right)}{\Phi_{ni}\left(\frac{1}{s} s\right)}$$

and

$$\begin{aligned} \Phi_{ni}\left(\frac{1}{n} s\right) + \Phi_{nii}\left(\frac{1}{n} s\right) &= 1 - M_{nii} + \sum_j \left(\Phi_{nij}(\sigma_n(s)) + \Phi_{niij}(\sigma_n(s))\right) \frac{s_j}{n} \\ &= \frac{1}{n}\left[\alpha_{nii} + \sum_j \left(\Phi_{nij}(\sigma_n(s)) + \Phi_{niij}(\sigma_n(s))\right) s_j\right], \end{aligned}$$

where $0 \leqq \sigma_n(s) \leqq \frac{1}{n} s$. The sequences α_{nij}, $\Phi_{niij}(\sigma_n(s))$ and $|\Phi_{nij}(\sigma_n(s))| \leqq M_{nij}$ are uniformly bounded with respect to all n and $s \in [0x]^r$ because of (2) and (3). From the proof of (iii) it follows that $\Phi_{ni}\left(\frac{1}{n} s\right) > \frac{1}{2}$ for sufficiently large n and all $s \in [0x]^r$ and, consequently, the functions $1/\Phi_{ni}\left(\frac{1}{n} s\right)$ are bounded uniformly with respect to all n and all $s \in [0x]^r$. Hence $n\left[\phi_{nii}\left(\frac{1}{n} s\right) - 1\right]$ is bounded uniformly with respect to all n and all $s \in [0x]^r$. This proves the assertion of (iv) for $i=j$.

(v) *To each* $t_0 \geqq 0$ *and* $x > 0$ *there exists* $k > 0$ *such that*

$$\left|\phi_{nijk}\left(t, \frac{1}{n} s\right)\right| \leqq k\, t$$

for all n, *all* $t = 0, 1, \ldots, [t_0 n]$, *all* i, j, k *and all* $s \in [0\, x]^r$.

Proof. Differentiating the relation $\phi_n(t+1, s) = \phi_n(\phi_n(t, s))$ (see (12)) we obtain

$$\phi_{n \cdot jk}(t+1, s) = w_{njk}(t, s) + m_n(t, s)\, \phi_{n \cdot jk}(t, s) \tag{14}$$

where $\phi_{n \cdot jk}(t, s)$ denotes the vector $(\phi_{nijk}(t, s))_{i=1, \ldots, r}$, $w_{njk}(t, s)$ the vector $\sum_{l, m} \phi_{n \cdot lm}(\phi_n(t, s))\, \phi_{nlj}(t, s)\, \phi_{nmk}(t, s)$ and $m_n(t, s)$ the matrix $(\phi_{nij}(\phi_n(t, s)))_{i,j}$.

From (14) we have

$$\phi_{n \cdot jk}(t, s) = \sum_{\tau=1}^{t} \prod_{v=1}^{\tau-1} m_n(t-v)\, w_{njk}(t-\tau). \tag{15}$$

It follows from (3) that there exists $x_1 > 0$ such that the functions $\phi_{nijk}(s)$ are bonunded uniformly with respect to all n and all $s \in [0\, x_1]^r$.

By (ii), $\phi_n\left(t, \frac{1}{n} s\right) \in [0\, x_1]^r$ for all sufficiently large n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$. Hence $\phi_{nijk}\left(\phi_n\left(t, \frac{1}{n} s\right)\right)$ are bounded uniformly with respect to all n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$.

By (i) and (iii), $\phi_{nij}\left(t, \frac{1}{n} s\right)$ are bounded uniformly with respect to all n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$. Consequently, there exists $k_1 > 0$ such that

$$\left|w_{njk}\left(t, \frac{1}{n} s\right)\right| \leqq k_1\, e \tag{16}$$

for all n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$.

By (ii), there exists $x_2 > 0$ such that

$$0 \leqq n\, \phi_{ni}\left(t, \frac{s}{n}\right) \leqq x_2$$

for all n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$. Hence, by (iv), there exists $k_2 > 0$ such that

$$0 \leqq \phi_{nij}\left(\frac{1}{n} \cdot n\, \phi_n\left(t, \frac{1}{n} s\right)\right) \leqq e_j^{(i)} + \frac{k_2}{n},$$

or

$$0 \leqq m_n\left(t, \frac{1}{n} s\right) \leqq E + \frac{k_2}{n} I \tag{17}$$

for all n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$.

By (15), (16) and (17) we have

$$\left|\phi_{n \cdot jk}\left(t, \frac{1}{n} s\right)\right| \leqq k_1 \sum_{\tau=0}^{t-1} \left(E + \frac{k_2}{n} I\right)^{\tau} e \tag{18}$$

for all n, all $t \leqq [t_0 n]$ and all $s \in [0\, x]^r$.

For the same reasons as in the proof of (i), the sequence $\left(E+\frac{k_2}{n}I\right)^t$ is bounded with respect to all n and all $t \leqq [t_0 n]$ and the assertion of (v) follows then from (18).

(vi) *To each $t_0 \geqq 0$ there exist n_0 and $k>0$ such that*

$$|M_n^{[tn]-[t'n]} - E| < k(t-t')I$$

for all $n \geqq n_0$ and all $0 \leqq t' \leqq t \leqq t_0$.

Proof.

$$M_n^{[tn]-[t'n]} - E = (M_n - E) \sum_{\tau=0}^{[tn]-[t'n]-1} M_n^{\tau}.$$

By (i),

$$|M_n^{[tn]-[t'n]} - E| \leqq \frac{1}{n}|A_n| \cdot ([tn]-[t'n])\, k_1 I$$

for some $k_1>0$, and the assertion of (vi) follows from the fact that the sequence $|A_n|$ is bounded and that $\frac{1}{n}([\tau n]-[\tau' n]) \xrightarrow[n\to\infty]{} \tau - \tau'$.

(vii) *To each $t_0 \geqq 0$ and $x>0$, there exist n_0 and $k>0$ such that*

$$|\psi_{ni}(t,s) - \psi_{ni}(t',s)| \leqq k|t-t'|$$

for all $n \geqq n_0$, all i, all $t \leqq t_0$, $t' \leqq t_0$ and all $s \in [0,x]^r$.

Proof. Assume $t'<t$. By (12)

$$\psi_{ni}(t,s) - \psi_{ni}(t',s) = n\,\phi_{ni}\left([tn]-[t'n],\, \phi_n\left([t'n], \frac{1}{n}s\right)\right)$$
$$- n\,\phi_{ni}\left([t'n], \frac{1}{n}s\right) = n \sum_j (M_{nij}^{[tn]-[t'n]} - e_j^{(i)})\, \phi_{nj}\left([t'n], \frac{1}{n}s\right)$$
$$+ \frac{n}{2}\sum_{j,k} \phi_{nijk}([tn]-[t'n], \sigma_n(t,t',s))\, \phi_{nj}\left([t'n], \frac{1}{n}s\right)\, \phi_{nk}\left([t'n], \frac{1}{n}s\right)$$

where

$$0 \leqq \sigma_n(t,t',s) \leqq \phi_n\left([t'n], \frac{1}{n}s\right). \tag{19}$$

By (ii)

$$\phi_{ni}\left([tn], \frac{1}{n}s\right) \leqq \frac{k_1}{n}x \tag{20}$$

for some $k_1>0$, all $t \leqq t_0$ and all $s \in [0\,x]^r$.

By (v), (19) and (20)

$$\phi_{nijk}([tn]-[t'n], \sigma_n(t,t',s)) \leqq ([tn]-[t'n]) \cdot k_2 \tag{21}$$

for all n, all $t, t' \leqq t_0$, all $s \in [0\,x]^r$ and some $k_2>0$. Using (vi), (20) and (21) we have finally

$$|\psi_{ni}(t,s) - \psi_{ni}(t',s)| \leqq k_3(t-t') + k_4 \frac{[tn]-[t'n]}{n}$$

for some positive constants k_3, k_4, for all sufficiently large n, all t, $t' \leqq t_0$ and all $s \in [0\,x]^r$.

(viii) *Let $Q_{n,m}$ be a double sequence of probability measures on R^r_+. Then there exist an infinite sequence $\mathscr{S}$ of natural numbers and finite measures Q_m on R^r_+ such that*

$$Q_{n,m} \xrightarrow[n\in\mathscr{S}]{n\to\infty} Q_m \quad \textit{for each } m \textit{ (weakly)}.$$

This assertion is well known for simple sequences. Its proof applies verbally to double sequences

(ix) *There exist an infinite sequence $\mathscr{S}$ of natural numbers and probability measures $Q_i(t,\cdot)$ on R^r_+ such that*

$$Q_{ni}([t]_n,\cdot) \xrightarrow[n\in\mathscr{S}]{n\to\infty} Q_i(t,\cdot) \quad \textit{(weakly)} \tag{22}$$

for each $t \geqq 0$.

Proof. Consider first rational $t\geqq 0$ only. In this case, the probability measures $Q_{ni}(t,\cdot)$ can be arranged into a double sequence Q_{nm} and by (viii), there exist finite measures $Q_i(t,\cdot)$ and an infinite sequence $\mathscr{S}$ such that (22) holds for all rational $t\geqq 0$. Put, for each rational $t\geqq 0$,

$$\psi_i(t,s) = -\log \int_{R^r_+} e^{-bs} Q_i(t,db) \qquad (s\in R^r_+). \tag{23}$$

It follows from (21) that

$$\psi_i(t,s) = \lim_{\substack{n\to\infty\\ n\in\mathscr{S}}} \psi_{ni}(t,s) \tag{24}$$

for all rational $t\geqq 0$ and all $s\in R^r_+$, $s>0$.

Consider now an irrational $t'>0$ and take any $\varepsilon>0$ and $s\in R^r_+$. There exists a rational $t\geqq 0$ such that $|t-t'|<\varepsilon$ and, by (vii),

$$\begin{aligned}|\psi_{ni}(t',s)-\psi_{mi}(t',s| &\leqq |\psi_{ni}(t',s)-\psi_{ni}(t,s)| + |\psi_{mi}(t',s)-\psi_{mi}(t,s)|\\ &\quad + |\psi_{ni}(t,s)-\psi_{mi}(t,s)| < k\varepsilon + |\psi_{ni}(t,s)-\psi_{m,i}(t,s)|\end{aligned}$$

for some $k>0$ and sufficiently large m, n. Using (24) we see that the limit $\lim_{\substack{n\to\infty\\ n\in\mathscr{S}}} \psi_{ni}(t,s)$ exists also for each irrational $t>0$, i.e. for all $t\geqq 0$ (and all $s\in R^r_+$, $s>0$). By a well known theorem, there exist then finite measures $Q_i(t,\cdot)$ such that (22) holds for all $t\geqq 0$, and if we define $\psi_i(t,s)$ by (23) for all $t\geqq 0$ this time, the relation (24) holds (for all $t\geqq 0$ and all $s\in R^r_+$, $s>0$). It remains to show that $Q_i(t,\cdot)$ are probability measures.

Take a fixed $t\geqq 0$. By (ii)

$$\psi_{ni}(t,s) = n\,\phi_{ni}\left([t\,n], \frac{1}{n}s\right) \leqq k\,\|s\|$$

for some $k>0$, all n and all s from a finite interval $[0\,x]^r$. Hence, by (24), $\psi_i(t,s)\leqq k\|s\|$ for all $s\in[0\,x]^r$, $s>0$, and since $\psi_i(t,s)$ is continuous in s, $\psi_i(t,0)=0$.

(x) *The relation* (24) *holds for all $s\in R^r_+$ and the functions $\psi_i(t,s)$ are continuous in (t,s).*

Proof. Since we have shown that $Q_i(t,\cdot)$ are also probability measures, the first part of the assertion follows from (22). By (vii),

$$|\psi_i(t,s)-\psi_i(t',s)|<k|t-t'|$$

for some $k>0$, all $t, t'\leqq t_0$ and all $s\in[0\,x]^r$ and each $\psi_i(t,s)$ is continuous in s. This proves the second part of the assertion.

In the next lemmas, the following notation will be used:

$$R_{ij}(t,s)=\alpha_{ij}-\sum_k \beta_{ijk}\psi_k(t,s), \qquad R(t,s)=\big(R_{ij}(t,s)\big)_{i,j},$$

$$S_m(t,s)=\int_{0\leqq t_1<\cdots t_m\leqq t} R(t_m,s)R(t_{m-1},s)\dots R(t_1,s)\,dt_1\dots dt_m,$$

$$S_0(t,s)=E,$$

$$\psi(t,s)=\big(\psi_1(t,s),\dots,\psi_r(t,s)\big).$$

(xi) *For each $t_0>0$ and $x>0$, the series*

$$\sum_{m=0}^{\infty} S_m(t,s)$$

is convergent uniformly with respect to $(t,s)\in[0\,t_0]\times[0\,x]^r$.

Proof. By (x), $R_{ij}(t,s)$ are bounded in $[0\,t_0]\times[0\,x]^r$. Hence,

$$|S_m(t,s)|\leqq\frac{k^m t_0^m}{m!}I \tag{25}$$

for some $k>0$, all m and all $(t,s)\in[0\,t_0]\times[0\,x]^r$.

(xii)

$$\psi(t,s)=\left(\sum_{m=0}^{\infty}S_m(t,s)\right)s.$$

Proof. By (13) (with $t=1$) we have

$$\phi_n(s)=[M_n+Z_n(s)]\,s \tag{26}$$

where $Z_n(s)$ is the matrix $\big(Z_{nij}(s)\big)_{ij}$ and

$$Z_{nij}(s)=\tfrac{1}{2}\sum_k \phi_{nijk}\big(\sigma_n(s)\big)\,s_k, \qquad 0\leqq\sigma_n(s)\leqq s.$$

Using $\phi_n(\tau+1,s)=\phi_n\big(\phi_n(\tau,s)\big)$ and (26), we obtain

$$\phi_n(t,s)=\prod_{\tau=1}^{t}\big(M_n+Z_n(\phi_n(t-\tau,s))\big)\,s.$$

For each $t\geqq 0$, put

$$R_n(t,s)=A_n+nZ_n\left(\phi_n\left([t\,n],\frac{1}{n}s\right)\right).$$

Then

$$\psi_n(t,s)=n\,\phi_n\left([t\,n],\frac{1}{n}\,s\right)=\prod_{\tau=1}^{[tn]}\left(E+\frac{1}{n}R_n\left(\frac{[t\,n]-\tau}{n},s\right)\right)s$$
$$=\left(\sum_{m=0}^{[tn]}S_{n,m}(t,s)\right)s,$$

where

$$S_{n,m}(t,s)=\frac{1}{n^m}\sum_{\substack{0\leqq t_1<\cdots<t_m\leqq[tn]-1\\ \text{(integers)}}}R_n\left(\frac{t_m}{n},s\right)\ldots R_n\left(\frac{t_1}{n},s\right)$$
$$=\int_{0\leqq t_1\leqq\cdots t_m\leqq[t]_n}R_n(t_m,s)\ldots R_n(t_1,s)\,dt,\ldots,dt_m$$

and $S_{n,0}(t,s)\equiv E$.

By (2), (3), (ii) and (x),

$$R_n(t,s)\xrightarrow[\substack{n\to\infty\\ n\in\mathscr{S}}]{}R(t,s)$$

for each $t\geqq 0$ and $s\in R^r_+$. Further, for each $t\geqq 0$ and s, the functions $R_n(\tau,s)$ are uniformly bounded with respect to n and $\tau\in[0\;t]$ because of (2), (3) and (ii). Hence

$$S_{nm}(t,s)\xrightarrow[\substack{n\to\infty\\ n\in\mathscr{S}}]{}S_m(t,s) \tag{27}$$

for each m, t, s. Also, by (2), (3) and (ii), to each t and s there exists $k_1>0$ such that

$$|S_{n,m}(t,s)|\leqq\frac{k_1^m\,t^m}{m!}\,I \tag{28}$$

for all n and m. Finally

$$\psi_n(t,s)-\left(\sum_{m=0}^{\infty}S_m(t,s)\right)s=\sum_{m=0}^{l}\left(S_{n,m}(t,s)-S_m(t,s)\right)\cdot s$$
$$+\left(\sum_{m=l+1}^{[tn]}S_{m,m}(t,s)-\sum_{m=l+1}^{\infty}S_m(t,s)\right)\cdot s$$

for any l, $0<l<[t\,n]$ and the assertion of (xii) follows from (25), (27) and (28).

(xiii) *The functions $\psi_i(t,s)$ satisfy* (5) *with initial conditions $\psi_i(0,s)=s_i$.*

Proof. Differentiating formally (xii) we obtain

$$\frac{\partial}{\partial t}\psi(t,s)=\left(\sum_{m=1}^{\infty}\frac{\partial}{\partial t}S_m(t,s)\right)s$$
$$=R(t,s)\left(\sum_{m=0}^{\infty}S_m(t,s)\right)s=R(t,s)\,\psi(t,s).$$

Since by (xi) the series on the right-hand side is convergent uniformly with respect to t in each finite interval, the term-by-term differentiation is justified and (5) holds. The initial conditions follow from

$$\psi_{ni}(0,s)=s_i.$$

(xiv) $\psi_n(t, s) \xrightarrow[n\to\infty]{} \psi(t, s)$ *for all t and s.*

Proof. By (x), the limit relation holds if n runs through the sequence $\mathscr{S}$. Let us assume that (xiv) does not hold. Then there exists a sequence $\mathscr{S}_1$ of natural numbers such that

$$\psi_{ni}(t_0, s_0) \xrightarrow[\substack{n\to\infty \\ n\in\mathscr{S}_1}]{} l \neq \psi_i(t_0, s_0) \tag{29}$$

for some i, t_0, s_0. Using the same method as before we can select a subsequence $\mathscr{S}_2 \subset \mathscr{S}_1$ such that

$$\psi_n(t, s) \xrightarrow[\substack{n\to\infty \\ n\in\mathscr{S}_2}]{} \bar{\psi}(t, s) \qquad \text{for all } t \text{ and } s, \tag{30}$$

and prove that the limit function $\bar{\psi}(t, s)$ satisfies (5) with the same initial conditions $\bar{\psi}(0, s) = s$. Hence, $\bar{\psi}(t, s) \equiv \psi(t, s)$ which contradicts (29) and (30).

In the last three lemmas, the functions $C_i(s)$ defined by (7) and the functions

$$C_{ni}(s) = n^2\left(\phi_{ni}\left(\frac{1}{n}s\right) - \frac{s_i}{n}\right)$$

will be used.

(xv) *For each i and s*

$$C_{ni}(s) \xrightarrow[n\to\infty]{} C_i(s).$$

Proof.

$$\phi_{ni}\left(\frac{1}{n}s\right) = \sum_j M_{nij}\frac{s_j}{n} + \frac{1}{2}\sum_{j,k}\phi_{nijk}(\sigma_n)\frac{s_j s_k}{n^2}$$

$$= \frac{s_i}{n} + \frac{1}{n^2}\left[\sum_j \alpha_{nij} s_j + \frac{1}{2}\sum_{jk}\phi_{nijk}(\sigma_n) s_j s_k\right],$$

where $0 \leqq \sigma_n < \frac{1}{n} s$. The assertion follows from (2) and (3).

(xvi) *To each t_0 and $x > 0$ there exists n_0 and $k > 0$ such that*

$$|\psi_{nij}(t, s) - \psi_{nij}(t', s)| \leqq k\,|t - t'|$$

for all $n \geqq n_0$, all i, j, all $t \leqq t_0$, $t' \leqq t_0$ and all $s \in [0\, x]^r$.

Proof. Assume $t' < t$. By (11) and (12)

$$\psi_{nij}(t, s) - \psi_{nij}(t', s) = \sum_k \phi_{nik}\left([t\,n] - [t'\,n], \phi_n\left([t'\,n], \frac{1}{n}s\right)\right)\phi_{nkj}\left([t'\,n], \frac{1}{n}s\right)$$

$$-\phi_{nij}\left([t'\,n], \frac{1}{n}s\right) = \sum_k \left(M_{nik}^{[tn]-[t'n]} - e_k^{(i)}\right)\phi_{nkj}\left([t'\,n], \frac{1}{n}s\right)$$

$$+\sum_{k,l}\phi_{nikl}([t\,n] - [t'\,n], \sigma_n(t, t', s))\,\phi_{nl}\left([t'\,n], \frac{1}{n}s\right)\phi_{nkj}\left([t'\,n], \frac{1}{n}s\right)$$

where

$$0 \leqq \sigma_n(t, t', s) \leqq \phi_n\left([t'\,n], \frac{1}{n}s\right).$$

The assertion of (xvi) can be now proved in a similar way as that of (vii), if, in addition, the inequality

$$\phi_{nkj}\left([t\,n], \frac{1}{n} s\right) \leqq c \tag{31}$$

is used. (31) holds for some $c>0$, all $t \leqq t_0$ and all $s \in [0\, x]^r$ and follows easily from (i) and (iii).

(xvii) (6) *holds for each* i.

Proof. We shall prove the assertion for $s>0$ only, but it would possible to show that it holds for all $s \in R_+^r$ (with the convention that $\psi_{ij}(s)$ means the right-hand side derivative if $s_j=0$).

By (12),

$$\begin{aligned}\phi_{ni}(t+1, s) &= \phi_{ni}(t, \phi(s))\\ &= \phi_{ni}(t, s) + \sum_j \phi_{nij}(t, s)(\phi_{nj}(s) - s_j)\\ &\quad + \tfrac{1}{2} \sum_{jk} \phi_{nijk}(t, \sigma_n(t, s))(\phi_{nj}(s) - s_j)(\phi_{nk}(s) - s_k)\end{aligned}$$

where

$$|\sigma_n(t, s) - s| \leqq |\phi_n(s) - s|. \tag{32}$$

Hence

$$\begin{aligned}\psi_{ni}(t, s) &= n\, \phi_{ni}\left([t\,n], \frac{1}{n} s\right)\\ &= s_i + \frac{1}{n} \sum_j C_{nj}(s) \sum_{\tau=0}^{[tn]-1} \psi_{nij}\left(\frac{\tau}{n}, s\right)\\ &\quad + \frac{1}{2n^3} \sum_{jk} C_{nj}(s)\, C_{nk}(s) \sum_{\tau=0}^{[tn]-1} \phi_{nijk}\left(\tau, \sigma_n\left(\tau, \frac{1}{n} s\right)\right).\end{aligned}$$

Since $Q_{ni}([t]_n, \cdot) \xrightarrow[n\to\infty]{} Q_i(t, \cdot)$ weakly,

$$\psi_{nij}(t, s) \to \psi_{ij}(t, s) \tag{33}$$

for each $t \geqq 0$ and $s \in R_+^r$, $s>0$. Further, by (31)

$$\psi_{nij}(t, s) = \phi_{nij}\left([t\,n], \frac{1}{n} s\right) \leqq c \qquad \text{for some } c>0, \text{ all } n \text{ and all } t$$

from any given finite interval. Hence

$$\frac{1}{n} \sum_{\tau=0}^{[tn]-1} \psi_{ni}\left(\frac{\tau}{n}, s\right) = \int_0^{[t]_n} \psi_{nij}(\tau, s)\, d\tau \xrightarrow[n\to\infty]{} \int_0^t \psi_{ij}(\tau, s)\, ds. \tag{34}$$

By (ii) and (32), $n\, \sigma_n\left(t, \frac{1}{n} s\right) \leqq x$ for some x, all t and all n and consequently, by (v),

$$\left| \sum_{\tau=0}^{[tn]-1} \phi_{nijk}\left(\tau, \sigma_n\left(\tau, \frac{1}{n} s\right)\right) \right| \leqq k \sum_{\tau=0}^{[tn]-1} \tau \leqq k\, t^2\, n^2 \tag{35}$$

for some $k>0$ and all n.

Using (xiv), (xv), (34) and (35) we see that

$$\psi_i(t, s) = s_i + \sum_j C_j(s) \int_0^t \psi_{ij}(\tau, s)\, d\tau. \tag{36}$$

By (33) and (xvi),

$$|\psi_{ij}(t, s) - \psi_{ij}(t', s)| \leqq k(t - t')$$

for all t, t' from a finite interval. Hence, $\psi_{ij}(t, s)$ is continuous in t and we can therefore differentiate (36) with respect to t. This proves the existence of $\frac{\partial}{\partial t} \psi_i(t, s)$ and (6).

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