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On Semi-Groups such that $T_t - I$ is Compact for Some $t > 0$

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This paper develops some aspects of the theory of semi-groups $(T_t;\ t \geqq 0)$ of bounded linear operators on a Banach space which have the property of being near the identity, in the sense that, for some positive values of t, $T_t - I$ is compact. In Section 1, the theory is developed for general semi-groups, the only restriction being the assumption of strong continuity at the origin: in Section 2, a particular case is considered, that in which (T_t) is a Markov semi-group of operators on l_1, and it is shown that a particularly simple theory results.

1.

Let B be a complex Banach space, and let $(T_t; t \geqq 0)$ be a semi-group of class C_0 of bounded linear operators on B; i.e. T_t is strongly continuous in t for $t \geqq 0$, and $T_0 = I$. The infinitesimal generator of the semi-group is denoted by A; A is defined by

$$A x = \lim_{t \to 0+} t^{-1}(T_t - I)\, x$$

for all x in B for which the limit exists. The operator A is in general closed and unbounded, with domain $\mathfrak{D}(A)$ dense in B. The resolvent operator of A is denoted by R_λ, where $R_\lambda = (\lambda I - A)^{-1}$ for all scalar λ such that $(\lambda I - A)^{-1}$ exists and is bounded. It is assumed that $\|T_t\| \leqq M e^{\omega t}$ for all $t \geqq 0$, for constants M, ω. Throughout this paper, C will denote the set defined in the following way:

$$C = \{t > 0;\ T_t - I \text{ compact}\}.$$

We are concerned with the theory of semi-groups for which $C \neq \emptyset$. As mentioned in the summary, such semi-groups can be regarded as being near the identity for some positive t. In fact, as the following theorem shows, it can be said more explicitly that such semi-groups are invertible.

Theorem 1. *If $C \neq \emptyset$, then T_t is invertible for all t.*

Proof. Let $\sigma(T_t)$ denote the spectrum of T_t, $P\sigma(T_t)$ its point spectrum, and $N(T_t)$ its nullspace.

Suppose the semi-group is not invertible, that is, that $0 \in \sigma(T_t)$ for some, and then for all, $t > 0$. Then, by the Riesz-Schauder theory on compact operators, and by the spectral mapping theorem, it follows that, for all $t \in C$, $0 \in P\sigma(T_t)$, and $N(T_t)$ is finite dimensional.

Let t_1 be any element of C, and let x be any non-zero member of $N(T_{t_1})$; define $t_2 = \frac{1}{2} t_1$.

Then, since

$$T_{t_2}(T_{t_2}x)=T_{t_1}x=0,$$

it follows that $0\in P\sigma(T_{t_2})$.

Proceeding in this way with $t_n=\frac{1}{2}t_{n-1}$, it follows that there exists a sequence $t_n\downarrow 0$ such that $0\in P\sigma(T_{t_n})$ for all $n\geqq 1$.

Let $A_n=N(T_{t_n})\cap\{x\colon \|x\|=1\}$: then, since $N(T_s)\subseteq N(T_t)$ for $s\leqq t$, and since the T_t are bounded, it follows that (A_n) is a decreasing sequence of non-void closed sets, and further, since $N(T_{t_1})$ is finite dimensional, A_1 is compact.

Therefore, there exists x such that $x\in A_n$ for all $n\geqq 1$, and so $\|T_{t_n}x-Ix\|=\|x\|=1$ for all $n\geqq 1$.

This contradicts the strong continuity of the semi-group; hence the semi-group is invertible, the desired result.

From the identity

$$(T_t-I)(T_s-I)=(T_{t+s}-I)-(T_s-I)-(T_t-I),$$

it follows readily that

$$s\in C,\quad t\in C\Rightarrow s+t\in C,$$
$$s\in C,\quad t\notin C\Rightarrow s+t\notin C;$$

these relations imply that C is the intersection of an additive subgroup of the reals with the positive real line.

We distinguish three mutually exclusive and exhaustive forms which C may take.

(i) $C=]0,\infty[$.

(ii) $C=\{nx;$ for some $x>0$, and $n=1,2\ldots\}$.

(iii) C is a dense subset of $]0,\infty[$ with empty interior.

Examples will be given to show that all three types of set may occur. Further, some relationships between the type of C-set and the structure of the semi-group will be derived. In particular, it will be shown that C takes the first of the above forms if, and only if, A, the infinitesimal generator, is compact; if C takes the third form, A is necessarily unbounded; but if C takes the second form, A may be bounded or unbounded.

First, examples are given showing that the above classification of C-sets is non-void, and also that sets of type (ii) can arise from semi-groups having bounded or unbounded infinitesimal generators.

Examples. Take $B=l_1$, the space of absolutely convergent sequences.

(i) $T_t=I$ for all t.

(ii) (a) A bounded: $T_t=\operatorname{diag}\{e^{it},e^{-it},e^{it},e^{-it}\ldots\}$ $(x=2\pi)$.

(b) A unbounded: $T_t=\operatorname{diag}\{e^{it},e^{2it},e^{3it},\ldots\}$ $(x=2\pi)$.

(iii) $T_t=\operatorname{diag}\{e^{it},e^{2it},e^{3!it},e^{4!it},\ldots\}$.

The following two theorems contain the stated assertions about the type of set C and the structure of the semi-group. In fact, when C is of the first type, something more can be said.

Theorem 2. *The following conditions are equivalent for a strongly continuous semi-group* $(T_t; t \geqq 0)$.

(i) $C =]0, \infty[$.

(ii) A *is compact.*

(iii) $\lambda R_\lambda - I$ *is compact for some (and then for all)* $\lambda > \omega$.

Proof. (a) (i) $\Rightarrow A$ is bounded.

By the spectral theory of compact operators, it follows that, for all $t>0$, $\mu \in \sigma(T_t)$, $\mu \neq 1 \Rightarrow \mu \in P\sigma(T_t)$: using this fact, together with the spectral theory for semi-groups developed in [3], Chapter 16, Section 7, it follows that, apart possibly from the point 1,

$$\sigma(T_t) = \{e^{xt}; x \in P\sigma(A), x \neq 0\}, \qquad \text{for all } t>0. \tag{1}$$

Again by the spectral theory of compact operators, it is true that, for any $\varepsilon > 0$, the set

$$\{\lambda \in \sigma(T_t); |\lambda - 1| > \varepsilon\} \qquad \text{is finite, for all } t>0. \tag{2}$$

Suppose that the set $\{\mathrm{Im}(x): x \in P\sigma(A)\}$ is not bounded: then, on combining (1) and (2), it follows that there must be a subset $\{a_n; n \geqq 1\}$ (considered without loss of generality to be of positive elements) of $\{\mathrm{Im}(x): x \in P\sigma(A)\}$ with the following properties:

(I) $a_n \uparrow \infty$ as $n \to \infty$;

(II) for any $\varepsilon > 0$, $t > 0$, only a finite number of elements of $\{t a_n\}$ differ from integer multiples of 2π by more than ε. It is now proved that no such sequence $\{a_n\}$ can exist: (I am indebted to Professor J.F.C. Kingman for the following argument.)

Taking some ε with $0 < \varepsilon < \frac{1}{2}$, and writing

$$G_\varepsilon = \bigcup_{n=1}^{\infty} [2\pi n - \varepsilon, 2\pi n + \varepsilon],$$

property (II) implies that

$$]0, \infty[\subseteq \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} a_n^{-1} G_\varepsilon.$$

Therefore, by Baire's Category Theorem, there exists N such that $\bigcap_{n=N}^{\infty} a_n^{-1} G_\varepsilon$ has an interior point: therefore $\bigcap_{n=N}^{\infty} a_n^{-1} G_\varepsilon$ must contain an interval $[a, b]$, say: but this is impossible, since, for all large enough n, $2\varepsilon a_n^{-1} < (b-a)$.

Therefore, no sequence $\{a_n\}$ with the above properties exists, and so $\{\mathrm{Im}(x): x \in P\sigma(A)\}$ is bounded, by $K > 0$, say.

It follows from (1) that, for all $t \in \left[0, \frac{\pi}{2K}\right]$, $\sigma(T_t)$ is contained in the right half plane.

Since, by Theorem 1, the semi-group is invertible, it can be extended to a strongly continuous group, $\{T_t; -\infty < t < \infty\}$. Consider the function $\|T_{-t} x\|$ for arbitrary $x \in B$: this is a continuous function on $\left[0, \frac{\pi}{2K}\right]$, and so is bounded above: hence, by the Banach-Steinhaus theorem,

$$\sup_{t\in\left[0,\frac{\pi}{2K}\right]} \|T_{-t}\| = H < \infty.$$

$$\Rightarrow \|T_{-t}x\| \leqq H\,\|x\|, \qquad t\in\left[0,\frac{\pi}{2K}\right]$$

$$\Rightarrow \|T_t x\| \geqq H^{-1}\,\|x\|, \qquad t\in\left[0,\frac{\pi}{2K}\right].$$

It follows easily from this that, for $t\in\left[0,\frac{\pi}{2K}\right]$,

$$\sigma(T_t)\cap\{z\colon |z|<H^{-1}\}=\emptyset.$$

Therefore, since the spectrum of T_t is also contained in the right half plane for t in this range, it follows that there exists an open, connected, unbounded set containing 0, and which does not intersect with $\sigma(T_t)$ for $t\in\left[0,\frac{\pi}{2K}\right]$.

Hence, by [3], Theorem 16.5.2, $T_t=\exp(At)$ for some bounded A; this completes the first part of the proof.

(b) (i) $\Rightarrow$ (ii).

By (a), T_t is uniformly continuous for $t\geqq 0$: therefore there exists $d>0$ such that

$$\|T_t-I\|\leqq \tfrac{1}{2} \qquad \text{for } t\leqq d.$$

Take $t\leqq d$; then

$$\left\|t^{-1}\int_0^t T_s\,ds - I\right\| = \left\|t^{-1}\int_0^t (T_s-I)\,ds\right\|$$

$$\leqq t^{-1}\int_0^t \|T_s-I\|\,ds \leqq \tfrac{1}{2}.$$

Hence $\int_0^t T_s\,ds$ is invertible for all $t\leqq d$.

Therefore, since

$$T_t-I=A\int_0^t T_s\,ds=\int_0^t T_s\,ds\,A,$$

it follows that

$$A=\left[\int_0^t T_s\,ds\right]^{-1}(T_t-I),$$

and is therefore compact.

(c) (ii) $\Rightarrow$ (i).

Since all the operators involved are bounded, it follows that

$$T_t-I=A\int_0^t T_s\,ds \qquad \text{for all } t>0$$

$$\Rightarrow T_t-I \text{ is compact for all } t>0.$$

(d) (ii) $\Leftrightarrow$ (iii).

The $\Rightarrow$ statement is contained in [3], Lemma 5.7.1.

Conversely, suppose $\lambda R_\lambda - I$ is compact for some $\lambda > \omega$. Then, for $x \in B$, and $t>0$, by the strong continuity of the semi-group at the origin, it is true that

$$T_t x - x = A \int_0^t T_s x\, ds$$

$$\Rightarrow R_\lambda (T_t - I)\, x = (\lambda R_\lambda - I) \int_0^t T_s x\, ds$$

i.e.,

$$R_\lambda (T_t - I) = (\lambda R_\lambda - I) \int_0^t T_s\, ds,$$

on dropping the term x, and making the obvious convention for the meaning of the integral;

$$\Rightarrow \lambda R_\lambda (T_t - I) - (T_t - I) + (T_t - I) = \lambda(\lambda R_\lambda - I) \int_0^t T_s\, ds$$

$$\Rightarrow T_t - I = -(\lambda R_\lambda - I)(T_t - I) + \lambda(\lambda R_\lambda - I) \int_0^t T_s\, ds$$

$\Rightarrow T_t - I$ is compact for all $t>0$; i.e., $C =]0, \infty[$,

$\Rightarrow A$ compact

$\Rightarrow \lambda R_\lambda - I$ is compact for all $\lambda > \omega$, as above.

The following theorem proves the last of the stated assertions about the relationships between the form of the set C and the structure of the semi-group.

Theorem 3. *If C is a dense subset of $]0, \infty[$ with no interior points, then A is unbounded.*

Proof. If A is bounded, then, by using the argument of part (b) of the proof of Theorem 2, but choosing $t \leqq d$ such that $t \in C$ (as is clearly possible) it follows that A is compact, which implies $C =]0, \infty[$, by Theorem 2.

The above theorems leave certain questions about the sets C unresolved: in particular, it is not known whether every additive subgroup of the real line is the C-set of some semi-group.

Theorem 1, which states that semi-groups (T_t) for which $C \neq \emptyset$ are in fact invertible, enables us to apply the following general result on invertible semi-groups.

Theorem 4. *Let $(T_t; t \geqq 0)$ be an invertible semi-group of class C_0 with unbounded infinitesimal generator. Then there exists a real number ρ such that, for any α, β with $0 \leqq \alpha < \beta < \infty$,*

$$\sup_{t \in [\alpha, \beta]} [\| T_t - I \| - 1 - \exp(\rho t)] \geqq 0.$$

Proof. Since the inverse semi-group $(T_{-t}; t \geqq 0)$ is a semi-group of class C_0, it follows that there exist constants K and ρ such that

$$\| T_{-t} \| \leqq K \exp(-\rho t) \quad \text{for all } t \geqq 0.$$

Hence $\| T_{-nt} \| \leqq K \exp(-\rho n t)$ for positive integer n, and $t \geqq 0$, and so the spectral radius of T_{-t} is not greater than $\exp(-\rho t)$ for all $t \geqq 0$. Therefore, since

$$\lambda \in \sigma(T_t) \quad \text{if and only if } \lambda^{-1} \in \sigma(T_{-t}),$$

it follows that

$$\sigma(T_t)\cap\{z:|z|<\exp(\rho t)\}=\emptyset \qquad \text{for all } t\geqq 0. \tag{3}$$

Suppose that the statement of the theorem is violated for this particular ρ: then there exist α, β with $0\leqq\alpha<\beta<\infty$, and $k>0$, such that

$$\|T_t-I\|\leqq 1+\exp(\rho t)-k \qquad \text{for all } t\in[\alpha,\beta].$$

Combining this with (3), it follows that

$$\sigma(T_t)\cap\{z:\mathrm{Re}(z)\leqq 0,\ -k<\mathrm{Im}(z)<k\}=\emptyset \qquad \text{for all } t\in[\alpha,\beta]. \tag{4}$$

Relations (3) and (4) together imply that there exists an open, unbounded, connected set containing zero, and which does not intersect with $\sigma(T_t)$ for $t\in[\alpha,\beta]$. Hence, by Theorem 16.5.2 of [3], (T_t) has bounded infinitesimal generator, which contradicts the assumption of the theorem.

Theorem 4 has the following corollary, a result which has recently been proved by Williams [4].

Corollary 1. *If $(T_t; t\geqq 0)$ is an invertible semi-group of class C_0, and*

$$\limsup_{t\to 0+}\|T_t-I\|<2,$$

then the semi-group has bounded infinitesimal generator.

Proof. If this does not hold, then, putting $\alpha=0$ and letting $\beta\to 0$ in the statement of the theorem, implies that

$$\limsup_{t\to 0+}\|T_t-I\|\geqq 2, \qquad \text{which is a contradiction.}$$

This corollary, together with Theorem 1, yields immediately the following result.

Corollary 2. *If $(T_t; t\geqq 0)$ is a semi-group of class C_0 with unbounded infinitesimal generator, and $C\neq\emptyset$, then*

$$\limsup_{t\to 0+}\|T_t-I\|\geqq 2.$$

2. The Markov Case

We now consider the particular case of the above theory in which (T_t) is replaced by the standard Markov semi-group (P_t) of operators on l_1. In this case we shall show that the theory takes a particularly simple form, and that two of the possible forms that C may take can be excluded.

We use the same notation as before, but substituting P_t for T_t, where P_t is a matrix of transition probabilities $p_{ij}(t)$ (where i and j are positive integers) satisfying the following conditions:

(i) $p_{ij}(t)\geqq 0$, and $\sum_j p_{ij}(t)=1$ for all $t\geqq 0$.

(ii) $p_{ij}(s+t)=\sum_k p_{ik}(s)\,p_{kj}(t)$, for all $s, t\geqq 0$.

(iii) $\lim_{t\to 0+} p_{ij}(t)=\delta_{ij}$.

P_t operates on the space l_1 of absolutely convergent sequences by the formula

$$[P_t x]_j = \sum_i x_i p_{ij}(t).$$

Two lemmas are established first.

Lemma 1. *If $C \neq \emptyset$, A is bounded.*

Proof. The following three properties of Markov semi-groups are needed:

(a) If $0<\varepsilon<\frac{1}{4}$, and $p_{ii}(t) \geqq 1-\varepsilon$, then $p_{ii}(s) \geqq 1-(\varepsilon)^{\frac{1}{2}}$ for $0<s<t$; ([1, 2]).

(b) From the criterion for the compactness of a matrix operator $Q=[q_{ij}]$ on l_1, namely that

$$\lim_{n\to\infty} \sup_i \sum_{j=n}^{\infty} |q_{ij}| = 0,$$

and from the fact that the P_t's are stochastic, it follows easily that

$$t \in C \Leftrightarrow \lim_{i\to\infty} p_{ii}(t) = 1.$$

(c) From the definition of the norm of an operator on l_1, it follows that

$$\|P_t - I\| = 2(1 - \inf_i p_{ii}(t)) \quad \text{for all } t>0.$$

Let $t \in C$: then, by (b), there exists $N>0$ such that

$$p_{ii}(t) \geqq \tfrac{8}{9} \quad \text{for all } i \geqq N.$$

Hence, by (a),

$$p_{ii}(s) \geqq \tfrac{2}{3} \quad \text{for all } i \geqq \mathrm{N}, \text{ and } 0<s<t.$$

This implies that

$$\liminf_{s\to 0+} [\inf_i p_{ii}(s)] \geqq \tfrac{2}{3} > 0,$$

so that, by (c)

$$\limsup_{s\to 0+} \|P_s - I\| < 2.$$

Hence A is bounded, by Corollary 2 to Theorem 4.

Lemma 2. *If $C \neq \emptyset$, then $\{t: \sup_j p_{ji}(t) = p_{ii}(t)$ for all $i\} \subseteq C$.*

Proof. Suppose $C \neq \emptyset$, and there exists $t \notin C$ such that

$$\sup_j p_{ji}(t) = p_{ii}(t) \quad \text{for all } i.$$

Then for any $s \geqq t$, it follows that

$$p_{ii}(s) = \sum_j p_{ij}(s-t)\, p_{ji}(t) \leqq p_{ii}(t), \quad \text{for all } i.$$

But $t \notin C$, and so $p_{ii}(t) \nrightarrow 1$ as $i \to \infty$. Hence

$$p_{ii}(s) \nrightarrow 1 \quad \text{as } i \to \infty$$

$$\Rightarrow s \notin C \quad \text{for all } s \geqq t$$

$$\Rightarrow C = \emptyset\text{: a contradiction.}$$

Using these two lemmas, it is possible to prove the following result, essentially a strengthened version of Theorem 2, showing that if $C \neq \emptyset$, then $C =]0, \infty[$.

Theorem 5. *The following conditions are equivalent for the Markov semi-group* (P_t):

(i) $P_t - I$ *is compact for some (and then for all)* $t>0$;

(ii) $\lim_{i\to\infty} p_{ii}(t)=1$ *for some (and then for all)* $t>0$;

(iii) $\lambda R_\lambda - I$ *is compact for some (and then for all)* $\lambda>0$;

(iv) A *compact*;

(v) $\lim_{i\to\infty} a_{ii}=0$ *(where* $A=[a_{ij}]$*).*

Proof. (a) It is first proved that $C\neq\emptyset$ implies $C=]0,\infty[$. By Lemma 1, $C\neq\emptyset$ implies that the semi-group is uniform: hence there exists $d>0$ such that $\|P_t - I\|\leqq 1$ for all $t\in[0,d]$, that is, such that $\inf_i p_{ii}(t)\geqq\frac{1}{2}$ for all $t\in[0,d]$.

Hence, since P_t is stochastic,

$$\sup_j p_{ji}(t)=p_{ii}(t) \qquad \text{for all } i, \text{ and for all } t\in[0,d].$$

Therefore $]0,d]\subseteq C$, by Lemma 2, and so

$$C=]0,\infty[.$$

(b) The equivalence of conditions (i), (iii) and (iv) now follows from Theorem 2. (i) is equivalent to (ii) by an earlier remark.

Finally, the matrix $A=[a_{ij}]$ is a matrix satisfying $a_{ij}\geqq 0$, $i\neq j$; $a_{ii}\leqq 0$; $\sum_j a_{ij}\leqq 0$, with equality if A is bounded. But if (iv) or (v) hold, A is necessarily bounded, and hence the equivalence of conditions (iv) and (v) follows immediately from the criterion for compactness of operators on l_1.

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References

1. Blackwell, D., Freedman, D.: On the local behaviour of Markov transition probabilities. Ann. math. Statistics **39**, 2123–2127 (1968).
2. Davidson, R.: Arithmetic and other properties of certain Delphic semi-groups. Z. Wahrscheinlichkeitstheorie verw. Geb. **10**, 146–172 (1968).
3. Hille, E., Phillips, R.S.: Functional analysis and semi-groups. Amer. math. Soc. Colloq. Publ. **31** (1957).
4. Williams, D.: On operator semi-groups and Markov groups. Z. Wahrscheinlichkeitstheorie verw. Geb. **13**, 280–285 (1969).

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