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Limit Theorems for Occupation Times of Markov Processes

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Contents

1. Introduction. Ergodic Limit Problems

Let $\{x(t,\omega): t\geqq 0\}$ be a Markov process with stationary transition probabilities

$$P(x|E,t)=\mathrm{pr}\{\omega; x(t,\omega)\in E | x(0,\omega)=x\} \tag{1}$$

and write

$$p(x|E,s)=\int_0^\infty e^{-st} P(x|E,t)\,dt. \tag{2}$$

Let Φ be any non-negative function for which the stochastic integrals

$$H(t,\omega)=\int_0^\infty \Phi\big(x(u,\omega)\big)\,du \tag{3}$$

are defined. The process $\{H(t,\omega): t\geqq 0\}$, or H-process, is our principal object of study. We define the stochastic process $\{T(v,\omega): v\geqq 0\}$ inverse to the H-process by

$$T(v,\omega)=\sup\{t: H(t,\omega)\leqq v\}. \tag{4}$$

Typically, Φ will be the indicator function of a set A of states of the Markov process, in which case $H(t,\omega)$ gives the occupation time for the set A over $[0,t]$, and $T(v,\omega)$ gives the time required to accumulate time v in A.

The fundamental condition of Darling and Kac is the following

Condition (A). There exists a positive function $h(s)\to\infty$ as $s\to 0$ and a positive constant C such that

$$[h(s)]^{-1}\int_{-\infty}^{\infty} \Phi(y)\,p(x|dy,s)\to C \qquad (s\to 0)$$

uniformly for x such that $\Phi(x)>0$.

Darling and Kac prove the following results:

Theorem A. *If*

$$h(s)=s^{-\alpha}L(s^{-1}) \tag{5}$$

where $0\leqq\alpha<1$ *and* L *varies slowly at infinity, then*

$$\mathrm{pr}\left\{\omega: [Ch(t^{-1})]^{-1}\int_0^t \Phi(x(u,\omega))\,du\leqq x\right\}\to W_\alpha(x) \quad (t\to\infty) \tag{6}$$

where

$$\int_0^\infty e^{-sx}\,W_\alpha(dx)=\sum_{n=0}^\infty(-s)^n/\Gamma(1+n\alpha) \tag{7}$$

is the Mittag-Leffler function with parameter α.

Theorem B. *If Condition* (A) *holds, and if there exists a positive norming function* $u(t)$ *such that*

$$\mathrm{pr}\left\{[u(t)]^{-1}\int_0^t \Phi(x(u,\omega))\,du\leqq x\right\}\to G(x) \quad (t\to\infty) \tag{8}$$

at points of continuity of the non-degenerate distribution G, *then* h *satisfies* (5) *and* $G(x)=W_\alpha(x/b)$ *for some positive constant* b, *and* $b\,u(t)\sim C\,h(t^{-1})$ $(t\to\infty)$.

We restrict attention henceforth to the case when Condition (A) is satisfied. Under this restriction, Theorems A and B give complete information on limits of one-dimensional distributions of the H-process (3). We proceed now to formulate and solve more general limit problems.

Let $\{S(t,\omega): t\geqq 0\}$ be a stochastic process and define the one-parameter family of stochastic processes $\{S_\lambda(t,\omega): t\geqq 0\}$, where $\lambda>0$, by

$$S_\lambda(t,\omega)=S(\lambda t,\omega)/s(\lambda) \tag{9}$$

where $s(\lambda)$ is a suitable positive norming function. Consider the possibility that as $\lambda\to\infty$ the processes S_λ may converge (in a sense to be defined) to a non-degenerate stochastic process S_∞. By the *ergodic limit problem* for S we mean the determination of

i) Necessary and sufficient conditions on S for the existence of such $s(\lambda)$, S_∞

ii) The class of possible limit processes S_∞

iii) The class of possible norming functions $s(\lambda)$.

When the convergence concept is convergence of one-dimensional distributions of S_λ to non-degenerate limit distributions at points of continuity of the latter, we call the ergodic limit problem *one-dimensional.* We define the *finite-dimensional* ergodic limit problem analogously.

If the path-functions of the processes concerned lie in a function-space D carrying a topology T, let P be the probability measure defined on Borel sets A in D by the process S:

$$P(A)=\mathrm{pr}\{\omega: [t\to S(t,\omega)]\in A\} \tag{10}$$

and define P_λ, P_∞ similarly. If

$$f(S_\lambda)\to f(S_\infty) \quad (P_\infty\text{-a.e.}) \tag{11}$$

for every T-continuous functional f on D, the process S_λ converges to S_∞ *weakly* under T. This convergence concept defines the *weak* ergodic limit problem.

The process S_∞ is called the (one-dimensional, finite-dimensional or weak) *ergodic limit* of the process S. If we say simply that there exists an ergodic limit we shall mean that there exists a finite-dimensional (and so also a one-dimensional) ergodic limit, and also a weak ergodic limit under the topology in question. Theorems A and B of Darling and Kac solve the one-dimensional ergodic limit problem for the H-process (3), when Condition (A) holds. A finite-dimensional and a weak ergodic limit problem were solved by Lamperti [15] who obtained as limit process a Markov process whose one-dimensional distributions were arc-sine laws.

If we let $\lambda \to 0$ instead of $\lambda \to \infty$, we obtain the *initial* limit problem (in its one-dimensional, finite-dimensional and weak forms).

Before formulating our main results we make the following definition. Let $\{X(t,\omega): t \geqq 0\}$, $\{Y(v,\omega): v \geqq 0\}$ be two stochastic processes whose path-functions are non-decreasing and right-continuous. If

$$X(t,\omega) = \sup\{v: Y(v,\omega) \leqq t\} \qquad (t \geqq 0) \tag{12}$$

or what is equivalent by the assumptions on the sample paths

$$Y(v,\omega) = \sup\{t: X(t,\omega) \leqq v\} \qquad (v \geqq 0) \tag{13}$$

we say the processes X and Y are *inverse*, and call either the inverse of the other. If the finite-dimensional distributions of the Y-process coincide with those of the process inverse to the X-process we say that X and Y are *distributionally inverse*.

If $0<\alpha<1$, let $\{Y_\alpha(v,\omega): v \geqq 0\}$ be the stationary process with independent non-negative increments such that

$$\mathscr{E}\exp(-s\,Y_\alpha(v,\omega)) = \exp(-v\,s^\alpha). \tag{14}$$

The process Y_α is the *stable subordinator* with parameter α.

In Section 6 we prove the following result:

Proposition 1(a). i) *If* $0<\alpha<1$, *there exists a unique stochastic process* $\{X_\alpha(t,\omega): t \geqq 0\}$ *satisfying the product moment density relations*

$$\begin{aligned} &\partial^k[\mathscr{E}[X_\alpha(t_1,\omega)\,X_\alpha(t_2,\omega)\ldots X_\alpha(t_k,\omega)]]/\partial t_1\,\partial t_2\ldots\partial t_k \\ &\quad = 1/[\Gamma(\alpha)]^k\,[t_1(t_2-t_1)\ldots(t_k-t_{k-1})]^{1-\alpha} \qquad (0<t_1<\cdots<t_k). \end{aligned} \tag{15}$$

ii) *The process* X_α *is distributionally inverse to the stable subordinator* Y_α.

iii) *The one-dimensional distributions of the process* X_α *are the Mittag-Leffler laws:*

$$\mathscr{E}\exp(-s\,X_\alpha(t,\omega)) = \sum_{n=0}^{\infty} [-s\,t^\alpha]^n/\Gamma(1+n\alpha). \tag{16}$$

Subject to the restriction that the Darling-Kac Condition (A) holds, we prove the following theorems, which are our main results.

Theorem 1a. *The process* $\{H(t,\omega): t \geqq 0\}$ *has an ergodic limit if and only if condition* (5) *holds. If the parameter* α *in* (5) *lies in* (0, 1), *the limit process is the* *process* $\{X_\alpha(t,\omega)\colon t\geqq 0\}$ *of Proposition* 1 *inverse to the stable subordinator* $\{Y_\alpha(v,\omega)\colon v\geqq 0\}$.

Theorem 2a. *The process* $\{T(v,\omega)\colon v\geqq 0\}$ *inverse to the H-process has an ergodic limit if and only if condition* (5) *holds. The limit process is then the stable subordinator* $\{Y_\alpha(v,\omega)\colon v\geqq 0\}$.

As in the Darling-Kac theorems, the parameter α must lie in the interval [0, 1] (the sequence $\{n!/\Gamma(1+n\alpha)\}_{n=0}^{\infty}$ in (7) is not a moment sequence for other values of α). The limiting cases $\alpha=0, 1$ are less interesting than the general case $0<\alpha<1$, since they exhibit degeneracies. Since they also require special treatment we postpone consideration of them until Section 7. When we speak of a non-degenerate limit process we shall thus be referring to the general case $0<\alpha<1$.

The implication (ii) implies (iii) in Proposition 1 was noted by Stone [20] (see also Feller [6], p. 428).

By (14), $Y_\alpha(\lambda v,\omega)/\lambda^{1/\alpha}$ has the same distribution as $Y_\alpha(v,\omega)$. Also, both $X_\alpha(t,\omega)$ and $X_\alpha(\lambda t,\omega)/\lambda^\alpha$ satisfy (15). Thus the processes

$$\{X_\alpha(\lambda t,\omega)/\lambda^\alpha\colon t\geqq 0\},\qquad \{X_\alpha(t,\omega)\colon t\geqq 0\}$$

are identically distributed for each $\lambda>0$, as are the processes

$$\{Y_\alpha(\lambda v,\omega)/\lambda^{1/\alpha}\colon v\geqq 0\},\qquad \{Y_\alpha(v,\omega)\colon v\geqq 0\}.$$

In particular, each of the processes X_α, Y_α is its own initial and ergodic limit.

The next result completes Proposition 1(a) by specifying the finite-dimensional distributions of X_α without reference to Y_α.

Proposition 1(b). iv) *The finite-dimensional distributions of the process* X_α *are given by*

$$\begin{aligned}
s_1\dots s_k\cdot\rho_1\dots\rho_k\cdot\int_0^\infty\dots\int_0^\infty &\exp\left(-\sum_1^k s_i t_i-\sum_1^k\rho_i x_i\right)\\
&\cdot\operatorname{pr}\{X_\alpha(t_i,\omega)\leqq x_i,\ 1\leqq i\leqq k\}\,dt_1\dots dt_k\cdot dx_1\dots dx_k\\
&=\sum_{r=0}^{k}(-1)^r\cdot\sum_{1\leqq i_1<\dots<i_r\leqq k}\sum\dots\sum\cdot\sum_\pi\rho_{\pi(i_1)}\cdots\rho_{\pi(i_r)}\\
&\cdot[\rho_{\pi(i_1)}+\dots+\rho_{\pi(i_r)}+(s_{\pi(i_1)}+\dots+s_{\pi(i_r)})^\alpha]^{-1}\\
&\cdot[\rho_{\pi(i_2)}+\dots+\rho_{\pi(i_r)}+(s_{\pi(i_2)}+\dots+s_{\pi(i_r)})^\alpha]^{-1}\dots[\rho_{\pi(i_r)}+(s_{\pi(i_r)})^\alpha]^{-1}
\end{aligned}$$

where π *runs over the* $r!$ *permutations of the set* $\{i_1,\dots,i_r\}$.

For $k=1$, it may be verified that (iv) reduces to (iii). The complicated functional form in (iv) is obtained from (ii) by use of the formula of inclusion and exclusion and a straightforward calculation using the independent increments property of the stable subordinator Y_α.

We have stated our main results, Theorems 1a and 2a, in terms of weak convergence under a topology implicit in the definition of ergodic limit. In Section 2 we define this topology, and prove an invariance principle which reduces the problem to convergence of finite-dimensional distributions. In Section 3 we define two new processes, the U- and V-processes, which together with our basic H- and T-processes form a tetrad $\{H,T,U,V\}$. The study of this tetrad is motivated by applications to Markov chains which will be developed elsewhere; a closely analogous tetrad features prominently in related work on regenerative phenomena (see Section 8). We also formulate Theorems 1b and 2b, the analogues of Theorems 1a and 2a for the U- and V-processes. The proof of Theorem 1a follows in Section 4. The close connection between the results of Theorems 1a and 2a is explained by the work of Section 5, which provides proofs of Theorems 2a, 1b and 2b. In Section 6 we identify the limit process X_α as the inverse of the stable subordinator Y_α, thus proving Propositions 1a, 1b. In fact (i) of Proposition 1a is proved earlier as a corollary to the work of Section 4, while two of the lemmas of Section 4 have their proofs postponed till Section 6. The degenerate limiting cases $\alpha=0, 1$ are dealt with in Section 7, where we also consider the important case $\alpha=\frac{1}{2}$ and its connection with Brownian motion. Finally we mention some connections with related work in Section 8.

2. Topological Preliminaries

In this section we define the topology under which our limit theorems hold, and use it to reduce the solution of our weak ergodic limit problems to the solution of the corresponding finite-dimensional ergodic limit problems.

The path-functions of the H-process are non-decreasing and continuous. By (1.3), the path-functions of the T-process are non-decreasing and right-continuous ($T(v,\omega)$ having a jump of size m at the point v whenever $H(t,\omega)$ is constant over an interval $[t_1, t_1+m]$ with $H(t_1,\omega)=v$). By (1.3), (1.4),

$$H(T(v,\omega),\omega)=v \qquad (v\geqq 0), \tag{1}$$

$$T(H(t,\omega),\omega)\geqq t \qquad (t\geqq 0). \tag{2}$$

Equality holds in (2) whenever t is a point of increase of $H(t,\omega)$.

For $0<T<\infty$, we write $C[0,T]$ for the space of all continuous real-valued functions defined on $[0,T]$, and $D[0,T]$ for the space of all functions right-continuous on $[0,T)$ with left-hand limits on $(0,T]$. Define $C[0,\infty)$, $D[0,\infty)$ similarly. Then the path-functions of the H-process lie in $C[0,\infty)$, and those of the T-process lie in $D[0,\infty)$.

The natural topology for $C[0,T]$ is the uniform or sup-norm topology U_T under which $x_n\to x$ whenever

$$\sup\{|x_n(t)-x(t)|: t\in[0,T]\}\to 0 \qquad (n\to\infty).$$

For $C[0,\infty)$ one defines the topology U_∞ (the topology of uniform convergence on compacta or the compact-open topology) under which $x_n\to x$ whenever

$$\sup\{|x_n(t)-x(t)|: t\in[0,T]\}\to 0 \qquad (n\to\infty) \text{ for all } 0<T<\infty.$$

From our point of view the natural topology for $D[0,T]$ is the J_1-topology defined by Skorokhod ([19]), under which $x_n\to x$ whenever there exists a sequence of continuous bijections λ_n from $[0,T]$ onto itself such that

$$\sup\{|\lambda_n(t)-t|: t\in[0,T]\}\to 0 \qquad (n\to\infty), \tag{3}$$

$$\sup\{|x_n(t)-x(\lambda_n(t))|: t\in[0,T]\}\to 0 \qquad (n\to\infty). \tag{4}$$

For a full discussion of this topology see Billingsley ([1]).

For $D[0, \infty)$, Stone ([21]) defines the topology J_1 under which $x_n \to x$ whenever there exists a sequence of continuous bijections λ_n from $[0, \infty)$ onto itself such that (3), (4) hold for each $T > 0$. The following theorem ([21]) extends Skorokhod's criterion for weak convergence under J_1 of stochastic processes with path-functions in $D[0, T]$ to $D[0, \infty)$.

Theorem (Stone).

If $\{X_n: n=0, 1, \ldots\}$ are stochastic processes on $D[0, \infty)$, X_n converges to the limit process X_∞ as $n \to \infty$ weakly under the J_1-topology on $D[0, \infty)$ if and only if

i) The finite-dimensional distributions of X converge to those of X_∞ as $n \to \infty$

ii)
$$\lim_{c \to 0} \limsup_{n \to \infty} \operatorname{pr}\{\Delta(c, X_n, T) > \varepsilon\} = 0 \tag{5}$$

for each $\varepsilon > 0$ and each $0 < T < \infty$, where

$$\Delta(c, X, T) = \sup\{\min(|X(t_1) - X(t)|, |X(t_2) - X(t)|): \\ 0 \leqq t - c < t_1 < t \leqq t_2 < t + c \leqq T\}. \tag{6}$$

In order to unify the treatment, we shall embed $C[0, \infty)$ in $D[0, \infty)$ and regard the H-process as having path-functions in $D[0, \infty)$. In $D[0, \infty)$, one sees by taking $\lambda_n(t) = t$ that for continuous functions x_n, x, $x_n \to x$ under J_1 is equivalent to $x_n \to x$ under U_∞ (see [1]). The restriction of the J_1-topology from $D[0, \infty)$ to $C[0, \infty)$ thus coincides with the U_∞-topology. Since the limit process X_∞ in Theorem 1a lies in $C[0, \infty)$, we may thus replace the J_1-topology in Theorem 1a by the U_∞-topology. Since U_∞ is the natural topology for $C[0, \infty)$, our results for the H-process are not at all weakened by the embedding procedure.

We call condition (5) the Skorokhod Δ-condition. In general, its verification is formidable. However, our next result shows that in the cases we shall consider, it follows quite simply from convergence of finite-dimensional distributions.

Theorem 3. *Let $\{X_n: n=0, 1, \ldots\}$ be a sequence of stochastic processes whose path-functions lie in $D[0, \infty)$. If*

i) *The finite-dimensional distributions of X_n converge as $n \to \infty$ to those of X_∞.*

ii) *The process X_∞ is continuous in probability.*

iii) *The processes X_n have monotone path-functions — then $X_n \to X$ weakly under the J_1-topology on $D[0, \infty)$.*

Under hypotheses (ii) and (iii), weak convergence under J_1 is thus equivalent to convergence of finite-dimensional distributions.

Proof. By a result of Lévy, (ii) implies that the process X_∞ is uniformly continuous in probability on compact sets. Hence

$$\operatorname{pr}\{\Delta(k^{-1}, X, T) > \varepsilon\} \to 0 \qquad (k \to \infty) \tag{7}$$

for each $\varepsilon > 0$, $0 < T < \infty$. Since X_n has monotone sample-paths, the variation of $X_n(t, \omega)$ over any interval is the difference in modulus of the values at the end-points. For any process $\{X(t, \omega): t \geqq 0\}$ with monotone path-functions, define

$$D(k, X, T) = \max\{|X(rT/k) - X((r-1)T/k)|: r = 1, 2, \ldots k\}. \tag{8}$$

Then $D(k, X, T) \leqq \Delta(k^{-1}, X, T)$, and if $D(k, X) < \varepsilon$, $\Delta(2k^{-1}, X, T) < 2\varepsilon$. So

$$\tfrac{1}{2}\Delta(2k^{-1}, X, T) \leqq D(k, X, T) \leqq \Delta(k^{-1}, X, T). \tag{9}$$

By (7) and (9) for $X=X_\infty$,

$$\operatorname{pr}\{D(k, X_\infty, T)>\varepsilon\}\to 0 \qquad (k\to\infty). \tag{10}$$

Since $D(k, X, T)$ involves only finitely many function values of $X(t,\omega)$, and since by hypothesis (i) we have convergence of finite-dimensional distributions,

$$\lim_{n\to\infty}\operatorname{pr}\{D(k, X_n, T)>\varepsilon\}=\operatorname{pr}\{D(k, X_\infty, T)>\varepsilon\}. \tag{11}$$

By (11) and (10),

$$\lim_{k\to\infty}\lim_{n\to\infty}\operatorname{pr}\{D(k, X_n, T)>\varepsilon\}=0. \tag{12}$$

By (12) and (9),

$$\lim_{k\to\infty}\lim_{n\to\infty}\operatorname{pr}\{\Delta(2k^{-1}, X_n, T)>2\varepsilon\}=0 \tag{13}$$

for every $\varepsilon>0$ and $0<T<\infty$, which shows that the Skorokhod Δ-condition is satisfied. By Stone's theorem, X_n converges to X_∞ weakly under the J_1-topology on $D[0,\infty)$, which proves the theorem.

By Theorem 3, we may reduce the solution of the weak ergodic limit problems in Theorems 1a and 2a to proving convergence of finite-dimensional distributions. For the stable subordinator Y_α is continuous in probability, since

$$\begin{aligned}\operatorname{pr}\{Y_\alpha(u+v)-Y_\alpha(u)>\varepsilon\}&=\operatorname{pr}\{Y_\alpha(v)>\varepsilon\}\\ &\leqq[1-\mathscr{E}\exp(-s\,Y_\alpha(v))]/[1-\exp(-s\,\varepsilon)]\\ &=[1-\exp(-v\,s^\alpha)]/[1-\exp(-s\,\varepsilon)]\to 0 \qquad (v\to 0).\end{aligned}$$

Also, since the sample paths of Y_α almost surely contain no intervals of constancy, the inverse process X_α has continuous sample-paths. Thus the hypothesis (ii) of Theorem 3 is satisfied in both cases.

Theorem 3 thus plays the role of an invariance principle in what follows (see [1]).

3. The U- and V-Processes

When $0\leqq\Phi\leqq 1$, we have from (1.3)

$$0\leqq H(t,\omega)\leqq t \qquad (t\geqq 0) \tag{1}$$

or by (1.4),

$$v\leqq T(v,\omega) \qquad (v\geqq 0). \tag{2}$$

In this case we define the U-process $\{U(v,\omega): v\geqq 0\}$ by

$$U(v,\omega)=T(v,\omega)-v \qquad (v\geqq 0). \tag{3}$$

Then $U(v,\omega)\geqq 0$. Also, by (2.1),

$$v_2-v_1=\int_{T(v_1,\omega)}^{T(v_2,\omega)}\Phi(x(u,\omega))\,du$$

and since

$$\Phi\leqq 1, \qquad v_2-v_1\leqq T(v_2,\omega)-T(v_1,\omega).$$

Thus $U(v,\omega)$ is non-decreasing. Since $T(v,\omega)$ is right-continuous, so is $U(v,\omega)$, and we may define the V-process $\{V(t,\omega): t\geqq 0\}$ inverse to the U-process by

$$V(t,\omega)=\sup\{v: U(v,\omega)\leqq t\}. \tag{4}$$

Lemma 1.

i) $$V(t,\omega)=\int_0^\infty \Phi(x(u,\omega))\, I\{u\colon u-H(u,\omega)\leqq t\}\,du, \tag{5}$$

ii) $$U(t,\omega)=\int_0^\infty [1-\Phi(x(u,\omega))]\, I\{u\colon H(u,\omega)\leqq t\}\,du. \tag{6}$$

Proof. To prove (i), write $\Sigma(t,\omega)$ for the right-hand side of (5). Since $0\leqq\Phi\leqq 1$, $u-H(u,\omega)=\int_0^u [1-\Phi(x(v,\omega))]\,dv$ is non-decreasing and continuous in u. We may thus define a random variable $a(t,\omega)$ by

$$a(t,\omega)=\sup\{u\colon u-H(u,\omega)\leqq t\}. \tag{7}$$

We assume $a(t,\omega)<\infty$; the case $a(t,\omega)=\infty$ is easily treated. Then

$$\begin{aligned}\Sigma(t,\omega)&=\int_0^{a(t,\omega)} \Phi(x(u,\omega))\,du\\ &=H(a(t,\omega),\omega)\end{aligned}$$

and

$$\{u\colon u-H(u,\omega)\leqq t\}=[0,a(t,\omega)]. \tag{8}$$

We next prove

$$\{u\colon u-H(u,\omega)\leqq t\}=[0,t+\Sigma(t,\omega)]. \tag{9}$$

For if $v-H(v,\omega)\leqq t$, $v\leqq a(t,\omega)$, whence

$$\Sigma(t,\omega)=H(a(t,\omega),\omega)\geqq H(v,\omega)\geqq v-t,$$

or $v\leqq t+\Sigma(t,\omega)$. Similarly if $v-H(v,\omega)>t$, $v>t+\Sigma(t,\omega)$, proving (9).

Since the H- and T-processes are inverse, $u-H(u,\omega)\leqq t$ is equivalent to $T(u-t,\omega)\leqq u$, or to $U(u-t,\omega)\leqq t$. By the definition of the V-process, this is equivalent to $u-t\leqq V(t,\omega)$. Thus

$$\{u\colon u-H(u,\omega)\leqq t\}=[0,t+V(t,\omega)]. \tag{10}$$

Combining (8), (9) and (10),

$$a(t,\omega)-t=V(t,\omega)=\Sigma(t,\omega),$$

proving (5).

To prove (ii): write $\Sigma^*(t,\omega)$ for the right-hand side of (6) (Σ^* is obtained from Σ by replacing Φ by $1-\Phi$). Since $\{u\colon H(u,\omega)\leqq t\}=[0,T(t,\omega)]$,

$$\begin{aligned}\Sigma^*(t,\omega)&=\int_0^{T(t,\omega)} [1-\Phi(u,\omega)]\,du\\ &=T(t,\omega)-H(T(t,\omega),\omega)\\ &=T(t,\omega)-t\\ &=U(t,\omega).\end{aligned}$$

This completes the proof of the lemma.

When Φ is the indicator function of a set A of states, $1-\Phi$ is the indicator function of the complement of A. By Lemma 1, replacing A by its complement corresponds to the operation of inversion.

The ergodic limit problem for the U- and V-processes has the following solution, exactly analogous to the solution for the H- and T-processes.

Theorem 1b. *When* $0 \leqq \Phi \leqq 1$, *Theorem* 1a *applies with* $H(t,\omega)$ *replaced by* $V(t,\omega)$.

Theorem 2b. *When* $0 \leqq \Phi \leqq 1$, *Theorem* 2a *applies with* $T(v,\omega)$ *replaced by* $U(v,\omega)$.

The term v in (3) is called the *deterministic drift*; thus the T- and U-processes differ only through the drift. We also say that the H- and V-processes differ only through the drift since the same is true of their inverses T and U.

One may then interpret Theorems 1b, 2b by saying that the deterministic drift is annihilated on passing to ergodic limits.

When Φ is the indicator function of a set A of states, $V(t,\omega)$ is the time spent in A when the time spent outside A reaches t. Heuristically, Theorem 1b indicates that for large t the system spends much more time outside A than inside A – as one would expect from (1.5) and the Darling-Kac theorem, since

$$H(t,\omega) \sim C h(t^{-1})/\Gamma(1+\alpha) \sim C t^{\alpha} L(t)/\Gamma(1+\alpha) \qquad (t \to \infty)$$

where $\alpha < 1$.

One observes throughout the proofs to follow a close connection between the one-dimensional and finite-dimensional ergodic limit problems (which explains why the Darling-Kac theorems possess these generalisations). This is essentially because the ergodic limit Y_α of the T- and U-processes has independent increments, and the one-dimensional distributions of such a process determine the finite-dimensional distributions.

We proceed to prove convergence of finite-dimensional distributions of the H-process in Section 4. In Section 5 we reduce the finite-dimensional ergodic limit problems for the processes T, U and V to the corresponding problem for the H-process. In Section 6 we prove Proposition 1 on the limit process, and in Section 7 we deal with the limiting cases $\alpha = 0, 1$. Finally we indicate some related results in Section 8.

We shall have many occasions in what follows to use the theory of regular variation. This was developed by Karamata ([8, 9]; see also Feller [6]) for continuous functions. The theory has been generalised to measurable functions by de Bruijn and others ([14]; see also [2]). We shall make use of Karamata's Tauberian theorem [9] to estimate moments and product moments, of various criteria for regular variation [6], and of asymptotic relations between pairs of regularly varying functions [2].

We shall refer repeatedly to the completely monotone functions $\exp(-s^{\alpha})$ and $\sum_{n=0}^{\infty} (-s)^n/\Gamma(1+n\alpha)$ (see [4]). For their inverse Laplace transforms see Pollard ([16, 17]).

4. The H-Process

Proof of Theorem 1a. Assume first that the weak ergodic limit problem for the H-process has a solution. Then the finite-dimensional, and hence also the one-dimensional, ergodic limit problems have solutions, and hence by the Darling-Kac Theorem B, (1.5) holds.

Conversely, assume (1.5). The Darling-Kac Theorem A shows that the one-dimensional ergodic limit problem has a solution. We show that the finite-dimensional ergodic limit problem also has a solution; by Section 2, this solves the weak ergodic limit problem also.

The Darling-Kac method is based on asymptotic estimates for moments. If

$$\mu_k(t)=\mathscr{E}[H(t,\omega)]^k \tag{1}$$

and

$$\tilde{\mu}_k(s)=\int_0^\infty e^{-st}\,d\mu_k(t) \tag{2}$$

is its Laplace-Stieltjes transform (LST), Darling and Kac proved

$$\tilde{\mu}_k(s)\sim k!\,C^k[h(s)]^k\sim k!\,C^k s^{-k\alpha}[L(s^{-1})]^k \quad (s\to 0) \tag{3}$$

or equivalently by the Tauberian theorem for LSTs ([9])

$$\mu_k(t)\sim k!\,t^{\alpha k}\,C^k[L(t)]^k/\Gamma(1+k\alpha) \quad (t\to\infty). \tag{4}$$

We use multi-dimensional LST arguments to estimate product moments.

Lemma 2. *When Condition* (A) *and* (1.5) *hold for* $0<\alpha<1$,

$$\begin{aligned}\lim_{\lambda\to\infty}\int_0^\infty\dots\int_0^\infty \exp\left(-\sum_1^k s_j t_j\right) d_{t_1,\dots,t_k}\frac{\mathscr{E}H(\lambda t_1)H(\lambda t_2)\dots H(\lambda t_k)}{C^k[h(\lambda^{-1})]^k}\\ =\sum_\pi [(s_{\pi(1)}+\dots+s_{\pi(k)})(s_{\pi(2)}+\dots+s_{\pi(k)})\dots(s_{\pi(k)})]^{-\alpha}\end{aligned} \tag{5}$$

where π runs through the permutations of the set $\{1,\dots,k\}$.

Proof. If

$$F_\lambda(s_1,\dots,s_k)=\iint_{0<t_1<\dots<t_k}\cdots\int \exp\left(-\sum_1^k s_j t_j\right) d_{t_1,\dots,t_k}\mathscr{E}H(\lambda t_1)\dots H(\lambda t_k)/C^k[h(\lambda^{-1})]^k \tag{6}$$

and if $G_\lambda(s_1,\dots,s_k)$ denotes the left-hand side in (5), then by symmetry,

$$G_\lambda(s_1,\dots,s_k)=\sum_\pi F_\lambda(s_{\pi(1)},\dots,s_{\pi(k)}). \tag{7}$$

We show that as $\lambda\to\infty$,

$$F_\lambda(s_1,\dots,s_k)\to[(s_1+\dots+s_k)(s_2+\dots+s_k)\dots(s_k)]^{-\alpha}. \tag{8}$$

The lemma follows from (7) and (8) by symmetry.

Write $\theta_j=s_j+\dots+s_k$. By the change of variable $u_i/\lambda=t_i-t_{i-1}$ ($t_0=0$), we may write $C^k[h(\lambda^{-1})]^k F_\lambda(s_1,\dots,s_k)$ as

$$\begin{aligned}\int_0^\infty\cdots\int_0^\infty \exp\left(\lambda^{-1}\sum_1^k\theta_j u_j\right)\\ \cdot\mathscr{E}\left[\Phi(x(u_1))\,\Phi(x(u_1+u_2))\dots\Phi(x(u_1+\dots+u_k))\right]du_1\dots du_k.\end{aligned} \tag{9}$$

In terms of the transition probabilities,

$$\begin{aligned}&\mathscr{E}\left[\Phi(x(u_1))\,\Phi(x(u_1+u_2))\dots\Phi(x(u_1+\dots+u_k))\right]\\ &=\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty \Phi(x_1)\dots\Phi(x_k)\,P(0|dx_1,u_1)\,P(x_1|dx_2,u_2)\dots P(x_{k-1}|dx_k,u_k).\end{aligned} \tag{10}$$

By (9) and (10), we may write $C^k[h(\lambda)]^k F_\lambda(s_1, \ldots, s_k)$ as

$$\int_{-\infty}^{\infty} \Phi(x_1) \int_0^{\infty} \exp(-\theta_1 u_1/\lambda) P(0|dx_1, u_1)\, du_1 \ldots \cdot \int_{-\infty}^{\infty} \Phi(x_k) \int_0^{\infty} \exp(-\theta_k u_k/\lambda) P(x_{k-1}|dx_k, u_k)\, du_k. \tag{11}$$

By (1.2),

$$\int_0^{\infty} \exp(-\theta_k u_k/\lambda) P(x_{k-1}|dx_k, u_k)\, du_k = p(x_{k-1}|dx_k, \theta_k u_k/\lambda). \tag{12}$$

By Condition (A),

$$\lim_{\lambda\to\infty} [C h(\theta_k/\lambda)]^{-1} \cdot \int_{-\infty}^{\infty} \Phi(x_k)\, p(x_{k-1}|dx_k, \theta_k u_k/\lambda) = 1 \tag{13}$$

uniformly in x_{k-1}. Combining (12) and (13), we obtain an asymptotic estimate for the result of the last two integrations in (11). Because of the uniformity in x_{k-1}, we may use this estimate and a further application of the same method to obtain the asymptotic estimate $C^2 h(\theta_k/\lambda)\, h(\theta_{k-1}/\lambda)$ for the result of the last four integrations in (11), uniformly in x_{k-2}. Iterating this procedure k times, we obtain finally

$$C^k[h(\lambda^{-1})]^k F_\lambda(s_1, \ldots, s_k) \sim C^k h(\theta_1/\lambda) \ldots h(\theta_k/\lambda) \qquad (\lambda\to\infty).$$

By (1.5), h varies regularly, and so

$$F_\lambda(s_1, \ldots, s_k) \to 1/[\theta_1\, \theta_2 \ldots \theta_k]^\alpha \qquad (\lambda\to\infty).$$

This proves (8) and completes the proof of the lemma.

The next result relates the expression on the right of (5) to the process X_α of Proposition 1. We postpone its proof to Section 6.

Lemma 3.

$$s_1, \ldots, s_k \cdot \int_0^{\infty} \cdots \int_0^{\infty} \exp\left(-\sum_1^k s_j t_j\right) \cdot \mathscr{E}[X_\alpha(t_1, \omega) \ldots X_\alpha(t_k, \omega)]\, dx_1, \ldots, dx_k = \sum_\pi 1/[(s_{\pi(1)} + \cdots + s_{\pi(k)})(s_{\pi(2)} + \cdots + s_{\pi(k)}) \ldots (s_{\pi(k)})]^\alpha. \tag{14}$$

By the continuity theorem for LSTs, Lemmas 2 and 3 show

$$\lim_{\lambda\to\infty} \frac{\mathscr{E} H(\lambda t_1, \omega) \ldots H(\lambda t_k, \omega)}{C^k[h(\lambda^{-1})]^k} = \mathscr{E} X_\alpha(t_1, \omega) \ldots X_\alpha(t_k, \omega). \tag{15}$$

Write $m_\alpha(\lambda; t_1, \ldots, t_k)$, $m_\alpha(t_1, \ldots, t_k)$ for the left- and right-hand sides of (15).

Lemma 4. *If* $D(\pi) = D(\pi, \alpha; t_1, \ldots, t_k)$ *is the domain*

$$\{(u_1, \ldots, u_k);\, 0 \leqq u_i \leqq t_i,\, i = 1, 2, \ldots, k,\, 0 < u_{\pi(1)} < u_{\pi(2)} < \cdots < u_{\pi(k)}\}, \tag{16}$$

then

$$m_\alpha(t_1, \ldots, t_k) = \iint\cdots\int_{D(\pi, \alpha;\, t_1, \ldots, t_k)} du_1 \ldots du_k \cdot [\Gamma(\alpha)]^{-k} \cdot [u_{\pi(1)}(u_{\pi(2)} - u_{\pi(1)}) \ldots (u_{\pi(k)} - u_{\pi(k-1)})]^{-(1-\alpha)}. \tag{17}$$

We defer the proof of Lemma 4 to Section 6. If $t=\max\{t_i: i=1,2,\ldots,k\}$, Lemma 4 shows that if we replace the restriction $0\leqq u_i\leqq t_i$ in (16), (17) by $0\leqq u_i\leqq t$, we obtain an upper bound for $m_\alpha(t_1,\ldots,t_k)$, namely $m_\alpha(t,\ldots,t)$. But $m_\alpha(t,\ldots,t)$ has the form of a convolution integral, and by a change of variable in (17) we obtain

$$\begin{aligned} m_\alpha(t,\ldots,t) &= k!\,[\Gamma(\alpha)]^{-k}\cdot \int\!\!\int_{0<u_1<\cdots<u_k<t}\!\!\cdots\int \frac{du_1\ldots du_k}{[u_1(u_2-u_1)\ldots(u_k-u_{k-1})]^{1-\alpha}} \\ &= k!\,[\Gamma(\alpha)]^{-k}\cdot \int\!\!\int_{\substack{v_1+\cdots+v_k\leqq t\\ v_i\geqq 0}}\!\!\cdots\int \frac{dv_1\ldots dv_k}{(v_1v_2\ldots v_k)^{1-\alpha}} \\ &= k!\,t^{k\alpha}/\Gamma(1+k\alpha) \end{aligned} \tag{18}$$

by Dirichlet's integral ([23] p. 258) or by direct evaluation.

Take positive parameters t_i, s_i $(1\leqq i\leqq k)$, and write $t=\max\{t_1,\ldots,t_k\}$. If $m_\alpha(\lambda;t,\ldots,t)$ has n arguments t, it is the n-th moment of $H(\lambda t,\omega)/Ch(1/\lambda)$. By (4) and (1.5),

$$m_\alpha(\lambda;t,\ldots,t)\sim\frac{n!}{\Gamma(1+n\alpha)}\cdot t^n[L(\lambda t)/L(\lambda)]^n.$$

Since L varies slowly, given $t>0$, $\varepsilon>0$ we can choose N so large that for $\lambda>N$,

$$\left|\frac{L(\lambda t)}{L(\lambda)}-1\right|<\varepsilon/2.$$

Then for $\lambda>N^*$,

$$m_\alpha(\lambda;t_{r_1},\ldots,t_{r_n})<n!\,t^{n\alpha}(1+\varepsilon)^n/\Gamma(1+n\alpha).$$

Thus for all sufficiently large λ,

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\sum_{r_1=1}^{k}\cdots\sum_{r_n=1}^{k} s_{r_1}\ldots s_{r_n}\,m_\alpha(\lambda;t_{r_1},\ldots,t_{r_n}) \\ &\quad<\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\sum_{r_1=1}^{k}\cdots\sum_{r_n=1}^{k} s_{r_1}\ldots s_{r_n}\cdot n!\,t^{n\alpha}(1+\varepsilon)^n/\Gamma(1+n\alpha) \\ &\quad=\sum_{n=0}^{\infty}[t^\alpha(1+\varepsilon)(s_1+\cdots+s_k)]^n/\Gamma(1+n\alpha)<\infty \end{aligned} \tag{19}$$

since the upper bound in (19) has the form of a Mittag-Leffler function, which is entire [4]. By definition of $m_\alpha(\lambda;t_{r_1},\ldots,t_{r_n})$ (see (15)) we may use (19) and Fubini's theorem to justify writing

$$\begin{aligned} &\mathscr{E}\exp\left(-\sum_1^k s_i H(\lambda t_i,\omega)/Ch(1/\lambda)\right) \\ &\quad=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\sum_{r_1=1}^{k}\cdots\sum_{r_n=1}^{k} s_{r_1}\ldots s_{r_n}\,m_\alpha(\lambda;t_{r_1},\ldots,t_{r_n}) \end{aligned} \tag{20}$$

for all sufficiently large λ.

Again by (19), we may use dominated convergence to let $\lambda\to\infty$ in (20); by (15) we obtain

$$\begin{aligned}&\lim_{\lambda\to\infty}\mathscr{E}\exp\left(-\sum_1^k s_i H(\lambda t_i,\omega)/Ch(1/\lambda)\right)\\&\quad=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\sum_{r_1=1}^{k}\cdots\sum_{r_n=1}^{k}s_{r_1}\dots s_{r_n}m_\alpha(t_{r_1},\dots,t_{r_n}).\end{aligned}\tag{21}$$

By (19), the right-hand side in (21) is jointly continuous in $s_1,\dots,s_k$ at $s_1=\dots=s_k=0$, and so is a k-dimensional Laplace-Stieltjes transform (LST), by the continuity theorem for LSTs. Also, if we put $s_k=0$ on the right-hand side of (21), we recover the same functional form with k replaced by $k-1$. This is the LST version of the Daniell-Kolmogorov consistency conditions, and so there exists a unique stochastic process $\{X_\alpha(t,\omega): t\geqq 0\}$ whose finite-dimensional distributions are given by

$$\begin{aligned}&\mathscr{E}\exp\left(-\sum_1^k s_i X_\alpha(t_i,\omega)\right)\\&\quad=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\sum_{r_1=1}^{k}\cdots\sum_{r_n=1}^{k}s_{r_1}\dots s_{r_n}m_\alpha(t_{r_1},\dots,t_{r_n}).\end{aligned}\tag{22}$$

By construction, X_α is determined by the product moments $m_\alpha(t_{r_1},\dots,t_{r_n})$. Since the boundary of the domain $D(\pi,\alpha;t_{r_1},\dots,t_{r_n})$ in Lemma 4 forms a set of n-dimensional Lebesgue measure zero, $m_\alpha(t_{r_1},\dots,t_{r_n})$ is itself determined by the integrand in (17) (although this is unbounded and is infinite on the boundary of each $D(\pi)$). We conclude that X_α is uniquely determined by the product moment density relations

$$\begin{aligned}&\partial^k\left[\mathscr{E}\left[X_\alpha(t_1,\omega)\dots X_\alpha(t_k,\omega)\right]\right]/\partial t_1\dots\partial t_k\\&\quad=1/[\Gamma(\alpha)]^k[t_1(t_2-t_1)\dots(t_k-t_{k-1})]^{1-\alpha}\qquad(0<t_1<\dots<t_k).\end{aligned}\tag{23}$$

This proves (i) of Proposition 1. Also, the finite-dimensional distributions of the process $\{H(\lambda t,\omega)/Ch(1/\lambda): t\geqq 0\}$ converge as $\lambda\to\infty$ to those of $\{X_\alpha(t,\omega): t\geqq 0\}$. By the remarks of Section 2, this suffices to prove Theorem 1a.

5. Inverse Processes and Ergodic Limits

We now have a complete solution of the ergodic limit problem for the H-process. We use it to solve the ergodic limit problems for the T-, U- and V-processes.

The path-functions of all our processes are monotone non-decreasing; we may thus assume that all our norming functions are non-decreasing (they will in any case be asymptotic to non-decreasing functions).

We write $g(\lambda)=Ch(\lambda^{-1})$ for the norming function of the H-process. Since g is monotone, we may define an inverse function f by

$$f(\lambda)=\sup\{\mu: g(\mu)\leqq\lambda\}.\tag{1}$$

When (1.5) holds, we have

$$\operatorname{pr}\{H(\lambda t_i,\omega)/g(\lambda)\leqq v_i,\ 1\leqq i\leqq k\}\to\operatorname{pr}\{X_\alpha(t_i,\omega)\leqq v_i,\ 1\leqq i\leqq k\}.\tag{2}$$

But $H(\lambda t, \omega) \leqq v g(\lambda)$ is equivalent to $\lambda t \leqq T(v g(\lambda), \omega)$ or by (1) to $f(\mu) t \leqq T(v \mu, \omega)$ if $g(\lambda) = \mu$.

Taking $k=1$ in (2), we see that

$$\operatorname{pr}\{T(v\mu, \omega)/f(\mu) < t\} \to \operatorname{pr}\{X_\alpha(t, \omega) > v\} \qquad (\mu \to \infty). \tag{3}$$

Now consider the ergodic limit problem for the T-process. If the norming function ϕ is so chosen that the one-dimensional distributions $\operatorname{pr}\{T(v\mu, \omega)/\phi(\mu) \leqq t\}$ are convergent, we may deduce (as in deriving (3)) that $\operatorname{pr}\{H(\lambda t, \omega)/\phi^{-1}(\lambda) \leqq v\}$ converges as $\lambda \to \infty$. Since (by the Darling-Kac theorem) the norming function of the H-process is determined uniquely up to asymptotic equivalence, we must have

$$\phi^{-1}(\lambda) \sim g(\lambda) \qquad (\lambda \to \infty). \tag{4}$$

From section (4) we know that the norming function g of the H-process varies regularly with exponent $\alpha \in (0, 1)$: there exists a slowly varying function L such that

$$g(\lambda) = \lambda^\alpha L(\lambda). \tag{5}$$

It was shown by de Bruijn [2] that this implies that the function f inverse to g varies regularly with exponent $1/\alpha$: there exists a slowly varying function M such that

$$f(\lambda) = \lambda^{1/\alpha} \cdot M(\lambda). \tag{6}$$

The functions L, M satisfy the symmetric relations

$$\begin{aligned} &L(\lambda^{1/\alpha} \cdot M(\lambda)) \cdot [M(\lambda)]^\alpha \to 1 \qquad (\lambda \to \infty) \\ &M(\lambda^\alpha \cdot L(\lambda)) \cdot [L(\lambda)]^{1/\alpha} \to 1 \qquad (\lambda \to \infty) \end{aligned} \tag{7}$$

or

$$\begin{aligned} &L(f(\lambda)) \cdot [f(\lambda)]^\alpha \sim \lambda \qquad (\lambda \to \infty) \\ &M(g(\lambda)) \cdot [g(\lambda)]^{1/\alpha} \sim \lambda \qquad (\lambda \to \infty). \end{aligned} \tag{8}$$

Assume now that the finite-dimensional distributions of the process $\{T(\lambda v, \omega)/f(\lambda): v \geqq 0\}$ converge as $\lambda \to \infty$ to those of a non-degenerate limit process $\{Y(v, \omega): v \geqq 0\}$. In particular, the one-dimensional distributions are convergent, and so f satisfies (4). We may thus assume that f is defined by (1), since two asymptotic norming functions are equivalent. Then

$$\operatorname{pr}\{T(\lambda v_i, \omega)/f(\lambda) \leqq t_i, 1 \leqq i \leqq k\} \to \operatorname{pr}\{Y(v_i, \omega) \leqq t_i, 1 \leqq i \leqq k\}.$$

But since the H- and T-processes are inverse,

$$\begin{aligned} &\operatorname{pr}\{T(\lambda v_i, \omega)/f(\lambda) \leqq t_i, 1 \leqq i \leqq k\} \\ &\quad = \operatorname{pr}\{\lambda v_i \leqq H(f(\lambda) t_i, \omega), 1 \leqq i \leqq k\} \\ &\quad = \operatorname{pr}\left\{\bigcap_{i=1}^k [H(\mu t_i, \omega) < v_i g(\mu)]^c\right\} \end{aligned} \tag{9}$$

(where A^c denotes the complement of A), writing $f(\lambda) = \mu$. By the formula of inclusion and exclusion, the right-hand side of (9) is

$$1 - S_1 + S_2 - \cdots \tag{10}$$

where

$$S_r = \sum_{1 \leqq j_1 < \cdots < j_r \leqq k} \cdots \sum \operatorname{pr}\{H(\mu t_{j_i}, \omega)/g(\mu) \leqq v_{j_i}, 1 \leqq i \leqq r\}. \tag{11}$$

By Section 4, $\operatorname{pr}\{H(\mu t_{j_i}, \omega)/g(\mu) \leqq v_{j_i}, 1 \leqq i \leqq r\}$ is convergent as $\mu \to \infty$ if and only if (1.5) holds, and its limit is then

$$\operatorname{pr}\{X(t_{j_i}, \omega) \leqq v_{j_i}, 1 \leqq i \leqq r\}$$

where $X = X_\alpha$ and α denotes the parameter in (1.5).

The process $\{T(\lambda v, \omega)/f(\lambda)\}$ thus has convergent finite-dimensional distributions if and only if (1.5) holds. The limit process Y then satisfies

$$\operatorname{pr}\{Y(v_i, \omega) \leqq t_i, 1 \leqq i \leqq k\} = 1 - S_1^* + S_2^* - \cdots \tag{12}$$

where

$$S_r^* = \sum_{1 \leqq j_1 < \cdots < j_r \leqq k} \cdots \sum \operatorname{pr}\{X(t_{j_i}, \omega) \leqq v_{j_i}, 1 \leqq i \leqq r\}.$$

By the formula of inclusion and exclusion, (12) implies

$$\operatorname{pr}\{Y(v_i, \omega) \leqq t_i, 1 \leqq i \leqq k\} = \operatorname{pr}\left\{\bigcap_{i=1}^{k} [X(t_i, \omega) < v_i]^c\right\}. \tag{13}$$

But if Y^* denotes the process inverse to X, the right-hand side of (13) is $\operatorname{pr}\{Y^*(v_i, \omega) \leqq t_i, 1 \leqq i \leqq k\}$, and thus Y and Y^* are identically distributed. Since Y is an ergodic limit of the T-process, Y^* is also an ergodic limit of T. We may thus assume that the ergodic limits X, Y of the H- and T-processes are inverse. We may say that *the inverse relationship between the processes H and T is preserved on taking ergodic limits.* The necessary and sufficient condition for the existence of non-degenerate ergodic limits is (1.5) in both cases.

Since $T(v, \omega) = v + U(v, \omega)$, the processes $\{T(\lambda v, \omega)/f(\lambda)\}$, $\{U(\lambda v, \omega)/f(\lambda)\}$ have identical ergodic limits whenever

$$\lambda/f(\lambda) \to 0 \qquad (\lambda \to \infty). \tag{14}$$

But by the de Bruijn relations (or (6)), f varies regularly with exponent $1/\alpha$. Thus (14) holds for $\alpha < 1$ (the exceptional case $\alpha = 1$ is treated later). In the general case, the solutions of the ergodic limit problems for the T- and U-processes coincide.

We may now adapt the analysis above to conclude that the V-process has an ergodic limit if and only if its inverse process, the U-process, has an ergodic limit, and that the limits are inverse. Combining the results just obtained, we see that the solutions of the ergodic limit problems for the H- and V-processes coincide. We have proved

Proposition 2. (i) *If two stochastic processes are inverse, each has an ergodic limit if and only if the other does, the limit processes are inverse, and the norming functions are asymptotically inverse.*

(ii) *Conditions* (1.5) *with* $0 < \alpha < 1$ *is necessary and sufficient for the existence of non-degenerate ergodic limits for each of the processes H, T, U and V. When* (1.5) *holds, the H- and V-processes have identical ergodic limits* X_α, *and the T- and U-processes have identical ergodic limits* Y_α *inverse to* X_α.

We obtain Theorems 1b, 2a and 2b immediately from Theorem 1a and Proposition 2.

6. Identification of the Limit Process

We know from Section 4 that if $0<\alpha<1$ there exists a unique stochastic process $\{X_\alpha(t,\omega): t\geqq 0\}$ satisfying

$$\frac{\partial^k}{\partial t_1\dots\partial t_k}\cdot\mathscr{E}\left[\prod_{i=1}^k X_\alpha(t_i,\omega)\right]=1/[t_1(t_2-t_1)\dots(t_k-t_{k-1})]^{1-\alpha}[\Gamma(\alpha)]^k \qquad (1)$$

$$(0<t_1<\dots<t_k).$$

We wish to prove that if Y_α is the stable subordinator of parameter $\alpha\in(0,1)$, and Z_α is its inverse, then Z_α satisfies (1). We shall obtain this result as a special case of a more general result applicable to any subordinator.

Let $\{Y(v,\omega): v\geqq 0\}$ be any homogeneous stochastic process with independent non-negative increments and path-functions satisfying $Y(0+,\omega)=0$ (a.s.). Such a process is called a *subordinator*. Then there exists a constant $c\geqq 0$ (the deterministic drift) and a measure μ on $(0,\infty]$ with

$$\int_{(0,\infty]}(1-e^{-x})\,\mu(dx) \qquad (2)$$

(the *Levy measure*) such that

$$\mathscr{E}\exp(-s\,Y(v,\omega))=\exp\Big\{-cs-\int_{(0,\infty]}(1-e^{-xs})\,\mu(dx)\Big\}. \qquad (3)$$

We shall consider separately the cases $c=0$ and $c>0$; absorbing an uninteresting scale factor it suffices to consider $c=0$ and $c=1$. We write

$$\Lambda(s)=\int_{(0,\infty]}(1-e^{-xs})\,\mu(dx) \qquad (4)$$

and

$$\Lambda^*(s)=s+\Lambda(s). \qquad (5)$$

We combine the two cases by writing

$$\mathscr{E}\exp(-s\,U(v,\omega))=\exp(-v\Lambda(s)), \qquad (6)$$

$$T(v,\omega)=v+U(v,\omega), \qquad (7)$$

$$\mathscr{E}\exp(-s\,T(v,\omega))=\exp(-v\Lambda^*(s)). \qquad (8)$$

Define the inverse processes K, L by

$$K(t,\omega)=\sup\{v: U(v,\omega)\leqq t\}, \qquad (9)$$

$$L(t,\omega)=\sup\{v: T(v,\omega)\leqq t\}. \qquad (10)$$

Lemma 5.

i) $$\int_0^\infty\dots\int_0^\infty\exp\left(-\sum_1^k s_i t_i\right)d_{t_1\dots t_k}\mathscr{E}\left[\prod_{i=1}^k K(t_i,\omega)\right]$$
$$=\sum_\pi 1/\Lambda(s_{\pi(1)}+\dots+s_{\pi(k)})\,\Lambda(s_{\pi(2)}+\dots+s_{\pi(k)})\dots\Lambda(s_{\pi(k)}).$$

ii) *Relation* (i) *holds with* K,Λ *replaced by* L,Λ^*.

Proof. We prove only (i). A proof will be contained in a forthcoming sequel to this paper on regenerative stochastic phenomena, but we include an outline proof here (by different methods) to make the present paper self-contained.

Consider the following $2k$-dimensional Laplace transforms:

$$\begin{aligned}\Psi(s,\rho)=\Psi(s_1,\ldots,s_k;\rho_1,\ldots,\rho_k)&=s_1\ldots s_k\cdot\rho_1\ldots\rho_k\cdot\int_0^\infty\ldots\int_0^\infty dt_1\ldots dt_k\\ &\cdot dx_1\ldots dx_k\cdot\exp\left(-\sum_1^k s_i t_i-\sum_1^k\rho_j x_j\right)\\ &\cdot\operatorname{pr}\{\omega: K(t_i,\omega)\leqq x_i,\ i=1,2,\ldots,k\}.\end{aligned}\tag{11}$$

Define $\Theta(s,\rho)$ similarly by replacing $\operatorname{pr}\{K(t_i)\leqq x_i\}$ by $\operatorname{pr}\{U(x_i)<t_i\}$. If

$$\begin{aligned}\Phi(s)=\int_0^\infty\ldots\int_0^\infty\exp\left(-\sum_1^k s_i t_i\right)d_{t_1,\ldots,t_k}\mathscr{E}\left[\prod_{i=1}^k K(t_i,\omega)\right],\\ \partial^k\Psi(s_1,\ldots,s_k;0,\ldots,0)/\partial\rho_1\ldots\partial\rho_k=\Phi(s_1,\ldots,s_k).\end{aligned}\tag{12}$$

Consider the 2^k possible expressions of the form

$$\operatorname{pr}\{\omega: K(t_1)\lesseqgtr x_1,\ldots,K(t_k)\lesseqgtr x_k\}.\tag{13}$$

If the choices of sign agree at each place except the i-th, and the two probabilities corresponding to the two choices of sign at the i-th place are summed, the result is independent of x_i, and so its transform is independent of ρ_i. This observation provides the inductive step needed to establish the relation

$$\partial^k\Psi(s_1,\ldots,s_k;0,\ldots,0)/\partial\rho_1\ldots\partial\rho_k=\partial^k\Theta(s_1,\ldots,s_k;0,\ldots,0)/\partial\rho_1\ldots\partial\rho_k.\tag{14}$$

Since the U-process is homogeneous and additive, a straightforward calculation shows that

$$\begin{aligned}\Theta(s,\rho)&=\int_0^\infty\ldots\int_0^\infty\rho_1\ldots\rho_k\cdot\exp\left(-\sum_1^k\rho_i x_i\right)\mathscr{E}\exp\left(-\sum_1^k s_i U(x_i,\omega)\right)dx_1\ldots dx_k\\ &=\sum_\pi F(\rho_{\pi(1)},\ldots,\rho_{\pi(k)}),\end{aligned}\tag{15}$$

where

$$F(\rho_1,\ldots,\rho_k)=\rho_1\ldots\rho_k/[\rho_1+\cdots+\rho_k+\Lambda(s_1+\cdots+s_k)]\ldots[\rho_k+\Lambda(s_k)].\tag{16}$$

Hence

$$\partial^k\Psi(s_1,\ldots,s_k;0,\ldots,0)/\partial\rho_1\ldots\partial\rho_k=\sum_\pi 1/\Lambda(s_{\pi(1)}+\cdots+s_{\pi(k)})\ldots\Lambda(s_{\pi(k)})\tag{17}$$

which proves the lemma.

Since the process $K(t,\omega)$ has non-decreasing sample-paths, the function

$$m(t)=\mathscr{E}K(t,\omega)\qquad(t\geqq 0)$$

is non-decreasing, and so determines a measure on $[0,\infty)$. Let ρ denote its density, when this exists.

Lemma 6.

$$s_1 \dots s_k \cdot \int_0^\infty \dots \int_0^\infty \exp\left(-\sum_1^k s_i t_i\right) G(t_1, \dots, t_k)\, dt_1 \dots dt_k$$
$$= \sum_\pi 1/\Lambda(s_{\pi(1)} + \dots + s_{\pi(k)}) \dots \Lambda(s_{\pi(k)}),$$

where

$$G(t_1, \dots, t_k) = \sum_\pi \underset{\substack{0 < u_{\pi(1)} < \dots < u_{\pi(k)} \\ 0 \le u_i \le t_i}}{\iint \dots \int} dm(u_{\pi(1)})\, dm(u_{\pi(2)} - u_{\pi(1)}) \dots dm(u_{\pi(k)} - u_{\pi(k-1)}).$$

Proof. We prove that for every π the summands coincide; it suffices to consider the identity permutation and use symmetry. By a change of variable, the summand on the left may be written

$$\int_0^\infty s_1 e^{-s_1 t_1}\, dt_1 \int_0^{t_1} dm(w_1) \cdot \int_0^\infty s_2 \exp(-s_2 t_2)\, dt_2 \int_0^{t_2 - w_1} dm(w_2) \dots$$
$$\cdot \int_0^\infty s_k \exp(-s_k t_k)\, dt_k \int_0^{t_k - (w_1 + \dots + w_{k-1})} dm(w_k).$$

By Fubini's theorem, the last two integrations give

$$\int_0^\infty dm(w_k) \int_{w_1 + \dots + w_{k-1} + w_k}^\infty s_k \exp(-s_k t_k)\, dt_k$$
$$= \int_0^\infty \exp(-s_k(w_1 + \dots + w_k))\, dm(w_k) = \exp(-s_k(w_1 + \dots + w_{k-1}))/\Lambda(s_k).$$

Similarly the next two integrations give

$$\exp(-s_{k-1}(w_1 + \dots + w_{k-2}))/\Lambda(s_{k-1} + s_k)\, \Lambda(s_k).$$

After k iterations, we obtain the result of the lemma.

Comparing the two previous results and using the uniqueness theorem for LSTs:

$$\mathscr{E}\left[\prod_{i=1}^k K(t_i, \omega)\right] = G(t_1, \dots, t_k).$$

When the measure m is absolutely continuous with density ρ,

$$\partial^k \left[\mathscr{E}\left(\prod_{i=1}^k K(t_i, \omega)\right)\right] \Big/ \partial t_1 \dots \partial t_k = \rho(t_1)\, \rho(t_2 - t_1) \dots \rho(t_k - t_{k-1}) \qquad (0 < t_1 < \dots < t_k).$$

All this holds for an arbitrary subordinator. For the particular stable subordinator U_α with inverse K_α:

$$\Lambda_\alpha(s) = s^{-\alpha} = \int_0^\infty e^{-st}\, dt/t^{1-\alpha}\Gamma(\alpha)$$

and so the measure m_α is absolutely continuous with density $\rho_\alpha(t) = 1/t^{1-\alpha}\Gamma(\alpha)$. Thus

$$\partial^k \left[\mathscr{E}\left(\prod_{i=1}^k K_\alpha(t_i, \omega)\right)\right] \Big/ \partial t_i \dots \partial t_k = 1/[\Gamma(\alpha)]^k [t_1(t_2 - t_1) \dots (t_k - t_{k-1})]^{1-\alpha}.$$

Since by (i) of Proposition 1 (proved in Section 4) the product moment densities (1.15) determine a unique stochastic process X_α, we conclude that the processes

$$\{K_\alpha(t,\omega): t \geqq 0\}, \qquad \{X_\alpha(t,\omega): t \geqq 0\}$$

have identical finite-dimensional distributions. Thus in view of the definition of K_α (put $U = U_\alpha$ in (9)), the process $X_\alpha(t,\omega)$ is distributionally inverse to the stable subordinator U_α. This proves part (ii) of Proposition 1. Note that we obtain Lemmas 3 and 4 of Section 4 by putting $U = U_\alpha$ in Lemmas 5 and 6.

Part (iii) of Proposition 1 may be proved easily by taking the double Laplace transform of the relation

$$\operatorname{pr}\{K_\alpha(t,\omega) \leqq x\} + \operatorname{pr}\{U_\alpha(x,\omega) \leqq t\} = 1$$

(compare Feller [5], p. 428, Stone [20]). Since U_α is the stable subordinator,

$$\int_0^\infty \int_0^\infty s\rho \cdot dt\,dx \cdot \exp(-st - \rho x) \cdot \operatorname{pr}\{U_\alpha(x,\omega) \leqq t\}$$
$$= \int_0^\infty \rho\,dx \cdot \exp(-\rho x)\exp(-x s^\alpha) = \rho/(\rho + s^\alpha).$$

Also,

$$\int_0^\infty s e^{-st}\,dt \cdot \sum_{n=0}^\infty (-\rho t^\alpha)^n / \Gamma(1 + n\alpha) = s^\alpha/(\rho + s^\alpha).$$

Thus by the uniqueness theorem for LSTs,

$$\int_0^\infty \rho e^{-\rho x}\,dx \cdot \operatorname{pr}\{K_\alpha(t,\omega) \leqq x\} = \sum_{n=0}^\infty (-\rho t^\alpha)^n / \Gamma(1 + n\alpha),$$

the Mittag-Leffler function obtained by Darling and Kac.

The proof of (iv) of Proposition 1 is a k-dimensional analogue of that of (iii). We use the formula of inclusion and exclusion to pass from the $2k$-dimensional Laplace transforms of the k-dimensional distributions of the stable subordinator Y_α ($k = 1, 2, \ldots$) given by (15), (16) with $U = Y_\alpha$, $\Lambda(s) = s^\alpha$, to the transforms of the finite-dimensional distributions of the inverse process X_α (compare Section 5). Details are omitted.

7. The Cases $\alpha = 0, 1, \frac{1}{2}$

The limiting cases $\alpha = 0, 1$ are of less interest since the limit processes which arise are (in different senses) degenerate. Since it was convenient earlier to postpone treatment of these cases, however, we now indicate briefly what changes they require. The case $\alpha = \frac{1}{2}$ is very interesting probabilistically because of its connection with Brownian motion.

When $\alpha = 1$, the Mittag-Leffler and the stable subordinator distributions both reduce to unit masses. The H- and V-processes and the T- and U-processes all converge to the trivial ergodic limit process $\{t: t \geqq 0\}$: thus

$$H(\lambda t,\omega)/\lambda L(\lambda) \to t \qquad (\lambda \to \infty) \text{ in probability},$$
$$T(\lambda v,\omega)/f(\lambda) \to v \qquad (\lambda \to \infty) \text{ in probability},$$

and similarly for V, U.

When $\alpha=0$, the norming function for the H-process varies slowly. One easily sees that in this case the ergodic limit formulation we have adopted gives no more information than the Darling-Kac theorems with $\alpha=0$. The ergodic limit process X_0 of the H- and V-processes thus satisfies

$$X_0(t,\omega)=X_0(\omega) \qquad (t \geqq 0)$$

where $X_0(\omega)$ has negative exponential distribution with unit parameter (this is also the Mittag-Leffler law with $\alpha=0$).

The methods used earlier for convergence of finite-dimensional distributions give easily

$$\mathscr{E} \exp\left(-\sum_1^k H(\lambda t_i,\omega)/L(\lambda)\right) \to 1/(s_1+\cdots+s_k) \qquad (\lambda\to\infty)$$

for all $t_i>0$. Using our knowledge of the limit process X_0 we obtain

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_1^k s_i x_i\right) d_{x_1,\ldots,x_k}[1-\exp(-\min(x_1,\ldots,x_k))]=1/(s_1+\cdots+s_k).$$

This interesting functional relation may be readily verified by induction.

The ergodic limit of the T- and U-processes is the stable subordinator of index 0, which has non-decreasing sample-paths and is determined by

$$\operatorname{pr}\{Y_0(v,\omega)=0\}=e^{-v}, \qquad \operatorname{pr}\{Y_0(v,\omega)=\infty\}=1-e^{-v}.$$

Then $\operatorname{pr}\{Y_0(v_i,\omega)=\infty, i=1,\ldots,k\}=1-\exp(-\min(v_1,\ldots,v_k))$, the functional form obtained above. Note that in this case the ergodic limits X_0, Y_0 are not inverse.

The process X_0 also appears as the ergodic limit of the occupation-time process $H^*_{B,\frac{1}{2}}(t,\omega)$ for *two-dimensional* Brownian motion over the bounded Borel set B. The norming function here is a multiple of $|B|\cdot\log t$, where $|\cdot|$ denotes Lebesgue measure. In a similar notation, Y_0 is the ergodic limit of $T^*_{B,\frac{1}{2}}(v,\omega)$, the process inverse to $H^*_{B,\frac{1}{2}}$ (Darling and Kac [3], Section 2).

When $\alpha=\frac{1}{2}$ the Mittag-Leffler law with parameter α reduces to the truncated normal law. The stable subordinator $U_{\frac{1}{2}}$ is (Ito-McKean [7]) the first-passage process for Brownian motion, and so its inverse $V_{\frac{1}{2}}$ is the supremum process for Brownian motion. As a corollary to Proposition 1, we see that if $\{x_0(t,\omega): t\geqq 0\}$ is standard Brownian motion started at the origin, and if $m_0(t,\omega)=\sup\{x_0(u,\omega): 0\leqq u\leqq t\}$, the process m_0 is characterised by the relations

$$\partial^k\left[\mathscr{E}\left(\prod_{i=1}^k m_0(t_i,\omega)\right)\right]\Big/\partial t_1\ldots\partial t_k=1/[t_1(t_2-t_1)\ldots(t_k-t_{k-1})]^{\frac{1}{2}}\pi^{k/2} \quad (0<t_1<\cdots<t_k).$$

We thus see that if B is any bounded linear Borel set, and if $H_{B,\frac{1}{2}}(t,\omega)$ is the occupation-time for the process x_0 in B over the time-interval $[0,t]$, the process $H_{B,\frac{1}{2}}$ has ergodic limit $V_{\frac{1}{2}}$, the supremum process for x_0. The process $T_{B,\frac{1}{2}}$ inverse to $H_{B,\frac{1}{2}}$ has ergodic limit $U_{\frac{1}{2}}$, the first-passage process for x_0.

8. Complements

The methods and results given above are very similar to those used by the author in obtaining analogous limit theorems in the context of regenerative stochastic phenomena (Kingman, [13]). We shall deal with these regenerative limit theorems in a forthcoming sequel to this paper. The two classes of limit theorems coincide when both are restricted to the important special case when the underlying process is a Markov chain.

In a similar spirit one may obtain analogous theorems on renewal theory and on Feller's recurrent events. These last theorems generalise those of Feller's pioneering work on fluctuation theory for recurrent events [5], in which the Mittag-Leffler laws first arose in probability theory.

The Darling-Kac theorems have been used by Kesten [11] to obtain an interesting Tauberian theorem for symmetric random walk on the line. Kesten's theorem may be formulated and generalised as an ergodic limit theorem in our sense. Analogous limit theorems for random walks with spherical symmetry (Kingman [12]) may also be obtained, containing these results on symmetric walks on the line as a limiting case. These questions will be considered in a forthcoming publication.

The limit theorems of Karlin and McGregor [10] on the one-dimensional ergodic limit problem for birth-and-death processes may be extended by our methods to finite-dimensional and weak ergodic limits.

Weak convergence theorems for occupation-time problems in the theory of queues under heavy traffic have been considered by Whitt ([22], Sections 9.4, 10.7). The methods used here yield extensions to these results.

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