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Weak Convergence of Multidimensional Empirical Processes for Strong Mixing Sequences of Stochastic Vectors

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1. Introduction

Let $\{\mathbf{X}_i\}=\{(X_{i1},\ldots,X_{ip})',\ -\infty<i<\infty\}$ be a strictly stationary sequence of stochastic $p(\geqq 1)$ dimensional vectors defined on a probability space $(\Omega,\mathscr{A},P)$ with each $X_{ij}(-\infty<i<\infty,\ 1\leqq j\leqq p)$ having the uniform distribution on the interval $[0,1]$. Let $\mathscr{M}_a^b$ denote the σ-algebra generated by $\mathbf{X}_i,\ a\leqq i\leqq b$. Suppose that the sequence satisfies one of the following conditions: for all $B\in\mathscr{M}_{k+n}^{\infty}$ with probability one

$$|P(B\mid\mathscr{M}_{-\infty}^{k})-P(B)|\leqq\phi(n)\downarrow 0\qquad(n\to\infty)\tag{1}$$

(the ϕ-mixing condition) and

$$\sup|P(A\cap B)-P(A)\,P(B)|\leqq\alpha(n)\downarrow 0\qquad(n\to\infty)\tag{2}$$

(the strong mixing (s.m.) condition). Here the supremum is taken over all $A\in\mathscr{M}_{-\infty}^{k}$ and $B\in\mathscr{M}_{k+n}^{\infty}$. The difference between the ϕ-mixing and the s.m. conditions is explained in [4] and [5].

Let

$$F_{[j]}(t)=P\{X_{ij}\leqq t\}=t,\qquad 0\leqq t\leqq 1,\qquad j=1,\cdots,p$$

and put

$$F(\mathbf{t})=P\{\mathbf{X}_i\leqq\mathbf{t}\},\qquad \mathbf{t}\in E^p\tag{3}$$

where $E^p=\{\mathbf{t}:\ \mathbf{0}\leqq\mathbf{t}\leqq\mathbf{1}\}$ is the p-dimensional unit cube, $\mathbf{0}=(0,\ldots,0)$, $\mathbf{1}=(1,\ldots,1)$ and $\mathbf{a}\leqq\mathbf{b}$ means that $a_j\leqq b_j$, $1\leqq j\leqq p$. Note that $F(\mathbf{t})=0$ if at least one coordinate of $\mathbf{t}$ is 0. For a sample $\mathbf{X}_1,\ldots,\mathbf{X}_n$ of size n, the empirical df is defined by

$$F_n(\mathbf{t})=n^{-1}\sum_{i=1}^{n}c(\mathbf{t}-\mathbf{X}_i),\qquad \mathbf{t}\in E^p,\qquad n\geqq 1\tag{4}$$

where $c(\mathbf{u})=1$ if and only if $\mathbf{u}\geqq\mathbf{0}$, and 0 otherwise. $F_n(\mathbf{t})=0$, when at least one coordinate of $\mathbf{t}$ is 0. Let $W_n=\{W_n(\mathbf{t}):\ \mathbf{t}\in E^p\}$ be the empirical processes defined by

$$W_n(\mathbf{t})=n^{\frac{1}{2}}[F_n(\mathbf{t})-F(\mathbf{t})],\qquad \mathbf{t}\in E^p,\qquad n\geqq 1.\tag{5}$$

For every $n\geqq 1$, the process W_n belongs to the space $D^p[0,1]$ of all real valued functions on E^p with no discontinuities of the second kind, and with $D^p[0,1]$ we associate the (extended) Skorokhod J_1-topology. Let $W=\{W(\mathbf{t}):\ \mathbf{t}\in E^p\}$ be the p-dimensional Gaussian process where

$$EW(\mathbf{t})=0, \qquad \mathbf{t}\in E^p \tag{6}$$

and for every $\mathbf{s}, \mathbf{t}\in E^p$,

$$\begin{aligned} R(\mathbf{s},\mathbf{t}) &= EW(\mathbf{s})\,W(\mathbf{t}) \\ &= E\{c(\mathbf{s}-\mathbf{X}_1)\,c(\mathbf{t}-\mathbf{X}_1)-F(\mathbf{s})\,F(\mathbf{t})\} \\ &\quad + \sum_{k=2}^{\infty} E\{c(\mathbf{s}-\mathbf{X}_1)\,c(\mathbf{t}-\mathbf{X}_k)+c(\mathbf{s}-\mathbf{X}_k)\,c(\mathbf{t}-\mathbf{X}_1)-2F(\mathbf{s})\,F(\mathbf{t})\}. \end{aligned} \tag{7}$$

(Note that by Theorem 18.5.4 in [5] the series on the right side of (7) converges when $\sum \alpha(n)<\infty$.)

For ϕ-mixing sequences and $p=1$, Billingsley [2] proved firstly that the weak convergence of W_n to W holds under the condition $\sum n^2\phi^{\frac{1}{2}}(n)<\infty$ and the author [9] proved the same result under the condition $\phi(n)=O(n^{-2})$. Further, for ϕ-mixing sequences and general $p\geqq 1$, Sen [7] proved that the weak convergence of W_n to W holds under the condition $\sum n\phi^{\frac{1}{2}}(n)<\infty$. We remark here that from the methods of the proofs of Theorem 2.1 in [7] and Theorem in [9], it is obvious that the Sen's theorem holds under the weaker condition $\phi(n)=O(n^{-2})$.

On the other hand, for s.m. sequences and $p=1$, among others, the author proved the weak convergence of W_n to W under the condition $\alpha(n)=O(n^{-3-\delta})$ $\cdot(\delta>0)$. (See [4] and [8].)

The object of this paper is to prove that the analogous results to Sen's theorems in [7] hold for p-dimensional stochastic vectors satisfying some s.m. conditions (Theorems 1 and 2).

2. Weak Convergence of Empirical Processes

The following theorem extends Theorem 1 in [4] and Theorem in [8] and Theorem 1 in [10].

Theorem 1. *Suppose that $\{\mathbf{X}_i\}$ is a strictly stationary s.m. sequence of stochastic vectors, defined in Section 1, with mixing coefficient $\alpha(n)$. Then, W_n converges in law (in the Skorohod J_1-topology on D^p [0, 1]) to the Gaussian process W, defined in Section 1, if the mixing coefficient $\alpha(n)$ satisfies one of the following conditions:*

(i) $\alpha(n)=O(n^{-5/2-\delta})$ *for some* $\delta>0$ *if* $p=1$;

(ii) $\alpha(n)=O(n^{-3p/2-\delta})$ *for some* $\delta>0$ *if* $p\geqq 2$.

Next, as in [7], we shall consider a sequence of stochastic processes

$$W_n^*=\{W_n^*(\mathbf{t},u);\ \mathbf{t}\in E^p,\ 0\leqq u\leqq 1\} \qquad (n\geqq 1), \tag{7}$$

defined on the D^{p+1} [0, 1] space, where

$$W_n^*(\mathbf{t},u)=[nu]^{\frac{1}{2}}\,W_{[nu]}(\mathbf{t})/n^{\frac{1}{2}} \qquad (0\leqq u\leqq 1;\ \mathbf{t}\in E^p) \tag{8}$$

and $[s]$ is the largest integer contained in s. Let

$$W^*=\{W^*(\mathbf{t},u);\ \mathbf{t}\in E^p,\ 0\leqq u\leqq 1\} \tag{9}$$

be the Gaussian function with $EW^*(\mathbf{t}, u)=0$ and

$$EW^*(\mathbf{s}, v)\, W^*(\mathbf{t}, u)=\min(u, v)\, R(\mathbf{s}, \mathbf{t}) \tag{10}$$

for every $\mathbf{s}, \mathbf{t} \in E^p$, $0 \leqq u, v \leqq 1$ where $R(\mathbf{s}, \mathbf{t})$ is defined by (7). Then, we have the following theorem.

Theorem 2. *Under the condition in Theorem 1, W_n^* converges in law (in the extended Skorokhod J_1-topology on D^{p+1} [0, 1] space) to W^*, defined above.*

From Theorem 2, the following corollary easily follows.

Corollary. *Let $\{N_\nu, \nu \geqq 1\}$ be a sequence of positive integer valued random variables such that as $\nu \to \infty$ $\nu^{-1} N_\nu \to \xi$ in probability where ξ is a positive random variable defined on the same probability space $(\Omega, \mathscr{A}, P)$. Then, under the conditions in Theorem 1 $\{W_{N_\nu}\}$ converges weakly to W as $\nu \to \infty$.*

3. Proofs

In what follows, by the letter K, we shall denote any positive quantity (not always the same) which is bounded and does not depend on n. Let the process $\{z_n\}$ be a strictly stationary sequence of Bernoullian random variables, centered at expectations satisfying some s.m. conditions. Let $\tau = E|z_1|^2$ and $\|z\|_r = \{E|z|^r\}^{1/r}$ for $r>1$. Then, $E|z_1| = 2\tau$ and $\|z\|_r \leqq \{E|z|\}^{1/r} \leqq 2\tau^{1/r}$ for $r>1$. Furthermore, let $S_n = z_1 + \cdots + z_n$.

Now, we shall prove the basic two lemmas.

Lemma 1. *If $\{z_n\}$ satisfies the s.m. condition with mixing coefficient $\alpha(n) = O(n^{-3/\gamma})$ $(0<\gamma<1)$, then*

$$ES_n^4 \leqq K(n^2 \tau^{4/3} + \tau^{1-\gamma} n \log n) \tag{11}$$

for all n sufficiently large.

Proof. We follow the proof of Lemma 2.1 in [6]. We denote by $\sum_n$ the summation over all $i, j, k \geqq 0$ for which $i+j+k \leqq n$, and let $\sum_n^{(1)}, \sum_n^{(2)}$ and $\sum_n^{(3)}$ be, respectively, the components of $\sum_n$ for which $i \geqq (j, k), j \geqq (i, k)$ and $k \geqq (i, j)$. Then we have

$$ES_n^4 \leqq 24n \{\textstyle\sum_n^{(1)} + \sum_n^{(2)} + \sum_n^{(3)}\}\, |E z_0 z_i z_{i+j} z_{i+j+k}|. \tag{12}$$

Since $\alpha(j) = O(j^{-3/\gamma})$, so it follows from Lemma 2.1 in [3] that the following inequalities hold for all n sufficiently large:

$$\begin{aligned} &\textstyle\sum_n^{(1)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\ &\quad \leqq 6 \textstyle\sum_n^{(1)} \{\alpha(i)\}^\gamma \|z_0\|_{(1-\gamma)^{-1}} \leqq K \tau^{1-\gamma} \sum_n^{(1)} \{\alpha(i)\}^\gamma \\ &\quad \leqq K \tau^{1-\gamma} \sum_{i=1}^{n} (i+1)^2 \{\alpha(i)\}^\gamma \leqq K \tau^{1-\gamma} \log n; \end{aligned} \tag{13}$$

$$\begin{aligned}
&\textstyle\sum_n^{(2)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\quad \leqq \textstyle\sum_n^{(2)} |E z_0 z_i|\, |E z_0 z_k| + 6 \sum_n^{(2)} \{\alpha(j)\}^\gamma \, \| z_0 z_i \|_{(1-\gamma)^{-1}} \\
&\quad \leqq 36 \textstyle\sum_n^{(2)} \{\alpha(i)\}^{1/3} \{\alpha(k)\}^{1/3} \, \| z_0 \|_{3/2}^2 + 6 \sum_n^{(2)} \{\alpha(j)\}^\gamma \tau^{1-\gamma} \\
&\quad \leqq K \tau^{4/3} \textstyle\sum_n^{(2)} \{\alpha(i)\}^{1/3} \{\alpha(k)\}^{1/3} + K \tau^{1-\gamma} \sum_n^{(2)} \{\alpha(j)\}^\gamma \\
&\quad \leqq K n \tau^{4/3} \left\{ \sum_{i=1}^{n} \{\alpha(i)\}^{1/3} \right\}^2 + K \tau^{1-\gamma} \sum_{j=1}^{n} (j+1)^2 \{\alpha(j)\}^\gamma \\
&\quad \leqq K (n \tau^{4/3} + \tau^{1-\gamma} \log n); \qquad (14)
\end{aligned}$$

$$\begin{aligned}
&\textstyle\sum_n^{(3)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\qquad \leqq 6 \textstyle\sum_n^{(3)} \{\alpha(k)\}^\gamma \, \| z_0 \|_{(1-\gamma)^{-1}} \\
&\qquad \leqq K \tau^{1-\gamma} \sum_{k=1}^{n} (k+1)^2 \{\alpha(k)\}^\gamma \leqq K \tau^{1-\gamma} \log n. \qquad (15)
\end{aligned}$$

Thus, (11) follows from (12), (13), (14) and (15), and the proof is completed.

Lemma 2. *If $\{z_n\}$ satisfies the s.m. condition with mixing coefficient $\alpha(n) = O(n^{-5/2-\delta})$ for some $\delta > 0$, then for all n sufficiently large*

$$E S_n^4 \leqq K (n^2 \tau^{6/5} + n^{2-\rho} \tau^{1-\gamma}) \qquad (16)$$

where γ is a number such that $2/(2+\delta) < \gamma < 1$ and $\rho = (5/2 + \delta)\gamma - 2$.

Proof. We use the same notations as in the proof of Lemma 1. Since for all n sufficiently large

$$\sum_{j=1}^{n} (j+1)^2 \{\alpha(j)\}^\gamma \leqq K n^{1-\rho} \quad \text{and} \quad \sum_{i=1}^{\infty} \{\alpha(i)\}^{2/5} < \infty,$$

so for all n sufficiently large

$$\begin{aligned}
&\textstyle\sum_n^{(1)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\quad \leqq K \tau^{1-\gamma} \sum_{i=1}^{n} (i+1)^2 \{\alpha(i)\}^\gamma \leqq K n^{1-\rho} \tau^{1-\gamma}, \\
&\textstyle\sum_n^{(2)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\quad \leqq K n \tau^{6/5} \left[\sum_{i=1}^{n} \{\alpha(i)\}^{2/5} \right]^2 + K \tau^{1-\gamma} \sum_{j=1}^{n} (j+1)^2 \{\alpha(j)\}^\gamma \\
&\quad \leqq K (n \tau^{6/5} + n^{1-\rho} \tau^{1-\gamma}), \\
&\textstyle\sum_n^{(3)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\quad \leqq K \tau^{1-\gamma} \sum_{k=1}^{n} (k+1)^2 \{\alpha(k)\}^\gamma \leqq K n^{1-\rho} \tau^{1-\gamma}.
\end{aligned}$$

Thus, we have the lemma.

Now, the proofs of Theorems 1 and 2 follow along the same line as of the proofs of Theorems 2.1 and 2.2 of Sen [7] using Lemmas 1 and 2 instead of using Lemma 2.1 of Sen [6], and so are omitted.

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