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Transformation of H^p-Martingales by a Change of Law

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1. Introduction

Let M be a local martingale such that $M_0=0$ and $\Delta M_t=M_t-M_{t-}>-1$ for every $t>0$, and consider the process Z defined by the formula

$$Z_t=\exp(M_t-\langle M^c, M^c\rangle_t/2)\prod_{s\leqq t}(1+\Delta M_s)\exp(-\Delta M_s), \quad t\geqq 0 \tag{1}$$

where M^c is the continuous part of M and $\langle M^c, M^c\rangle$ is the continuous increasing process such that $(M^c)^2-\langle M^c, M^c\rangle$ is a local martingale. If Z is a uniformly integrable martingale such that $Z_\infty>0$, then $d\hat{P}=Z_\infty dP$ is a probability measure equivalent to the underlying measure dP. For example, as is shown in the next section, if $M\in BMO$ and $\Delta M_t\geqq -1+\delta$ for some δ with $0<\delta\leqq 1$, the process Z satisfies this property (see [4] or [5]). Let $\mathscr{L}$ (resp. $\hat{\mathscr{L}}$) denote the class of all local martingales X with $X_0=0$ relative to dP (resp. $d\hat{P}$). Then the mapping ϕ: $\mathscr{L}\to\hat{\mathscr{L}}$ given by

$$\phi(X)_t=X_t-\int_0^t Z_s^{-1}\,d[X,Z]_s \tag{2}$$

is well-defined and linear ([9], p. 376). Furthermore, if M is continuous, then $[\phi(X), \phi(X)]=[X,X]$ under dP and $d\hat{P}$ ([10], p. 884). But, unfortunately, $\phi(X)$ is not always uniformly integrable even if $X\in H^p$ for all $p\geqq 1$. Now, let $\hat{H}^p$ denote the H^p class associated with $d\hat{P}$. Similarly, $\widehat{BMO}$ denotes the class of all BMO-martingales with respect to $d\hat{P}$. In [7], assuming the sample continuity of M and dealing only with continuous local martingales, we showed that $\widehat{BMO}$ is isomorphic to BMO under the mapping ϕ, and that Φ: $X\to Z^{-1}\cdot\phi(X)$ is an isomorphism of H^1 onto $\hat{H}^1$. Here, $Z^{-1}\cdot\phi(X)$ denotes the stochastic integral of the process $Z^{-1}=(Z_t^{-1})$ relative to the $\hat{P}$-local martingale $\phi(X)$.

In this paper we shall remove this continuity condition and prove the following.

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Theorem. *Let* $1<p<\infty$ *and* $0<\delta\leqq 1$. *If* $M\in BMO$ *and* $\Delta M_t\geqq -1+\delta$ *for every* t, *then* Φ_p: $X\to Z_-^{-1/p}\cdot\phi(X)$ *is an isomorphism of* H^p *onto* $\hat{H}^p$. *Furthermore,* BMO *and* $\widehat{BMO}$ *are isomorphic under the mapping* ϕ. *In particular, if* M *is continuous, then we have*

$$\exp\{-\|M\|_{BMO}^2/(2p)\}\leqq\|\Phi_p\|\leqq\exp\{\|M\|_{BMO}^2/(2q)\} \tag{3}$$

where $p^{-1}+q^{-1}=1$ *and* $\|\Phi_p\|$ *denotes the norm of* Φ_p *as an operator from* H^p *to* $\hat{H}^p$.

From the definition of ϕ it follows at once that $\Phi_p(X)=\phi(Z_-^{-1/p}\cdot X)$. The isomorphism between H^1 and $\hat{H}^1$ is established in [3].

2. Preliminaries

1. Notations and Definitions

Let (Ω, F, P) be a complete probability space, given an increasing right continuous family $(F_t)_{0\leqq t<\infty}$ of sub σ-fields of F such that F_0 contains all null sets. If X is a process with left limits, X_- denotes the process (X_{t-}). For a semi-martingale X, let $[X,X]$ be the increasing process defined in [9]. By (2), any P-local martingale X is a $\hat{P}$-semi-martingale and the process $[X,X]$ under dP is equal to the one under $d\hat{P}$ (see Proposition 3 in [8]). Throughout, for a semi-martingale X and a locally bounded predictable process Ψ, we denote by $\Psi\cdot X$ the stochastic integral of Ψ relative to X. For $1\leqq p<\infty$, let H^p denote the class of all local martingales X over (F_t) such that $\|X\|_{H^p}=E[[X,X]_\infty^{p/2}]^{1/p}<\infty$. If $1<p<\infty$, then H^p coincides with the class of all L^p-bounded martingales. Let us denote by $\|X\|_{BMO}$ the smallest positive constant C such that C^2 dominates a.s., $E[[X,X]_\infty-[X,X]_{T-}|F_T]$ for every stopping time T. BMO is the class of those martingales X which satisfies $\|X\|_{BMO}<\infty$, and it is a Banach space with norm $\|\cdot\|_{BMO}$. As is well-known nowadays, BMO is the dual space of H^1. Z always denotes the process defined by the formula (1). It is the unique solution of the stochastic integral equation

$$Z_t=1+\int_0^t Z_{s-}\,dM_s,$$

which was pointed out by C. Doléans-Dade [2].

Let now $1<p<\infty$. We say that Z satisfies (B_p) if for every stopping time T

$$Z_T^{1/p}\leqq K_p E[Z_\infty^{1/p}|F_T] \tag{4}$$

where K_p is a positive constant depending on p only. If Z is a positive uniformly integrable martingale, then the inequality (4) is clearly valid for $0<p\leqq 1$ with $K_p=1$. If $1<p<p'$, then $(B_{p'})$ implies (B_p), which follows from the Jensen inequality. (B_p) is in fact no other than the condition $(b_{1/p}^-)$ which is stated in [3].

The reader is assumed to be familiar with the martingale theory as expounded in [9]. Throughout the paper, let us denote by C a positive constant and by C_p a positive constant depending on p only, both letters are not necessarily the same in each occurrence.

2. Preliminary Lemmas

In the next lemma we need not assume the uniform integrability of Z.

Lemma 1. *Let $0<\delta\leqq 1$. If $M\in BMO$ and $\Delta M_t\geqq -1+\delta$ for every t, then Z satisfies (B_p) for all $p>1$. More precisely, for every stopping time T we have*

$$Z_T^{1/p}\leqq \exp\{\|M\|_{BMO}^2/(2p\delta)\}\, E[Z_\infty^{1/p}|F_T]. \tag{5}$$

Proof. Assume that $M\in BMO$ and $\Delta M_t\geqq -1+\delta$. By an elementary calculation,

$$(1+x)\exp(-x)\geqq \exp\{-x^2/(2\delta)\}$$

for $x\geqq -1+\delta$, and so we have

$$(1+\Delta M_t)\exp(-\Delta M_t)\geqq \exp\{-(\Delta M_t)^2/(2\delta)\}$$

for every t. Then $Z_T>0$ a.s., for any stopping time T and

$$\begin{aligned} Z_\infty/Z_T &\geqq \exp\{(M_\infty-M_T)-(\langle M^c,M^c\rangle_\infty-\langle M^c,M^c\rangle_T)/2-\sum_{t>T}(\Delta M_t)^2/(2\delta)\}\\ &\geqq \exp\{(M_\infty-M_T)-([M,M]_\infty-[M,M]_{T-})/(2\delta)\}. \end{aligned}$$

Therefore, by the Jensen inequality

$$\begin{aligned} E[(Z_\infty/Z_T)^{1/p}|F_T] &\geqq \exp\{-E[[M,M]_\infty-[M,M]_{T-}|F_T]/(2p\delta)\}\\ &\geqq \exp\{-\|M\|_{BMO}^2/(2p\delta)\}, \end{aligned}$$

which completes the proof.

If $\Delta M_t\geqq -1$, then the process Z given by (1) is a non-negative local martingale, and so $Z_\infty\in L^1$. Therefore, if $M\in BMO$ and $\Delta M_t\geqq -1+\delta$ for some δ with $0<\delta\leqq 1$, then Z is a positive uniformly integrable martingale, because we have $Z_T\leqq CE[Z_\infty|F_T]$ for any stopping time T by (5) and the Jensen inequality.

In particular, if M is continuous, then, letting $\delta=1$, we get

$$Z_T^{1/p}\leqq \exp\{\|M\|_{BMO}^2/(2p)\}\, E[Z_\infty^{1/p}|F_T]. \tag{6}$$

In what follows, the process Z is assumed to be a uniformly integrable martingale such that $Z_\infty>0$. Now, let us consider the process $\hat{Z}$ defined by $\hat{Z}_t=1/Z_t$, which is a uniformly integrable martingale with respect to the weighted probability measure $d\hat{P}=Z_\infty dP$. Let $\hat{M}=-\phi(M)$. Then, under $d\hat{P}$, $\hat{Z}$ is the solution of the equation: $\hat{Z}_t=1+\int_0^t \hat{Z}_{s-}\,d\hat{M}_s$. Thus the mapping $\hat{\phi}$: $\hat{\mathscr{L}}\to\mathscr{L}$ given by the formula

$$\hat{\phi}(X')_t = X'_t - \int_0^t \hat{Z}_s^{-1}\, d[X', \hat{Z}]_s, \qquad X' \in \hat{\mathscr{L}} \tag{7}$$

is well-defined. We now remark that, if X is a semi-martingale under either probability measure, then the stochastic integral $\Psi \cdot X$ with respect to dP coincides with the one relative to $d\hat{P}$ ([9], p. 379). For simplicity, let $(\hat{B}_p)$ denote the (B_p) condition associated with $d\hat{P}$.

Lemma 2. *Let* $p^{-1}+q^{-1}=1$ *with* $1<p<\infty$. *Then* Z *satisfies* (B_p) *if and only if* $\hat{Z}$ *satisfies* $(\hat{B}_q)$.

Proof. We denote by $\hat{E}[\cdot]$ the expectation over Ω with respect to $d\hat{P}$. It is easy to see that for every $\hat{P}$-integrable random variable Y we have

$$\hat{E}[Y|F_T] = E[Z_\infty Y|F_T]/Z_T \quad \text{a.s., under } dP \text{ and } d\hat{P}. \tag{8}$$

Now, we assume that Z satisfies (B_p). Then, dividing both sides of (4) by Z_T, we get

$$\hat{Z}_T^{1/q} \leqq K_p E[Z_\infty^{1/p}|F_T]/Z_T$$

and by (8) the right hand side is equal to $K_p \hat{E}[Z_\infty^{1/q}|F_T]$. Consequently, $\hat{Z}$ satisfies $(\hat{B}_q)$. The proof of the converse is similar and so is omitted.

Lemma 3. *The mapping* $\phi\colon \mathscr{L} \to \hat{\mathscr{L}}$ *is bijective.*

Proof. Let $X \in \mathscr{L}$ and $\hat{M} = -\phi(M)$. Under dP, $\phi(X)$ and $\hat{M}$ are semi-martingales. Clearly, $\phi(X)^c = X^c$ and $\hat{M}^c = -M^c$. In addition,

$$\Delta\phi(X)_t = (1+\Delta M_t)^{-1}\Delta X_t \quad \text{and} \quad \Delta\hat{M}_t = -(1+\Delta M_t)^{-1}\Delta M_t.$$

Then, combining these facts, we can find that

$$\phi(X) = X + [X, \hat{M}], \tag{9}$$

$$X = \phi(X) + [\phi(X), M]. \tag{10}$$

Therefore, $[X, \hat{M}] + [\phi(X), M] = 0$ under dP. The similar results holds under $d\hat{P}$. It follows at once from (10) that ϕ is injective. To show that it is surjective, let $X' \in \hat{\mathscr{L}}$ and $X = \hat{\phi}(X')$ where $\hat{\phi}$ is the mapping defined by (7). Then, by using (10) for $\hat{\phi}$ and ϕ we have

$$\begin{aligned} X' &= \hat{\phi}(X') + [\hat{\phi}(X'), \hat{M}] \\ &= \{\phi(X) + [\phi(X), M]\} + [X, \hat{M}] \\ &= \phi(X). \end{aligned}$$

This implies that ϕ is surjective. $\hat{\phi}$ is the inverse of ϕ.

To prove that $\widehat{BMO}$ is isomorphic to BMO, we need the next two lemmas.

Lemma 4. *If* $\|X\|_{BMO}<1$, *then for any stopping time* T *we have*

$$E[\exp([X,X]-[X,X]_{T-})|F_T]\leqq(1-\|X\|^2_{BMO})^{-1}. \tag{11}$$

This inequality was obtained by Garsia [1] for discrete parameter martingales. For the proof, see [6].

Lemma 5. *Let* $0<\delta\leqq 1$, *and assume that* $-1+\delta\leqq \Delta M_t\leqq C$ *for every* t. *Then* $M\in BMO$ *if and only if* Z *satisfies the condition*

$$Z_T E[Z_\infty^{-1/(p-1)}|F_T]^{p-1}\leqq C_p \tag{A$_p$}$$

for some $p>1$, *where* T *is arbitrary stopping time.*

For the proof, see [3], [4] or [5].

3. Proof of Theorem

Assume that $M\in BMO$ and $\Delta M_t\geqq -1+\delta$ for some δ with $0<\delta\leqq 1$. Then, by Lemma 1, Z is a uniformly integrable martingale which satisfies (B_p) for all $p>1$. Firstly we show that the inequality

$$C_1[X,X]\leqq[\phi(X),\phi(X)]\leqq C_2[X,X] \tag{12}$$

is valid for every $X\in\mathscr{L}$ under either probability measure, where $C_1=(1+\|M\|_{BMO})^{-2}$ and $C_2=\delta^{-2}$. Recall that, if M is continuous, then $[\phi(X),\phi(X)]=[X,X]$ for all $X\in\mathscr{L}$. To see (12), let us assume dP to be the underlying measure. Then $\phi(X)$ is a semi-martingale and by the definition of [,] we have

$$[\phi(X),\phi(X)]_t=\langle\phi(X)^c,\phi(X)^c\rangle_t+\sum_{s\leqq t}(\Delta\phi(X)_s)^2.$$

But we know that $\phi(X)^c=X^c$,

$$\Delta\phi(X)_t=(1+\Delta M_t)^{-1}\Delta X_t \quad\text{and}\quad \delta\leqq 1+\Delta M_t\leqq 1+\|M\|_{BMO}.$$

Therefore, by combining these facts, we can obtain (12). Similarly, the same conclusion follows under $d\hat{P}$.

Now, let $p^{-1}+q^{-1}=1$ with $1<p<\infty$, and we are going to prove that Φ_p: $X\to Z_-^{-1/p}\cdot\phi(X)$ is a continuous mapping of H^p into $\hat{H}^p$. Let $X\in H^p$ and $Y'\in\hat{H}^q$. Without loss of generality we may assume that Y' is bounded, for the class of all bounded martingales is dense in $\hat{H}^q$. It follows from Lemma 5 that Z satisfies (A_{p_0}) for some $p_0>1$, and so $Z_\infty^{-1/(p_0-1)}\in L^1(dP)$. Then, by the Hölder inequality with exponents p_0 and $q_0=p_0/(p_0-1)$ we get

$$E[[Y',Y']_\infty^{q/2}]\leqq E[Z_\infty^{-1/(p_0-1)}]^{1/q_0}\hat{E}[[Y',Y']_\infty^{qp_0/2}]^{1/p_0}.$$

Moreover, by (12) we have

$$E[[\phi(X),\phi(X)]_\infty^{p/2}]^{1/p}\leqq C\|X\|_{H^p}.$$

Thus the increasing process $\left\{\int_0^t |d[\phi(X), Y']_s|\right\}$ is integrable with respect to dP, and by using the stopping argument we may assume that the process $\left\{\int_0^t Z_{s-}^{-1/p} |d[\phi(X), Y']_s|\right\}$ is also integrable. Since $(1+\|M\|_{BMO})^{-1} Z_s \leqq Z_{s-}$ and Z satisfies (B_q), we have

$$\begin{aligned}
&\hat{E}[|[\Phi_p(X), Y']_\infty|] \\
&\quad \leqq E\left[Z_\infty \int_0^\infty Z_{s-}^{-1/p} |d[\phi(X), Y']_s|\right] \\
&\quad \leqq (1+\|M\|_{BMO})^{1/p} E\left[\int_0^\infty Z_s^{1/q} |d[\phi(X), Y']_s|\right] \\
&\quad \leqq (1+\|M\|_{BMO})^{1/p} K_q E\left[\int_0^\infty E[Z_\infty^{1/q}|F_s]\, |d[\phi(X), Y']_s|\right].
\end{aligned}$$

But the expectation on the right hand side is $E\left[Z_\infty^{1/q} \int_0^\infty |d[\phi(X), Y']_s|\right]$, which is smaller than $E[Z_\infty^{1/q}[\phi(X), \phi(X)]_\infty^{1/2}[Y', Y']_\infty^{1/2}]$. Then, applying the Hölder inequality with exponents p and q to this term, we can find that

$$\begin{aligned}
\hat{E}[|[\Phi_p(X), Y']_\infty|] &\leqq C_p E[[\phi(X), \phi(X)]_\infty^{p/2}]^{1/p} \|Y'\|_{\hat{H}^q} \\
&\leqq C_{p,\delta} \|X\|_{H^p} \|Y'\|_{\hat{H}^q}.
\end{aligned}$$

Therefore, the inequality $\|\Phi_p(X)\|_{\hat{H}^p} \leqq C_{p,\delta} \|X\|_{H^p}$ follows from the usual duality argument. It is clear that Φ_p is linear and injective. We now remark that, if M is continuous, then the process Z is, of course, continuous and $[X, X] = [\phi(X), \phi(X)]$ under dP and $d\hat{P}$, so that by (6) we can obtain the right side of (3).

On the other hand, by Lemma 2, $\hat{Z}$ satisfies $(\hat{B}_p)$ for all $p>1$. Moreover, as is proved in [3] and [5], $\hat{M} = -\phi(M)$ belongs to $\widehat{BMO}$ and $\Delta\hat{M}_t = -(1+\Delta M_t)^{-1}\Delta M_t \geqq -1+(1+\|M\|_{BMO})^{-1}$, so that the mapping $\hat{\Phi}_p$: $\hat{H}^p \to H^p$ defined by $\hat{\Phi}_p(X')_t = \int_0^t \hat{Z}_{s-}^{-1/p} d\hat{\phi}(X')_s$, $X' \in \hat{H}^p$, is linear and continuous. And it is easy to see that $\hat{\Phi}_p$ is the inverse of Φ_p. Consequently, H^p and $\hat{H}^p$ are isomorphic with the mapping Φ_p.

Finally, we shall establish the isomorphism between BMO and $\widehat{BMO}$. For that, we show that, if Z satisfies (A_p), then the inequality

$$\|X\|_{BMO} \leqq C_p \|\phi(X)\|_{\widehat{BMO}} \tag{13}$$

is valid for all $X \in \mathscr{L}$. $\|\phi(X)\|_{\widehat{BMO}} = 0$ implies $X=0$, so that we may assume $0 < \|\phi(X)\|_{\widehat{BMO}} < \infty$. Now let T be any stopping time, and set $a = (2p\|\phi(X)\|_{\widehat{BMO}}^2)^{-1}$. Then from Lemma 4

$$\hat{E}[\exp\{ap([\phi(X), \phi(X)]_\infty - [\phi(X), \phi(X)]_{T-})\}|F_T] \leqq 2$$

follows, and by the definition of (A_p) $E[(Z_T/Z_\infty)^{1/(p-1)}|F_T] \leqq C_p$. Applying now the left side of (12) and the Hölder inequality we have

$$
\begin{aligned}
&E[[X,X]_\infty - [X,X]_{T-} | F_T] \\
&\quad \leqq (C_1 a)^{-1} E[\exp\{a([\phi(X),\phi(X)]_\infty - [\phi(X),\phi(X)]_{T-})\} | F_T] \\
&\quad \leqq C_p \|\phi(X)\|^2_{\widehat{BMO}} E[(Z_T/Z_\infty)^{1/(p-1)} | F_T]^{(p-1)/p} \\
&\qquad \times \hat{E}[\exp\{ap([\phi(X),\phi(X)]_\infty - [\phi(X),\phi(X)]_{T-})\} | F_T]^{1/p} \\
&\quad \leqq C_p \|\phi(X)\|^2_{\widehat{BMO}}.
\end{aligned}
$$

Thus (13) is proved. On the other hand, as $\hat{M} \in \widehat{BMO}$ and $1 + \Delta \hat{M}_t \geqq (1 + \|M\|_{BMO})^{-1}$, for some $p > 1$ the process $\hat{Z}$ satisfies the (A_p) condition associated with $d\hat{P}$ by Lemma 5. Therefore, the inequality

$$\|X'\|_{\widehat{BMO}} \leqq C_p \|\hat{\phi}(X')\|_{BMO} \tag{14}$$

is valid for all $X' \in \hat{\mathscr{L}}$. Then, setting $X' = \phi(X)$ in (14), we get $\|\phi(X)\|_{\widehat{BMO}} \leqq C_p \|X\|_{BMO}$. Hence, BMO and $\widehat{BMO}$ are isomorphic under the mapping ϕ. This completes the proof.

In [7], it is pointed out that the assumption "$M \in BMO$" cannot be omitted for the validity of the theorem.

Finally, we remark that the spaces H^2 and $\hat{H}^2$ are isometrically isomorphic under the mapping Φ_2 whenever M is continuous and the process Z is a uniformly integrable martingale with $Z_\infty > 0$.

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