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n-Person Games with Only 1, $n-1$, and n-Person Coalitions*

W. F. LUCAS**

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Abstract. A symmetric solution is presented for any von Neumann-Morgenstern n-person game when the only coalitions that are not completely defeated contain $n-1$ or n players.

1. Introduction

The main mathematical problem in the VON NEUMANN-MORGENSTERN theory of n-person games in characteristic function form [5] is to show the existence and nature (or non-existence) of solution sets. This paper describes a solution of a symmetric nature for an arbitrary n-person game in which only coalitions with 1, $n-1$, and n players enters into the problem. Results for similar games in the theory of bargaining sets (see bibliography by MASCHLER in [2]) are given in [3], and results for such games in the solution theory for n-person games in partition function form will appear in a separate paper (see abstract by author in [2]).

In order to be complete a brief review of the basic definitions for a VON NEUMANN-MORGENSTERN n-person game is given, where the games are assumed to be in 0,1 normal form. Let $N = \{1, \ldots, n\}$ be a set of n players $1, \ldots, n$, where $n > 2$. First, assume there exist a real valued *characteristic function* v defined on the set 2^N of all subsets of N, that is, v assigns the real number $v(M)$ to each coalition (subset) M of N, and assume that $v(\emptyset) = 0$. There is no loss in generality with respect to solution theory (see p. 68 in [4]), if one further assumes that v is superadditive, that is, $v(M_1 \cup M_2) \geqq v(M_1) + v(M_2)$ whenever $M_1 \cap M_2 = \emptyset$. Second, define the set A consisting of all *imputations* $\boldsymbol{x} = (x_1, \ldots, x_n)$ which satisfy $x_i \geqq 0$ for all $i \in N$, and $\sum_{i \in N} x_i = 1$. Third, an imputation $\boldsymbol{x}$ is said to *dominate* an imputation $\boldsymbol{y}$ *with respect to* a nonempty coalition M, denoted by $\boldsymbol{x} \operatorname{dom}_M \boldsymbol{y}$, if

$$x_i > y_i \qquad \text{for all } i \in M ,$$

and

$$\sum_{i \in M} x_i \leqq v(M) .$$

An $\boldsymbol{x}$ satisfying this latter inequality is called *effective* for M. One further says $\boldsymbol{x}$ *dominates* $\boldsymbol{y}$, denoted by $\boldsymbol{x} \operatorname{dom} \boldsymbol{y}$, if there is a nonempty M such that $\boldsymbol{x} \operatorname{dom}_M \boldsymbol{y}$. For $\boldsymbol{x} \in A$ or $B \subset A$, let $\operatorname{dom}_M \boldsymbol{x} = \{\boldsymbol{y} \in A \mid \boldsymbol{x} \operatorname{dom}_M \boldsymbol{y}\}$, $\operatorname{dom} \boldsymbol{x} = \{\boldsymbol{y} \in A \mid \boldsymbol{x} \operatorname{dom} \boldsymbol{y}\}$,

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** Mathematics Research Center, The University of Wisconsin, Madison.

$\mathrm{dom}_M B = \bigcup_{x \in B} \mathrm{dom}_M \boldsymbol{x}$, and $\mathrm{dom}\, B = \bigcup_{x \in B} \mathrm{dom}\, \boldsymbol{x}$. Fourth, a subset K of A is called a *solution* if

$$K \bigcap \mathrm{dom}\, K = \emptyset ,$$

and

$$K \bigcup \mathrm{dom}\, K = A .$$

This paper considers those games in which only coalitions with $n - 1$ and n players can have nonzero values. Thus assume that

$$\begin{aligned} & v(N) = 1 , & \\ (1) \qquad & 0 \leqq v(N - i) \leqq 1 & i = 1, \ldots, n , \\ & v(M) = 0 & \text{for all } M \subset N, |M| < n - 1, \end{aligned}$$

where $|M|$ denotes the number of players in M, and where i stands for either the player i or the coalition containing the one player i. So a game is determined by the n values $v(N - i)$. To simplify the notation, let

$$d_i = 1 - v(N - i) \qquad i = 1, \ldots, n .$$

Then $0 \leqq d_i \leqq 1$ and the n numbers d_i also determine the game. The only type of domination by imputations is with respect to the coalitions of $n - 1$ players, and $\boldsymbol{x}\ \mathrm{dom}_{N-i}\ \boldsymbol{y}$ means $x_j > y_j$ for

$$\text{all} \quad j \in N - i \quad \text{and} \quad \sum_{j \in N - i} x_j \leqq v(N - i) .$$

The former condition implies $x_i < y_i$ and the latter condition is equivalent to $x_i \leqq d_i$.

2. A Solution

A solution for any game which satisfies (1) is

$$\begin{aligned} K = \bigcup_{r=0}^{[n/2]} \bigcup_{\sigma_r} \{\boldsymbol{x} \in A \,|\, & x_p \geqq d_p, p = i(1), i(2), \ldots, i(2r); \\ & x_q \leqq d_q, q = i(2r+1), i(2r+2), \ldots, i(n); \\ & x_{i(s-1)} - d_{i(s-1)} = x_{i(s)} - d_{i(s)}, s = 2, 4, \ldots, 2r\} \end{aligned}$$

where $[n/2]$ is the greatest integer in $n/2$ and each inner union is taken over the

$$\frac{n!}{(n-2r)!\,r!\,2^r} \qquad \text{permutations } \sigma_r = (i(1), i(2), \ldots, i(n))$$

of $(1, 2, \ldots, n)$ which give distinct terms. In other words an imputation $\boldsymbol{x}$ is in the solution K if and only if all $x_p - d_p$ that are positive are equal in pairs. For $r = 0$ one gets the term

$$C = \{\boldsymbol{x} \in A \,|\, x_q \leqq d_q , \ q = 1, 2, \ldots, n\}$$

which is the core, and for $r = [n/2]$ one gets the term

$$\begin{aligned} K_{[n/2]} = \bigcup_{\sigma_{[n/2]}} \{\boldsymbol{x} \in A \,|\, & x_p \geqq d_p, \ p = i(1), i(2), \ldots, i(2[n/2]); \\ & x_{i(n)} \leqq d_{i(n)} \quad \text{if } n \text{ is odd}; \\ & x_{i(s-1)} - d_{i(s-1)} = x_{i(s)} - d_{i(s)} , \\ & s = 2, 4, \ldots, 2[n/2]\} . \end{aligned}$$

Let
$$Z = \{x \in A \mid x_p \geqq d_p, p = 1, 2, \ldots, n\}.$$
Then $K_Z = K_{[n/2]} \cap Z$ is the solution on the "reduced" imputation simplex Z to the corresponding (n, k) simple majority games when $k = n - 1$ which was given by BOTT in [1]. So the solution K is the natural generalization of Bott's solution when $k = n - 1$. K is also the natural generalization to the solution of an arbitrary 3-person game which is nondiscriminatory (when $Z \neq \emptyset$) and which has "symmetric" line segments for bargaining curves. Note that if

$$\sum_{j \in N} d_j < 1 \quad \text{then} \quad C = \emptyset \quad \text{and} \quad Z \neq \emptyset, \quad \text{if}$$
$$\sum_{j \in N} d_j > 1 \quad \text{then} \quad C \neq \emptyset \quad \text{and} \quad Z = \emptyset, \quad \text{and if}$$
$$\sum_{j \in N} d_j = 1 \quad \text{then} \quad C = Z = \boldsymbol{d} = (d_1, \ldots, d_n).$$

Geometrically one has a simple game in the interior part Z of A and n truncated pyramid games (see p. 81 of [4]) in the regions $S_h = \{x \in A \mid x_h \leqq d_h\}$ which extend off each of the faces of Z. A trace, x_h = constant, in S_h gives an $(n-1)$-person game of the type being considered and this trace of K is the corresponding solution for this new game. In Z the solution K is symmetric with respect to all permutations of the $x_i - d_i$, and in S_h the solution K is symmetric with respect to all permutations of $x_i - d_i$ with $i \neq h$. Note that if $Z \neq \emptyset$ then the dimension of K is smallest in the interior part Z of A and the dimension increases as one goes more toward the exterior parts, that is, as more $x_i \leqq d_i$.

3. The Proof

To prove that K is a solution one must first prove that $K \cap \operatorname{dom} K = \emptyset$. Since $K \cap \operatorname{dom} K = [(K - C) \cup C] \cap \operatorname{dom} [(K - C) \cup C] \subset [(K - C) \cap \operatorname{dom} (K - C)] \cup [(K - C) \cap \operatorname{dom} C] \cup [C \cap \operatorname{dom} K]$, it is sufficient to prove that $K \cap \operatorname{dom} C = \emptyset$, $C \cap \operatorname{dom} K = \emptyset$, and $(K - C) \cap \operatorname{dom} (K - C) = \emptyset$.

If $K \cap \operatorname{dom} C = \emptyset$ fails to hold, then there exists $\boldsymbol{a} \in C$ and $\boldsymbol{b} \in K$ such that $\boldsymbol{a} \operatorname{dom}_{N-k} \boldsymbol{b}$ for some $k \in N$. Since $\boldsymbol{a}$ is effective for $N - k$, $\sum_{j \in N} a_j = \sum_{j \in N} b_j = 1$, and $a_i > b_i$ for all $i \neq k$; one gets $d_k \leqq a_k < b_k$, which implies $\boldsymbol{b} \notin C$. Since $\boldsymbol{a} \in C$, one also gets $b_i < a_i \leqq d_i$ for $i \neq k$. It follows that $\boldsymbol{b}$ has exactly one coordinate with $b_j > d_j$ (namely $j = k$), which implies $\boldsymbol{b} \notin K - C$. Thus $\boldsymbol{b} \notin C \cup (K - C) = K$, which is a contradiction.

If $C \cap \operatorname{dom} K = \emptyset$ were not true, than there exists $\boldsymbol{a} \in K$ and $\boldsymbol{b} \in C$ such that $\boldsymbol{a} \operatorname{dom}_{N-k} \boldsymbol{b}$. As in the proceeding case one gets $b_k > a_k \geqq d_k$. This implies that $\boldsymbol{b} \notin C$, which is a contradiction.

Next assume that $(K - C) \cap \operatorname{dom} (K - C) \neq \emptyset$. Then there exists $\boldsymbol{a}$ and $\boldsymbol{b}$ in $K - C$ such that $\boldsymbol{a} \operatorname{dom}_{N-k} \boldsymbol{b}$, which implies

(2) $$a_i - d_i > b_i - d_i \quad \text{for all} \quad i \in N - k.$$

However, $\boldsymbol{b} \in K - C$ implies that all the $b_i - d_i$ that are positive are equal in pairs, and since $\boldsymbol{b} \notin C$ there is at least one such $b_i - d_i > 0$. Likewise the positive $a_i - d_i$ are equal in pairs, and since $\boldsymbol{a}$ is effective for $N - k$, $d_k \leqq a_k < b_k$ or

(3) $$0 \leqq a_k - d_k < b_k - d_k.$$

It follows that (if one lets $k = i(1)$) there exists distinct players

$$i(1), i(2), \ldots, i(2r),$$
$$i(2r+1), \ldots, i(n) \quad \text{with} \quad r > 0 \quad \text{so that}$$

$$\begin{aligned} & a_{i(s)} - d_{i(s)} = a_{i(s+1)} - d_{i(s+1)}, \\ (4) \qquad & b_{i(s-1)} - d_{i(s-1)} = b_{i(s)} - d_{i(s)} \end{aligned}$$

for $s = 2, 4, \ldots, 2r$, where either

$$\text{(a)} \qquad i(2r+1) = i(1)$$

or

$$\text{(b)} \qquad b_{i(2r+1)} - d_{i(2r+1)} \leqq 0 .$$

In case (a) the relations (2), (3), and (4) imply

$$\begin{aligned} & a_{i(1)} - d_{i(1)} \geqq b_{i(2r-1)} - d_{i(2r-1)}, \\ & a_{i(t+1)} - d_{i(t+1)} > b_{i(t-1)} - d_{i(t-1)} && t = 2, 4, \ldots, 2r-2, \\ & a_{i(t)} - d_{i(t)} > b_{i(t)} - d_{i(t)} && t = 2, 4, \ldots, 2r, \\ & a_j - d_j > b_j - d_j && j = 2r+1, 2r+2, \ldots, n . \end{aligned}$$

In case (b) the relations (2), (3), and (4) imply

$$\begin{aligned} & a_{i(1)} - d_{i(1)} \geqq b_{i(2r+1)} - d_{i(2r+1)} \\ & a_{i(t+1)} - d_{i(t+1)} > b_{i(t-1)} - d_{i(t-1)} && t = 2, 4, \ldots, 2r, \\ & a_{i(t)} - d_{i(t)} > b_{i(t)} - d_{i(t)} && t = 2, 4, \ldots, 2r, \\ & a_j - d_j > b_j - d_j && j = 2r+2, 2r+3, \ldots, n . \end{aligned}$$

Summing the equations above for either case (a) or (b) gives

$$\sum_{i \in N} (a_i - d_i) > \sum_{i \in N} (b_i - d_i) \quad \text{or} \quad \sum_{i \in N} a_i > \sum_{i \in N} b_i ,$$

which is a contradiction.

It follows that $(K - C) \cap \text{dom}\,(K - C) = \emptyset$, and this completes the proof that $K \cap \text{dom}\, K = \emptyset$.

Finally one has to prove that $K \cup \text{dom}\, K = A$. Assume that $\boldsymbol{b} \in A - K$. Since $\boldsymbol{b} \notin C \subset K$, there exists i such that $b_i - d_i > 0$. Also, there exists k such that $0 < b_k - d_k \neq b_j - d_j$ for an odd number of $j \neq k$, because if all such positive $b_j - d_j$ could be set equal in pairs, then $\boldsymbol{b} \in K - C$. So, by permuting the subscripts on the b_i and d_i if necessary, one can assume that

$$\begin{aligned} & b_j - d_j \geqq b_{j+1} - d_{j+1} && j = 1, 2, \ldots, n-1, \\ & b_k - d_k > 0 \\ (5) \qquad & b_k - d_k > b_{k+1} - d_{k+1} \\ & b_q - d_q \geqq 0 > b_{q+1} - d_{q+1} \end{aligned}$$

where k is odd and $k \leqq q$. The following three cases will be considered.

(i) $\quad q \geqq 3$ is odd,

(ii) $\quad q = 1$,

(iii) $\quad q$ is even.

In case (i) let

$$(n-1)\,\varepsilon_1 = (b_k - d_k) - \max(b_{k+1} - d_{k+1}, 0) > 0\,,$$
$$\varepsilon_2 = -(b_{q+1} - d_{q+1}) > 0\,,$$
$$\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0\,,$$
$$(q-1)\,\delta = (b_1 - d_1) - \sum_{i=1}^{(q-1)/2} [(b_{2i} - d_{2i}) - (b_{2i+1} - d_{2i+1})] - (n-1)\,\varepsilon \geqq 0\,.$$

Next define $\boldsymbol{a}$ by

$$\begin{aligned} a_1 - d_1 &= 0\,, \\ a_{2i} - d_{2i} &= a_{2i+1} - d_{2i+1} \\ &= b_{2i} - d_{2i} + \varepsilon + \delta \qquad i = 1, 2, \ldots, (q-1)/2\,, \\ a_j &= b_j + \varepsilon \qquad j = q+1, q+2, \ldots, n\,. \end{aligned}$$

Then $\boldsymbol{a}$ satisfies $a_i \geqq 0$ for all $i \in N$ and $\sum_{i\in N} (a_i - d_i) = \sum_{i\in N} (b_i - d_i)$ or $\sum_{i\in N} a_i = \sum_{i\in N} b_i = 1$, and so $\boldsymbol{a} \in A$. Also $\boldsymbol{a} \in K$ since the positive $a_i - d_i$ are equal in pairs. Furthermore $\boldsymbol{a}$ dom_{N-1} $\boldsymbol{b}$, because $a_i - d_i > b_i - d_i$ for all $i \neq 1$, and $a_1 = d_1$ implies $\boldsymbol{a}$ is effective for $N-1$. Thus $\boldsymbol{b} \in \mathrm{dom}\, K$. If one had to permute the subscripts of the b_i and d_i to get it in form (5), then the inverse permutation will give the corresponding $\boldsymbol{a}$ which is clearly still in K. This completes the proof for case (i).

Now consider case (ii) where $q = 1$.
Define $\boldsymbol{a}$ by

$$\begin{aligned} a_1 - d_1 &= 0\,, \\ a_2 - d_2 &= b_2 - d_2 + \varepsilon + \delta_2\,, \\ a_3 - d_3 &= b_3 - d_3 + \varepsilon + \delta_3\,, \\ a_j &= b_j + \varepsilon \qquad j = 4, 5, \ldots, n\,, \end{aligned}$$

where ε is the same as in case (i) and where δ_2 and δ_3 are defined by

$$\delta_2 + \delta_3 = (b_1 - d_1) - (n-1)\,\varepsilon \geqq 0$$

and

$$a_2 - d_2 = a_3 - d_3 \quad \text{if} \quad \delta_2 + \delta_3 \geqq (b_2 - d_2) - (b_3 - d_3)$$

or

$$\delta_2 = 0 \quad \text{if} \quad \delta_2 + \delta_3 < (b_2 - d_2) - (b_3 - d_3)\,.$$

Again $\boldsymbol{a}$ satisfies $a_i \geqq 0$ for all $i \in N$ and $\sum_{i\in N} (a_i - d_i) = \sum_{i\in N} (b_i - d_i)$, and so $\boldsymbol{a} \in A$. Also $\boldsymbol{a} \in K$ since $a_2 - d_2$ and $a_3 - d_3$ are either equal or nonpositive and all other $a_i - d_i \leqq 0$. Clearly $\boldsymbol{a}$ dom_{N-1} $\boldsymbol{b}$, and thus $\boldsymbol{b} \in \mathrm{dom}\, K$.

In case (iii) where q is even, let

$$n\,\varepsilon_1 = (b_k - d_k) - (b_{k+1} - d_{k+1}) > 0\,,$$
$$\varepsilon_2 = -(b_{q+1} - d_{q+1}) > 0\,,$$
$$\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0\,,$$
$$2\,\delta = (b_1 - d_1) - (b_q - d_q) - n\,\varepsilon - \sum_{i=1}^{(q-2)/2} [(b_{2i} - d_{2i}) - (b_{2i+1} - d_{2i+1})] \geqq 0\,.$$

Define $\boldsymbol{a}$ by

$$\begin{aligned} a_1 - d_1 &= a_q - d_q = b_q - d_q + \varepsilon + \delta > 0\,, \\ a_{2i} - d_{2i} &= a_{2i+1} - d_{2i+1} \\ &= b_{2i} - d_{2i} + \varepsilon \qquad i = 1, 2, \ldots, (q-2)/2\,, \\ a_j &= b_j + \varepsilon \qquad j = q+1, q+2, \ldots, n\,. \end{aligned}$$

Again one can show that $\boldsymbol{a} \in A$, $\boldsymbol{a} \in K$, and $\boldsymbol{a}$ dom_{N-1} $\boldsymbol{b}$.

So $\boldsymbol{b} \in \mathrm{dom}\, K$, which proves case (iii). This completes the proof that $K \cup \mathrm{dom}\, K = A$, and therefore K is *a* solution.

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Mathematics Research Center
University of Wisconsin
Madison, Wisconsin