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# Spatially Homogeneous Markov Operators*

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## Introduction

In this paper we continue the study of Markov operators begun in [2, 3]. Among all measure-preserving transformations the rotations on a compact abelian group play a special canonical role. (Cf. [7] and [6], pp. 46—50.) It turns out that among all Markov operators with a finite invariant measure those which act on functions on a compact abelian group and are spatially homogeneous, in a sense to be defined below, play essentially the same role. We show that every such operator $T$ has an integral representation over the set of translation operators. Using this representation we then investigate the spectrum of $T$ and prove a theorem analogous to the representation theorem of P. R. HALMOS and J. VON NEUMANN [7] for measure-preserving transformations with a discrete (pure-point) spectrum.

## 1. Markov Operators and the Translation Group

Let $(X, \mathscr{F}, m)$ be a finite measure space, and let us denote by $L_2$ the Banach space of square-integrable, complex-valued functions on $X$. We shall say that an operator $T$ on $L_2$ is *positive* if $f \geqq 0 \Rightarrow Tf \geqq 0$ $(f \in L_2)$. A *Markov operator* (with invariant measure $m$) is a positive linear operator $T$ on $L_2$ satisfying $T1 = T^*1 = 1$. It was shown in [2] that the set $M$ of all such operators is a convex set, which is compact in the weak operator topology. Moreover, there is a one-to-one affine correspondence between $M$ and the set of all *doubly stochastic measures* on $X \times X$, i.e. positive measures $\lambda$ on $(X \times X,\ \mathscr{F} \times \mathscr{F})$ such that $\lambda(A \times X) = \lambda(X \times A) = m(A)$ for $A \in \mathscr{F}$. This correspondence is given by

$$(f, Tg) = \int_{X \times X} f(x)\overline{g(y)}\,\lambda(dx, dy) \qquad (f, g \in L_2). \tag{1}$$

Moreover, the group $\Phi$ of all invertible measure-preserving transformations $\varphi$ of $(X, \mathscr{F}, m)$ is canonically embedded in $M$ by setting $T_\varphi f(x) = f(\varphi x)$. The induced topology on $\Phi$ is the weak topology of [6].

Let $X$ be a compact, abelian group, let $\mathscr{F}$ be the class of Borel subsets of $X$, and let $m$ be normalized Haar measure on $X$ $(m(X) = 1)$. The group $X$ can be embedded as a subgroup into $\Phi \subseteqq M$ by setting

$$T_y f(x) = f(x - y)\ (x, y \in X, f \in L_2).$$

Our first result is that this is a topological, as well as algebraic, embedding.

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**Theorem 1.** *The relative topology on $X$ induced by the weak operator topology of $M$ coincides with the given topology of $X$.*

*Proof.* Since $M$ is compact in the weak operator topology, any Hausdorff topology on $M$ which is weaker than the weak operator topology must coincide with the latter. Thus a subbase for the relative topology on $X$ consists of all sets

$$\begin{aligned} N(y_0; f, g, \varepsilon) &= \{y \in X : |(f, T_y g) - (f, T_{y_0} g)| < \varepsilon\} \\ &= \{y \in X : |\int f(x)[\overline{g(x-y)} - \overline{g(x-y_0)}]\, m(dx)| < \varepsilon\}, \end{aligned}$$

where $y_0 \in X$, $\varepsilon > 0$, and $f$ and $g$ are *continuous* functions on $X$. For each such choice of $f$ and $g$ the function $h(x, y) = f(x)\,\overline{g(x-y)}$ is uniformly continuous on $X \times X$. Hence there exists a neighborhood $N_\varepsilon$ of $0 \in X$ in the given topology such that

$$\begin{aligned} y - y_0 \in N_\varepsilon &\Rightarrow |f(x)\overline{g(x-y)} - f(x)\overline{g(x-y_0)}| < \varepsilon \quad (x \in X) \\ &\Rightarrow |\int f(x)[\overline{g(x-y)} - \overline{g(x-y_0)}]\, m(dx)| < \varepsilon. \end{aligned}$$

Thus $y_0 + N_\varepsilon \subset N(y_0; f, g, \varepsilon)$, and the given topology on $X$ is stronger than the weak operator topology. Since $X$ is compact in its given topology, the two must coincide. ∎

According to the Choquet representation theorem (see, for instance, [*9*]), for each $T \in M$ there exists a regular probability measure $\mu$ on the Borel subsets of $M$ such that

$$(f, Tg) = \int_M (f, Sg)\, \mu(dS) \qquad (f, g \in L_2), \tag{2}$$

and such that $\mu$ vanishes on any Baire set not containing extreme points of $M$. Unfortunately, no completely adequate description of the set of extreme points of $M$ is known even when $X$ is the unit interval (circle group), although J. Lindenstrauss [*8*] has given a necessary and sufficient condition for the doubly stochastic measure $\lambda$ to be extremal. It is easily seen that the set of extreme points of $M$ contains $\Phi$ and hence $X$.

In the remainder of this paper we discuss operators $T$ having a representation of the form (2) with $\mu$ concentrated on the set of translation operators. According to Theorem 1, $\mu$ can be thought of as an ordinary Borel measure on $X$, and equation (2) becomes

$$(f, Tg) = \int_X \int_X f(x)\overline{g(x-y)}\, m(dx)\, \mu(dy) \qquad (f, g \in L_2). \tag{3}$$

We shall see that the condition that $T$ have such a representation has a natural probabilistic interpretation, when $T$ is associated with a Markov transition function, and a natural geometric interpretation.

## 2. Spatially Homogeneous Operators

**Definition.** A doubly stochastic measure $\lambda$ on $X \times X$ (or the associated Markov operator $T$) is said to be *spatially homogeneous* if

$$\lambda((A + x) \times (B + x)) = \lambda(A \times B) \tag{4}$$

for all $x \in X$, $A, B \in \mathscr{F}$.

We shall denote the set of all spatially homogeneous Markov operators by $\Sigma$.

In order to see the probabilistic significance of spatial homogeneity, let us suppose that $T$ is given by a *Markov transition function:*

$$T f(x) = \int_X f(y) P(x, dy) \qquad (f \in L_2), \tag{5}$$

where $P(x, B)$ is a measurable function of $x \in X$ for each $B \in \mathscr{F}$ and a probability measure in $B \in \mathscr{F}$ for each $x \in X$. It follows from (1) and (5) that

$$\lambda(A \times B) = \int_A P(x, B) m(dx) \qquad (A, B \in \mathscr{F}). \tag{6}$$

Then $\lambda$ is spatially homogeneous if and only if

$$P(x + y, B + y) = P(x, B) \qquad m - \text{a.e.}(x) \tag{7}$$

for each $y \in X$, $B \in \mathscr{F}$.

Note. It can be shown that $T$ always has a representation of the form (5) if $X$ is metrizable. In this case it can be shown rather easily that $T$ is a (left) centralizer in the sense of J. G. Wendel [*10*], and our Theorem 3 is a special case of his Theorem 1. For the details see [*1*].

We now turn to the geometric interpretation of spatial homogeneity.

**Theorem 2.** *The Markov operator $T$ is spatially homogeneous iff it commutes with all translations.*

*Proof.* For $A \in \mathscr{F}$ let $\chi_A$ denote the characteristic function of $A$. According to (1), we may rewrite (4) as

$$(\chi_A, T\chi_B) = (T_x \chi_A, T T_x \chi_B) = (\chi_A, T_{-x} T T_x \chi_B)$$

for all $A, B \in \mathscr{F}$, $x \in X$. However, this is equivalent to

$$T = T_{-x} T T_x \quad \text{or} \quad T_x T = T T_x$$

for all $x \in X$.

**Corollary.** *The convex set $\Sigma$ is closed, and hence compact, in the weak operator topology.*

## 3. Integral Representation of Spatially Homogeneous Operators

We begin by proving a lemma.

**Lemma.** *The set of extreme points of $\Sigma$ coincides with $X$.*

*Proof.* For each doubly stochastic measure $\lambda \in \Sigma$ let us define $\tilde{\lambda}$ on the Borel sets of $X$ by

$$\tilde{\lambda}(B) = \lambda\{(x, y) : x - y \in B\}. \tag{8}$$

Since the mapping $(x, y) \to x - y$ is a continuous mapping of $X \times X$ onto $X$, $\tilde{\lambda}$ is a probability measure on $X$. The correspondence $\lambda \to \tilde{\lambda}$ is thus an affine mapping of $\Sigma$ into the set of all probability measures on $X$. Let us show that it is one-to-one.

The mapping $(x, y) \to (x, x - y)$ is a homeomorphism of $X \times X$ onto itself. For fixed $\lambda \in \Sigma$ and $B \in \mathscr{F}$ let us define

$$\nu^B(A) = \lambda\{(x, y) : x \in A, x - y \in B\}. \tag{9}$$

From the spatial homogeneity of $\lambda$ it follows that

$$\nu^B(A + z) = \lambda\{(x + z, y + z) : x \in A, x - y \in B\} = \nu^B(A).$$

By the uniqueness of Haar measure $\nu^B$ must be a multiple of $m$. Comparing (8) and (9) we see that

$$\nu^B(A) = \tilde{\lambda}(B)\, m(A)\,. \tag{10}$$

Comparing (9) and (10) and noting that the mapping $(x, y) \to (x, x - y)$ is its own inverse, we see that

$$\lambda(A \times B) = (m \times \tilde{\lambda})\{(x, y) : x \in A,\, x - y \in B\} \tag{11}$$

for all $A, B \in \mathscr{F}$. It follows from (11) that the correspondence $\lambda \to \tilde{\lambda}$ is one-to-one.

Moreover, if $\tilde{\lambda}$ is any probability measure on $X$, then (11) determines a measure $\lambda \in \Sigma$ satisfying (8). That is, the correspondence $\lambda \to \tilde{\lambda}$ is onto.

It follows that $\lambda$ is an extreme point of $\Sigma$ iff $\tilde{\lambda}$ is an extremal probability measure on $X$, which is true iff $\tilde{\lambda}$ is concentrated at a point $z \in X$,

$$\tilde{\lambda}(B) = \chi_B(z)\,.$$

Finally, it follows from (8) and (11) that this is true iff $\lambda$ is concentrated on the graph of the translation $x \to x - z$, that is

$$\lambda(C) = m\{x : (x, x - z) \in C\}\,.$$

This completes the proof. ▮

**Theorem 3.** *Suppose that* $T \in \Sigma$ *and let* $\lambda$ *be the associated doubly stochastic measure on* $X \times X$. *Then there is a unique probability measure* $\mu$ *on* $X$ *such that*

$$\lambda(A \times B) = \int_X m(A \cap (B + y))\, \mu(dy) \tag{12}$$

*for all* $A, B \in \mathscr{F}$. *Moreover,*

$$Tf(x) = \int_X f(x - y)\, \mu(dy) \qquad m - a.e. \tag{13}$$

*for all* $f \in L_2$.

*Proof*. Clearly, (12) holds for all $A, B \in \mathscr{F}$ iff (3) holds for all $f, g \in L_2$. Moreover, by the Fubini theorem (3) is equivalent to (13). Thus existence follows from the Lemma and the Choquet theorem. (Since $X$ is closed, we actually need only the Krein-Milman and Riesz representation theorems.)

Since (13) may be read $Tf = f * \mu$, uniqueness of $\mu$ follows from properties of the Fourier transform. Thus if $\hat{f}$ and $\hat{\nu}$ denote Fourier and Fourier-Stieltjes transforms of $f \in L_2$ and the regular Borel measure $\nu$, we have

$$\begin{aligned} f * \nu = 0\,(f \in L_2) &\Rightarrow \hat{f}\hat{\nu} = 0\,(f \in L_2) \\ &\Rightarrow \hat{\nu} = 0 \\ &\Rightarrow \nu = 0\,. \end{aligned}$$ ▮

From the Lemma, the Krein-Milman theorem and the fact that the closed convex hull of a set of operators is the same in the weak operator topology and the strong operator topology ([*5*], p. 477), we obtain the following result, which is closely related to Theorem 4 of [*10*].

**Corollary.** $\Sigma$ *is the closed convex hull of* $X$ *in the strong operator topology.*

## 4. Spectral Properties of $T$ and the Isomorphism Theorem

Let $\hat{X}$ denote the dual group of $X$. Then $\hat{X}$ is discrete, and by the Pontryagin duality theorem we can identify its dual with $X$. For $x \in X$ and $\hat{x} \in \hat{X}$ we shall write $\hat{x}(x) = \langle x, \hat{x}\rangle$. We shall indicate integration with respect to Haar measure by $\int dx$ and $\int d\hat{x}$ on $X$ and $\hat{X}$, respectively.

**Theorem 4.** *Suppose that $T \in \Sigma$ has the representation* (13). *Then the spectrum of $T$ is $C = \{\hat{\mu}(\hat{x}) : \hat{x} \in \hat{X}\}$, $T$ has pure point spectrum, and the proper space corresponding to $\lambda \in C$ is the subspace spanned by $\{\hat{x} \in \hat{X} : \hat{\mu}(\hat{x}) = \lambda\}$.*

*Proof.* If $\hat{x} \in \hat{X}$, then $\hat{x} \in L_2$, and we have by (13) that

$$\begin{aligned} T\hat{x}(x) &= \int_X \langle x - y, \hat{x}\rangle \,\mu(dy) \\ &= \int_X \langle x, \hat{x}\rangle \overline{\langle y, \hat{x}\rangle}\,\mu(dy) \\ &= \hat{\mu}(\hat{x})\,\hat{x}(x)\,. \end{aligned}$$

Thus $\hat{x}$ is a proper function of $T$ corresponding to the proper value $\hat{\mu}(\hat{x})$.

On the other hand, if

$$Tf(x) = \int_X f(x-y)\,\mu(dy) = \lambda f(x) \qquad \text{a.e.,}$$

then

$$\begin{aligned} \lambda \hat{f}(\hat{x}) &= \int_X \overline{\langle x, \hat{x}\rangle} \int_X f(x-y)\,\mu(dy)\,dx \\ &= \int_X \int_X \overline{\langle x, \hat{x}\rangle} f(x-y)\,dx\,\mu(dy) \\ &= \int_X \overline{\langle y, \hat{x}\rangle} \int_X \overline{\langle x-y, \hat{x}\rangle} f(x-y)\,dx\,\mu(dy) \\ &= \int_X \overline{\langle y, \hat{x}\rangle}\,\mu(dy) \int_X \overline{\langle x, \hat{x}\rangle} f(x)\,dx \\ &= \hat{\mu}(\hat{x})\hat{f}(\hat{x})\,. \end{aligned}$$

If $f \neq 0$, then $\hat{f}$ does not vanish identically. It follows that $\lambda = \hat{\mu}(\hat{x}) \in C$ for some $\hat{x} \in \hat{X}$, and that

$$f(x) = \int_{\hat{X}} \langle x, \hat{x}\rangle \hat{f}(\hat{x})\,d\hat{x}$$

belongs to the subspace spanned by the set of $\hat{x}$ such that $\hat{f}(\hat{x}) \neq 0$, i.e.

$$\{\hat{x} \in \hat{X} : \hat{\mu}(\hat{x}) = \lambda\}\,.$$

Finally, since $\hat{X}$ spans $L_2$, it follows that $T$ has pure point spectrum. ∎

Note. The spectral possibilities for $T \in \Sigma$ are not as simple as in the case of a measure-preserving transformation with pure point spectrum. For instance, the proper values do not in general form a group, but only the range of a positive-definite function on some discrete group. Nor do the proper values need to have absolute value 1, although, as for any $T \in M$, they are contained in the unit disc. Moreover, the proper values are not necessarily simple even if $T$ is ergodic (in which case 1 is a simple proper value).

Consider, for instance, the symmetric random walk on the group of integers modulo $n$. In this case, $X = \hat{X} = \{0, 1, \ldots, n-1\}$ with

$$\langle k, j\rangle = e^{2\pi i jk/n},$$

and

$$Tf(k) = \tfrac{1}{2} f(k-1) + \tfrac{1}{2} f(k+1) = \sum_l f(k-l)\,\mu(l).$$

Thus

$$\begin{aligned}\hat{\mu}(j) &= \sum_k \overline{\langle k, j\rangle}\,\mu(k)\\ &= \tfrac{1}{2}\overline{\langle 1, j\rangle} + \tfrac{1}{2}\overline{\langle -1, j\rangle}\\ &= \cos(2\pi j/n).\end{aligned}$$

If follows that 1 and possibly $-1$ are simple proper values, while others are double. A similar situation holds for the Brownian motion on the circle group.

Finally, if $\mu = m$ so that

$$Tf(x) = \int_X f(y)\,dy = \text{const.},$$

then 1 is a simple proper value, while 0 has infinite multiplicity.

It is true, however, that the linearly independent proper functions of $T \in \Sigma$ may be chosen to have constant absolute value 1 and to form a group under pointwise multiplication. We shall show that these properties characterize spatially homogeneous Markov operators up to spatial homomorphism.

**Definition.** Let $T_1$ and $T_2$ be bounded linear operators on $L_2(X_1, \mathscr{F}_1, m_1)$ and $L_2(X_2, \mathscr{F}_2, m_2)$, respectively. We say that $T_1$ and $T_2$ are *spectrally isomorphic* if there exists an invertible isometry $U$ of $L_2(X_1, \mathscr{F}_1, m_1)$ onto $L_2(X_2, \mathscr{F}_2, m_2)$ such that $T_1 = U^{-1} T U$. They are *spatially isomorphic* if $Uf(x) = f(\psi x)$ $(f \in L_2(X_1, \mathscr{F}_1, m_1))$, where $\psi: X_2 \to X_1$ is an invertible measure-preserving transformation (modulo sets of measure 0).

P. R. Halmos and J. von Neumann have shown [7] that $T_\varphi \in \Phi$ is spatially isomorphic to a translation in a compact group iff it has pure point spectrum. The next theorem extends this result. The proof is a modification of that of Halmos and von Neumann.

**Theorem 5.** *A Markov operator $T$ on $L_2(X)$ of the finite measure space $(X, \mathscr{F}, m)$ is spatially isomorphic to a spatially homogeneous operator $\tilde{T}$ on a compact abelian group $G$ iff there exists a complete orthonormal system $C$ for $L_2(X)$ consisting of bounded proper functions of $T$ and forming a group under pointwise multiplication.*

Note. If $T$ is spatially isomorphic to an ergodic measure-preserving transformation, then it can be shown (see, for instance, [1]) that $T$ is also given by a measure-preserving transformation. For such a transformation our condition is equivalent to requiring that $T$ have pure point spectrum (see [6], p. 34). Thus Theorem 5 is indeed an extension of the above-mentioned theorem.

*Proof.* For $X = G$ the necessity follows from Theorem 4. More generally, the isometry $U_\psi$ which implements the isomorphism carries $\tilde{G}$ onto an orthonormal basis $C$ for $L_2(X)$ and preserves the boundedness and multiplicative properties of that set.

To prove the sufficiency we let the group $C$ have the discrete topology and denote its dual group by $G$. Then $G$ is compact. Under the identification of $C$ with $\hat{G}$ the set $C$ becomes a complete orthonormal system in $L_2(G)$. We shall indicate the corresponding isometric isomorphism of $L_2(X)$ and $L_2(G)$ by $f \to \tilde{f} = Wf$. Thus, in particular $W$ maps $C$ onto $\hat{G}$. Now let $\tilde{T} = WTW^{-1}$. We shall show that $\tilde{T}$ is spatially homogeneous, and that $W = U_\psi$ is induced by a measure-preserving transformation $\psi$.

For each $\tau \in C$ we have

$$\tilde{T}(W\tau) = W(T\tau) = \lambda_\tau(W\tau),$$

where $\lambda_\tau$ is the proper value of $T$ corresponding to the proper function $\tau$. Thus $W\tau$ is a proper function of $\hat{T}$ corresponding to the same $\lambda_\tau$.

Let $g \in L_2(X)$. Then $\tilde{g} = Wg$ has the Fourier expansion

$$\tilde{g}(x) = \int_C \hat{\tilde{g}}(\tau) \langle \tau, x \rangle \, d\tau \quad (x \in G)$$

and so

$$T\tilde{g}(x) = \int_C \hat{\tilde{g}}(\tau) \lambda_\tau \langle \tau, x \rangle \, d\tau \qquad (x \in G). \tag{14}$$

If $T_y (y \in G)$ is defined as in §1, and if $h = T_y\tilde{g}$ for fixed $y \in G$, then

$$\begin{aligned} \hat{h}(\tau) &= \int_G h(x) \overline{\langle \tau, x \rangle} \, dx \\ &= \int_G \tilde{g}(x-y) \overline{\langle \tau, x \rangle} \, dx \\ &= \int_G \tilde{g}(z) \overline{\langle \tau, x+y \rangle} \, dx \\ &= \overline{\langle \tau, y \rangle} \, \hat{\tilde{g}}(\tau) \; (\tau \in C). \end{aligned} \tag{15}$$

Combining (14) and (15) gives

$$\begin{aligned} \tilde{T} T_y g(x) &= \int_C \overline{\langle \tau, y \rangle} \, \hat{\tilde{g}}(\tau) \lambda_\tau \langle \tau, x \rangle \, dx \\ &= \int_C \hat{\tilde{g}}(\tau) \lambda_\tau \langle \tau, x-y \rangle \, dx \\ &= T_y \tilde{T} \tilde{g}(x) \qquad (x \in G). \end{aligned}$$

Thus $\tilde{T}$ commutes with each $T_y (y \in G)$, and according to Theorem 2 $\tilde{T}$ is spatially homogeneous.

It remains to show that the isometry $W$ of $L_2(X)$ onto $L_2(G)$ is a spatial isomorphism. Since the restriction of $W$ to $C$ is a group isomorphism of $C$ onto $\hat{G}$, the equality

$$W(fg) = (Wf)(Wg) \tag{16}$$

holds for all $f, g \in C$. By linearity (16) holds for all $f, g$ in the linear space spanned by $C$. If $B \in \mathscr{F}$, let $f$ be a fixed linear combination of elements of $C$, and let $g_n \to \chi_B$ in $L_2(X)$. Then $f$ and $Wf$ are bounded functions so that $fg_n \to f\chi_B$ and $(Wf)(Wg_n) \to (Wf)(W\chi_B)$. Thus $W(f\chi_B) = (Wf)(W\chi_B)$. Similarly, letting $f_n \to \chi_B$ gives $W(\chi_B^2) = W(\chi_B) = (W\chi_B)^2$. Thus $W\chi_B \geqq 0$. It follows that $Wf \geqq 0$ whenever $f \geqq 0$, i.e. $W$ is a positive operator. Clearly, $W1 = 1$ $(= W^*1)$. Since $W$ is an invertible isometry, we have by a trivial modification of Theorem 5 of [2] that $W = U_\psi$ for some invertible measure-preserving transformation $\psi$. ∎

**Corollary 5.1.** *If $T$ satisfies the hypotheses of Theorem 5, then there exists a probability measure $\mu$ on $X$ and a measurable family $\{\varphi_x : x \in X\}$ of measure-preserving transformations of $X$ with pure point spectrum such that*

$$T f(x) = \int_X f(\varphi_y x)\, \mu(dy) \qquad a.e. \tag{17}$$

*for each $f \in L_2(X)$.*

*Proof.* This is simply a change of variables in (13) with $T\varphi_y = U_\psi T_{\psi y} U_\psi^{-1}$ so that $\varphi_y$ has pure point spectrum. ∎

**Corollary 5.2.** *If $T$ satisfies the hypotheses of Theorem 5, then $T$ is spatially isomorphic to its adjoint $T^*$.*

*Proof.* If $T$ and $\tilde{T}$ are isomorphic, then so are $T^*$ and $\tilde{T}^*$. Thus we may assume that $T$ is spatially homogeneous. Then

$$T f(x) = \int_X f(x - y)\, \mu(dy)$$

and

$$T^* f(x) = \int_X f(x + y)\, \mu(dy).$$

Let $\psi(x) = -x$. Then

$$\begin{aligned} U_\psi(T^* f)(x) &= T^* f(-x) \\ &= \int_X f(-x + y)\, \mu(dy) \\ &= T(U_\psi f)(x). \end{aligned}$$

Thus $T^* = U_\psi^{-1} T U_\psi$ as asserted. ∎

Remark. J. R. CHOKSI has recently shown [4] that for non-ergodic transformations with pure point spectrum spectral isomorphism need not imply spatial isomorphism. However, the above proof of Theorem 5 does not depend on ergodicity as in the case of [7]. Thus our theorem applies to non-ergodic transformations.

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