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Sample Quadratic Variation of Sample Continuous, Second Order Martingales

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Introduction

A great deal of work concerning the quadratic variation of a stochastic process has been done in the last few years.

The problems dealt with have taken the following form:

Let $\{X(t), F(t);\ t \in T\}$ be a stochastic process on the probability space $(\Omega, \mathscr{F}, P)$ where $T = [0, 1]$, $X(t)$ is $F(t)$ measurable, and $F(s) \subset F(t) \subset F$ for $s, t \in T$ with $s \leqq t$. Let $\pi_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{N_n}^{(n)} = 1\}$ be a partition of $[0, 1]$ for each $n \geqq 1$. We assume π_{n+1} is a refinement of π_n and $\max\limits_j (t_{j+1}^{(n)} - t_j^{(n)}) \to 0$ as $n \to \infty$. Let

$$\Delta^2 X(t_j^{(n)}) = [X(t_{j+1}^{(n)}) - X(t_j^{(n)})]^2.$$

If $\{Y(t), F(t); t \in T\}$ is a stochastic process, we let

$$S_n(t) = \sum_t Y(\xi_j^{(n)})\, \Delta^2 X(t_j^{(n)})$$

where $\sum\limits_t$ means sum out to the last j with $t_{j+1}^{(n)} \leqq t$, and $\xi_j^{(n)} \in [t_j^{(n)}, t_{j+1}^{(n)}]$.

The problem is of course to determine when the limit of the sequence of processes $\{S_n(t), F(t); t \in T\}$ exist in probability, almost surely or in the mean, and to find this limit when it exists.

In this paper we assume the X process is a second order martingale and $Y \equiv 1$, or Y is a sample continuous process. In this case we obtain probability and in some cases mean limits.

The main theorems are Theorem 1.1 and Theorem 1.2. These limit theorems are more general in some cases than those in [4] and [6]. However, some of the limit theorems in [4] and [6] are stronger in the sense that, due to the special nature of the processes involved, certain mean and a.s. convergence is obtained where our limits are in probability.

Section 1

In this section we state some known results and develop the body of the paper.

Lemma 1.1. [3]. *Approximation theorem for sample continuous processes:*

Let $\{X(t), F(t); t \in T\}$ be an a.s. sample continuous process. There is a sequence of stopping times $\{\tau_\nu | \nu \geqq 1\}$, such that if $\{X_\nu(t), F(t);\ t \in T\}$ is the X-process stopped at τ_ν, then i) *each X_ν is sample equicontinuous and uniformly bounded by ν.*

* The theorems in Section 1 are essentially contained in a thesis written under Professor Herman Rubin in 1963. This paper was partially supported by NSF G-24500, Ns G-568.

ii) *There is a set* $\Lambda \in \mathscr{F}$, $P(\Lambda) = 0$, *such that if* $\omega \notin \Lambda$, *then there exists* $\nu(\omega)$ *such that* $X_\nu(t) = X(t)$ *for all* $t \in T$, *if* $\nu \geqq \nu(\omega)$.

Lemma 1.2. [*1, 2, 3*]. *Submartingale decomposition theorem:*

If $\{X(t), F(t); t \in T\}$ *is an a.s. sample continuous submartingale, then it has a unique decomposition,*

$$P([X(t) = X_1(t) + X_2(t) \text{ for all } t \in T]) = 1$$

where X_1 *is an a.s. sample continuous martingale,* X_2 *has a.e. sample function monotone non-decreasing and continuous with*

$$X_2(t) = P \lim \sum_t E[\Delta X(t_j^{(n)}) \mid F(t_j^{(n)})],$$

if and only if

$$\lim_{n \to \infty} n P([\sup_{t \in T} |X(t)| \geqq n]) = 0.$$

In particular, if the X-process has a.e. sample function non-negative, then the condition is always satisfied.

Note: The decomposition theorem was first proved by Meyer [*1*], and the given condition for an a.s. sample continuous submartingale was given by Johnson and Helms [*2*].

Let $\{X(t), F(t); t \in T\}$ be an a.s. sample continuous second order martingale. Let

$$Z(t) = [X(t)]^2.$$

Then $\{Z(t), F(t); t \in T\}$ is a non-negative sample continuous submartingale and hence by Lemma 1.2, it has the unique decomposition

$$P([Z(t) = Z_1(t) + Z_2(t) \text{ for all } t \in T]) = 1$$

where Z_1 is a sample continuous martingale and Z_2 has a.e. sample function monotone non-decreasing and continuous. We observe that $Z_2(0) = 0$ a.s.

Theorem 1.1. *With* X *and* Z *as just defined*

$$Z_2(t) = \int_0^t dZ_2(t) = \begin{cases} \text{l.i.m.}\, \Sigma_t \Delta^2 X(t_j^{(n)}) \\ \text{l.i.m.}\, \Sigma_t E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})] \end{cases}$$

⟨*where l.i.m. indicates limit in the mean*⟩.

Proof. We observe that

$$\begin{aligned} E\{\sum_t \Delta^2 X(t_j^{(n)})\} &= E\{\Sigma_t E[\Delta^2 X(t_j^{(n)}) \mid F(t_j^{(n)})]\} \\ &= E\{\sum_t E[X^2(t_{j+1}^{(n)}) - X^2(t_j^{(n)}) \mid F(t_j^{(n)})]\} \\ &= E\{\sum_t E(\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})]\} \\ &= E\{\sum_t E[\Delta Z_2(t_j^{(n)}) \mid F(t_j^{(n)})]\} \\ &= E\{\sum_t \Delta Z_2(t_j^{(n)})\} = E\{Z_2(t_{jt}^{(n)}) - Z_2(0)\} \\ &= E\{Z_2(t_{jt}^{(n)})\}, \text{ where } t_{jt}^{(n)} \text{ is the last } t_j^{(n)} \leqq t. \end{aligned}$$

Then, since a.e. sample function of the Z_2-process is monotone non-decreasing, $Z_2(t_{jt}^{(n)}) \to Z_2(t)$ as $n \to \infty$, and the monotone convergence theorem is applicable. Hence, for each $t \in T$,

$$\lim_{n\to\infty} E\{\sum_t \Delta^2 X(t_j^{(n)})\} = \lim_{n\to\infty} E\{\sum_t E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})\} = E\{Z_2(t)\}.$$

Since the sequences are non-negative, it is sufficient (see HALMOS [7], p. 112) to show that the probability limits exist and are as stated in the theorem. From Lemma 1.2 however, we have

$$P\lim \sum_t E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})] = Z_2(t).$$

We now establish that

$$P\lim \sum_t \Delta^2 X(t_j^{(n)}) = P\lim \sum_t E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})].$$

Let $\{X_\nu(t), F(t); t \in T\}$ be the sequence of processes as given in Lemma 1.1. Then each X_ν is a uniformly bounded, a.s. sample equicontinuous martingale. Letting $Z_\nu = X_\nu^2$, we have the decomposition $Z_\nu = Z_{1\nu} + Z_{2\nu}$, as given by Lemma 1.2, so that for each $\nu \geqq 1$,

$$Z_{2\nu}(t) = P\lim \sum_t E[\Delta Z_\nu(t_j^{(n)}) \mid F(t_j^{(n)})].$$

Consider now

$$\begin{aligned} &E\{|\sum_t \Delta^2 X_\nu(t_j^{(n)}) - E[\Delta Z_\nu(t_j^{(n)}) \mid F(t_j^{(n)})]|^2\} \\ &= E\{|\sum_t \Delta^2 X_\nu(t_j^{(n)}) - E[\Delta^2 X_\nu(t_j^{(n)}) \mid F(t_j^{(n)})]|^2\} \\ &= E\{|\Sigma_t \Delta^4 X_\nu(t_j^{(n)}) - E^2[\Delta^2 X_\nu(t_j^{(n)}) \mid F(t_j^{(n)})]|\} \\ &\leqq E\{\sum_t \Delta^4 X_\nu(t_j^{(n)})\} \leqq E\{\max_j \Delta^2 X_\nu(t_j^{(n)})\, \Sigma \Delta^2 X_\nu(t_j^{(n)})\} \\ &\leqq \varepsilon_n E\{|X(1) - X(0)|^2\}, \quad \text{where} \\ &\quad \varepsilon_n = \operatorname*{ess.sup.}_{\omega} \max_j \Delta^2 X_\nu(t_j^{(n)}) \to 0 \end{aligned}$$

as $n \to \infty$ because of the uniform sample equicontinuity of the X_ν process.

Thus for any $\nu \geqq 1$,

$$P\lim \sum_t \Delta^2 X_\nu(t_j^{(n)}) = P\lim \sum_t E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})] = Z_{2\nu}(t).$$

It is easily established [3], that $P\lim Z_{2\nu}(t) = Z_2(t)$ for all $t \in T$, and that

$$P\lim \sum_t \Delta^2 X_\nu(t_j^{(n)}) = \sum_t \Delta^2 X(t_j^{(n)})$$

the convergence being *uniform* in n. It follows that

$$P\lim \sum_t \Delta^2 X(t_j^{(n)}) = P\lim Z_{2\nu}(t) = Z_2(t)$$

and the theorem is completed.

Definition 1.1. A process $\{X(t), F(t); t \in T\}$ is called a quasi-martingale if there exist processes $\{X_i(t), F(t); t \in T\}$, $i = 1, 2$, such that

$$P([X(t) = X_1(t) + X_2(t) \text{ for all } t \in T]) = 1$$

where X_1 is a martingale and X_2 has a.e. sample function of bounded variation (b.v.) on T.

Corollary 1.1. *If X is a quasi-martingale with X_1 a sample continuous second order martingale and X_2 having a.e. sample function continuous, and if*

$$Z(t) = X_1^2(t) = Z_1(t) + Z_2(t)$$

is as defined previously, then

$$P\lim \sum_t \Delta^2 X(t_j^{(n)}) = Z_2(t).$$

Proof. We have

$$P\lim \sum_t \Delta^2 X(t_j^{(n)}) = P\lim \sum_t \Delta^2 X_1(t_j^{(n)}) + P\lim \sum_t \Delta^2 X_2(t_j^{(n)}) + P\lim \sum_t \Delta X_1(t_j^{(n)}) \Delta X_2(t_j^{(n)}) = Z_2(t).$$

Theorem 1.2. *Let X be a quasi-martingale satisfying the condition of Corollary 1.1 and let $Z(t) = X_1^2(t) = Z_1(t) + Z_2(t)$ be as given there. If $\{Y(t), F(t);\ t \in T\}$ is a.s. sample continuous, then*

$$R\int_0^t Y(s)\, dZ_2(s) = \begin{cases} P\lim \sum_t Y(t_j^{(n)})\, \Delta^2 X(t_j^{(n)}) & (1.1) \\ P\lim \sum_t Y(t_j^{(n)})\, E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})] & (1.2) \end{cases}$$

where $R\int$ denotes the ordinary Riemann-Stieltjes integral which exists a.s. under the stated conditions.

Proof. It is clear from Theorem 1.1 and Corollary 1.1 that if either of the probability limits exist, then so does the other and they are equal. Hence it is sufficient to show that

$$\begin{aligned} R\int_0^t Y(s)\, dZ_2(s) &= P\lim \sum_t Y(t_j^{(n)})\, E[\Delta Z(t_j^{(n)}) \mid F(t_j^{(n)})] \\ &= P\lim \sum_t Y(t_j^{(n)})\, E[\Delta Z_2(t_j^{(n)}) \mid F(t_j^{(n)})]. \end{aligned}$$

1. Assume Y and Z_2 are uniformly bounded, then

$$\begin{aligned} & E\{|\sum_t Y(t_j^{(n)})\, (\Delta Z_2(t_j^{(n)}) - E[\Delta Z_2(t_j^{(n)}) \mid F(t_j^{(n)})])|^2\} \\ &= E\{\sum_t |Y(t_j^{(n)})|^2\, |\Delta Z_2(t_j^{(n)}) - E[\Delta Z_2(t_j^{(n)}) \mid F(t_j^{(n)})]|^2\} \\ &\leqq M^2 E\{\sum_t \Delta^2 Z_2(t_j^{(n)})\} \\ &\leqq M^2 E\{\max_j |\Delta Z_2(t_j^{(n)})|\, \Sigma\, |\Delta Z_2(t_j^{(n)})|\} \\ &= M^2 E\{\max_j |\Delta Z_2(t_j^{(n)})|\, Z_2(1)\}. \end{aligned}$$

But this expression goes to zero as $n \to \infty$ because of the uniform boundedness and continuity of Z_2.

2. We now observe that if X_ν is X stopped at τ_ν as given in Lemma 1.1, then X_ν is again a quasi-martingale with $X_\nu = X_{1\nu} + X_{2\nu}$ where $X_{i\nu}$ is X_i stopped at τ_ν. Also

$$X_{1\nu}^2 = Z_\nu = Z_{1\nu} + Z_{2\nu}$$

where $Z_{i\nu}$ is Z_i stopped at τ_ν.

Thus if X_ν and Y_ν are X and Y stopped at τ_ν, what has been proved in 1, gives us the desired result for each X_ν and Y_ν. The result now follows from the approximation.

Note: One can show that (1.1) holds even if $t_j^{(n)}$ is replaced by $\xi_j^{(n)}$ with $t_j^{(n)} \leqq \xi_j^{(n)} \leqq t_{j+1}^{(n)}$ simply by using the continuity of the Y-process.

Section 2: Some Applications

We let $\{W(t), F(t);\, t \in T\}$ be a Brownian motion process. We will denote by $D \int_0^t \Phi(s, \omega)\, W(d\,s, \omega)$. The stochastic integral as defined in [5].

Lemma 2.1. (Theorem 5.3, page 449, Doob). *Let $\{X(t), F(t);\, t \in T\}$ be a second order a.s. sample continuous martingale. If there is a measurable, a.s. positive process $\{\Phi(t), F(t);\ t \in T\}$ such that for $t_1, t_2 \in T$ with $t_1 < t_2$*

$$E\{|X(t_2) - X(t_1)|^2 \,|\, F(t_1)\} = E\Big(\int_{t_1}^{t_2} |\Phi(t)|^2\, dt \,\Big|\, F(t_1)\Big)\ a.s.$$

then there is a Brownian motion process $\{W(t), F(t);\, t \in T\}$ such that

$$X(t) = X(a) + D\int_0^t \Phi(s)\, W(ds)\ \ a.s.$$

From this theorem and what we have proven in Section 1, we get the following theorem.

Theorem 2.1. *Let $\{X(t), F(t);\, t \in T\}$ be a second order a.s. sample continuous martingale. Let $X^2 = Z = Z_1 + Z_2$ be the decomposition of X^2 as given in Lemma 1.2. If for a.e. ω, $Z_2(t)$ is absolutely continuous w.r.t. Lebesgue measure, and if*

$$Z_2'(t) = \frac{d}{dt} Z_2(t)$$

is a.s. positive (it is a.s. non-negative), then there is a Brownian motion process $\{W(t), F(t);\, t \in T\}$ such that

$$X(t) = X(0) + D\int_0^t [Z_2'(s)]^{1/2}\, W(ds)\ \ a.s.$$

Since $Z_2(t) = P\lim \sum_t \Delta^2 X(t_j^{(n)})$, we can choose a sequence of partitions such that

$$Z_2(t, \omega) = \text{a.s.} \lim \sum_t \Delta^2 X(t_j^{(n)})$$

and one may ask if $Z_2(t, \omega)$ is always absolutely continuous w.r.t. Lebesgue measure. However if one takes a Brownian motion process with $\operatorname{Var}(X(t)) = C(t)$, where $C(t)$ is the Cantor function, and $X(0) = 0$ a.s., then the resulting $Z_2(t)$ is just $C(t)$. One may have some difficulty proving directly that if

$$G(t) = \lim \Sigma_t \Delta^2 g(t_j^{(n)}),$$

where $g(t)$ is continuous, then $G(t)$ need not be absolutely continuous w.r.t. Lebesgue measure. In [4] and [6], the limit $Z_2(t, \omega)$ is always a.s. sample absolutely continuous w.r.t. Lebesgue measure because the martingale processes considered are exactly those given in Lemma 2.1.

With Theorem 2.1 we obtain a theorem similar to that proved in [6].

Theorem 2.2. *Assume that $\{X(t), F(t); t \in T\}$ is a diffusion process given by the integral equation*

$$X(t) = X(0) + \int_0^t m[s, X(s)]\, ds + D \int_0^t \sigma[s, X(s)]\, W(ds)$$

where $W(t)$ is a Brownian motion process.

Then

a) $P \lim \sum_t \Delta^2 X(t_j^{(n)}) = \int_0^t \sigma^2[s, X(s)]\, ds$ *and*

b) *if $\{Y(t), F(t); t \in T\}$ is an a.s. sample continuous process*

$$P \lim \sum_t Y(t_j^{(n)})\, \Delta^2 X(t_j^{(n)}) = \int_0^t Y(s)\, \sigma^2[s, X(s)]\, ds.$$

Proof. We need only observe that X is a quasi-martingale satisfying the conditions of Corollary 1.1 with

$$X_1(t) = D \int_0^t \sigma[s, X(s)]\, W(ds),\ X_2(t) = X(0) + \int_0^t m[s, X(s)]\, ds.$$

As was shown in [6], the limit in 2. is actually in the mean if one uses the sufficient conditions on $m(.,.)$ and $\sigma(.,.)$ given in [5] to insure the existence of a solution of the diffusion equation.

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