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Dichotomies for Probability Measures Induced by $(f_1, f_2, \ldots)$-expansions

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Introduction

The classical paper by Rényi [9] on ergodic properties of transformations associated with f-expansions has given a great impetus to the investigation of various algorithms by which a sequence of integers, termed as the digits of x, is attached to any x in the unit interval. Most of these algorithms are stationary, i.e. each digit $a_n(x)$ is determined, independently of n, by making use of a system with fixed components. The recent monograph [11] by Schweiger is aimed at presenting such stationary algorithms from a unified point of view. As for the non-stationary algorithms, only two types have been considered. One of them, constructed with the help of a sequence of functions f_n, was introduced by Krabill and Reichaw [6] in order to generalize f-expansions of real numbers. The other one is defined in terms of a sequence (α_n, γ_n) of pairs of functions, and forms the core of the work [3] by Galambos. However, as already remarked in [13], this last algorithm may be described as well by a suitable sequence of piecewise linear functions in place of a sequence of pairs of functions.

The present paper deals essentially with an algorithm of the type considered in [6]. In Section 1 we rigorously define this algorithm, and give a detailed description of the set of all realizable sequences of digits. On account of this description, in Section 2 we indicate necessary and sufficient conditions under which there exists a probability λ on the Borel subsets of $[0,1)$ that makes the digits independent random variables with prescribed distributions. The central part of the paper is Section 3. Here, we present dichotomy properties for the probability measures constructed in Section 2 which proceed from the zero-one law. In Section 4, under some necessary and sufficient condition, we find a way of reducing our algorithm to that leading to f-expansions. This enables us to precise Theorem 4 of the preceding section.

Finally, it should be pointed out that, in order to simplify the exposition, we deal only with sequences of increasing functions f_n. However, the techniques developed in this paper enable one to obtain the correspondents of all results below when working with appropriate sequences of decreasing functions.

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1. $(f_1, f_2, \ldots)$-expansions of Numbers in $[0, 1)$

1.1. Assume that we are given a sequence T_n, $0 < T_n \leqq \infty$, $n \geqq 1$, and a sequence f_n, $n \geqq 1$, of continuous functions on $[0, \infty)$ such that f_n is increasing on $[0, T_n)$, f_n maps $[0, T_n)$ onto $[0, 1)$, and $f_n(t) = 1$, $t > T_n$. For $t_n \geqq 0$, $n \geqq 1$, we use the following notation throughout the paper. Whatever $k \geqq 1$, set $\underline{x}_{k,1}(t_k) = f_k(t_k)$, and define inductively

$$\underline{x}_{k,n+1}(t_k, t_{k+1}, \ldots, t_{k+n}) = f_k(t_k + \underline{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n})), \qquad n \geqq 1.$$

For any $k \geqq 0$ and $n \geqq 1$, put

$$\bar{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n}) = \underline{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n-1}, t_{k+n} + 1).$$

Then, the monotony of f_n, $n \geqq 1$, implies that

$$\begin{aligned} &\underline{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n}) \leqq \underline{x}_{k+1,n+1}(t_{k+1}, \ldots, t_{k+n}, t_{k+n+1}) \\ &\leqq \bar{x}_{k+1,n+1}(t_{k+1}, \ldots, t_{k+n}, t_{k+n+1}) \leqq \bar{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n}), \\ &k \geqq 0, \quad n \geqq 1, \end{aligned}$$

and so there exist

$$\underline{x}_{k+1}(t_{k+1}, t_{k+2}, \ldots) = \lim_{n \to \infty} \underline{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n}),$$

$$\bar{x}_{k+1}(t_{k+1}, t_{k+2}, \ldots) = \lim_{n \to \infty} \bar{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n}), \qquad k \geqq 0.$$

Moreover, on account of the continuity of f_n, $n \geqq 1$, we get

$$\begin{aligned} &\underline{x}_{k+1}(t_{k+1}, t_{k+2}, \ldots) \\ &= \underline{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n-1}, t_{k+n} + \underline{x}_{k+n+1}(t_{k+n+1}, t_{k+n+2}, \ldots)), \\ &k \geqq 0, \quad n \geqq 1, \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\bar{x}_{k+1}(t_{k+1}, t_{k+2}, \ldots) \\ &= \underline{x}_{k+1,n}(t_{k+1}, \ldots, t_{k+n-1}, t_{k+n} + \bar{x}_{k+n+1}(t_{k+n+1}, t_{k+n+2}, \ldots)), \\ &k \geqq 0, \quad n \geqq 1. \end{aligned} \tag{2}$$

We also set

$$\underline{x}_n(t_1, \ldots, t_n) = \underline{x}_{1,n}(t_1, \ldots, t_n), \quad \bar{x}_n(t_1, \ldots, t_n) = \bar{x}_{1,n}(t_1, \ldots, t_n),$$

$$n \geqq 1,$$

and

$$\underline{x}(t_1, t_2, \ldots) = \underline{x}_1(t_1, t_2, \ldots), \quad \bar{x}(t_1, t_2, \ldots) = \bar{x}_1(t_1, t_2, \ldots).$$

1.2. Using the above sequences of functions, we can associate with any $x \in [0,1)$ a sequence $a_n(x)$, $n \geqq 1$, of non-negative integers, termed as the digits of x, and a sequence $r_n(x)$, $n \geqq 0$, of numbers in $[0,1)$, termed as the remainders of x, defined recursively by way of the algorithm

$$r_0(x) = x$$

$$r_{n-1}(x) = f_n(a_n(x) + r_n(x)), \qquad n \geqq 1. \tag{3}$$

For any $n \geqq 1$, $a_n(x)$ and $r_n(x)$ depend only on $f_1, \ldots, f_n$. When we wish to emphasize this functional dependence, we write them in the form $a_n(f_1, \ldots, f_n; x)$ and $r_n(f_1, \ldots, f_n; x)$. The equations

$$r_{k+n}(f_1, \ldots, f_{k+n}; x) \\ = r_n(f_{k+1}, \ldots, f_{k+n}; r_k(f_1, \ldots, f_k; x)), \qquad k \geqq 1, \; n \geqq 1, \tag{4}$$

and

$$a_{k+n}(f_1, \ldots, f_{k+n}; x) \\ = a_n(f_{k+1}, \ldots, f_{k+n}; r_k(f_1, \ldots, f_k; x)), \qquad k \geqq 1, \; n \geqq 1, \tag{5}$$

which proceed from (3), are of prime importance in proving Theorem 4 below.

Now, whatever $n \geqq 1$, denote I_n as the set of the non-negative integers which are strictly less than T_n. Obviously, $a_n \in I_n$, $n \geqq 1$, and therefore we can define the mapping a from $[0,1)$ into $\prod_{n \geqq 1} I_n$ by

$$a(x) = (a_1(x), a_2(x), \ldots), \qquad x \in [0,1).$$

If necessary, we specify the dependence of $a(x)$ on the underlying sequence $f_1, f_2, \ldots$ by writing it as $a(f_1, f_2, \ldots; x)$. If a is injective, then any $x \in [0,1)$ may be recovered from the sequence of its digits. More exactly, in this case we have the representation (see Corollary 3 below)

$$x = \lim_{n \to \infty} \underline{x}_n(a_1(x), \ldots, a_n(x)) = \lim_{n \to \infty} \bar{x}_n(a_1(x), \ldots, a_n(x)), \qquad x \in [0,1), \tag{6}$$

which is called the $(f_1, f_2, \ldots)$-*expansion* of x. (For $f_n = f$, $n \geqq 1$, we get the classical f-expansion of x.) A sufficient condition ensuring the injectivity of a is as follows. There exist γ_n, $n \geqq 1$, such that $\prod_{n \geqq 1} \gamma_n = 0$, and

$$f_n(t') - f_n(t) \leqq \gamma_n(t' - t), \qquad 0 \leqq t < t'. \tag{7}$$

This can be shown, by using Proposition 1 below, through repeated application of (1), (2) and (7). However, nothing as strong as this condition is needed to guarantee the injectivity of a.

Finally, let us indicate that, whatever $(i_1, i_2, \ldots) \in \prod_{n \geqq 1} I_n$, we put

$$\Delta_n(i_1, \ldots, i_n) = \{x : a_1(x) = i_1, \ldots, a_n(x) = i_n\}, \qquad n \geqq 1,$$

and

$$\Delta(i_1, i_2, \ldots) = \{x\colon a_1(x) = i_1, a_2(x) = i_2, \ldots\}.$$

1.3. In what follows, we describe the set of all realizable sequences of digits with respect to (w.r.t.) the algorithm (3), i.e. the image $a([0,1))$ of $[0,1)$ under a. This description is essentially contained in the conditions of Theorem 2 below. We begin with the following result.

Proposition 1. *For* $(i_1, i_2, \ldots) \in \prod_{n \geqq 1} I_n$, *we have*

$$\Delta(i_1, i_2, \ldots) = \begin{cases} [\underline{x}(i_1, i_2, \ldots), \bar{x}(i_1, i_2, \ldots)] & \text{if } (i_1, i_2, \ldots) \in W \\ [\underline{x}(i_1, i_2, \ldots), \bar{x}(i_1, i_2, \ldots)) & \text{if } (i_1, i_2, \ldots) \notin W, \end{cases} \tag{8}$$

where

$$W = \{(i_1, i_2, \ldots) \in \prod_{n \geqq 1} I_n \colon \bar{x}(i_1, i_2, \ldots) < \bar{x}_n(i_1, \ldots, i_n), n \geqq 1\}.^1$$

Proof. If $a_n(x) = i_n$, $n \geqq 1$, then iteration of (3) yields that $x = \underline{x}_n(i_1, \ldots, i_{n-1}, i_n + r_n(x))$, $n \geqq 1$, and thus, by the monotony of f_n and the inequalities $0 \leqq r_n(x) < 1$, $n \geqq 1$, we get

$$\underline{x}_n(i_1, \ldots, i_n) \leqq x < \bar{x}_n(i_1, \ldots, i_n), \quad n \geqq 1. \tag{9}$$

Therefore, we always have

$$\underline{x}(i_1, i_2, \ldots) \leqq x \leqq \bar{x}(i_1, i_2, \ldots). \tag{10}$$

Now, if $(i_1, i_2, \ldots) \notin W$, then there exists $k \geqq 1$ such that $\bar{x}(i_1, i_2, \ldots) = \bar{x}_k(i_1, \ldots, i_k)$ and so, in view of (9), the last inequality of (10) is strengthened.

Conversely, if x belongs to the right-hand side of (8), then (9) holds on account of the definition of W. In particular, for $n = 1$, we get $f_1(i_1) \leqq x < f_1(i_1 + 1)$, and thus $a_1(x) = i_1$. Now, if $a_n(x) = i_n$, $1 \leqq n \leqq m$, then $x = \underline{x}_m(i_1, \ldots, i_{m-1}, i_m + r_m(x))$ and, by virtue of (9), it follows that $f_{m+1}(i_{m+1}) \leqq r_m(x) < f_{m+1}(i_{m+1} + 1)$. Consequently, $a_{m+1}(x) = i_{m+1}$, and thus we have proved inductively that $a_n(x) = i_n$, $n \geqq 1$, q.e.d.

The following corollaries are easy to verify.

Corollary 1. *We have*

$$a([0,1)) = W \cup \{(i_1, i_2, \ldots) \in W' \colon \underline{x}(i_1, i_2, \ldots) < \bar{x}(i_1, i_2, \ldots)\},$$

where $W' = (\prod_{n \geqq 1} I_n) - W$.

Corollary 2. *If a is injective, then* $a([0,1)) = W$.

Corollary 3. *a is injective if and only if* (6) *holds.*

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[1] Sometimes, we will write $W_{f_1, f_2, \ldots}$ to emphasize that W depends on $f_1, f_2, \ldots$.

Now, for each $k \geqq 1$, we introduce a sequence $j_{k,n}$, $n \geqq 1$, defined recursively by $j_{k,n} = \sup\{i \in I_{k+n-1}: \underline{x}_{k,n}(j_{k,1}, \ldots, j_{k,n-1}, i) < 1\}$ and $j_{k,n+1} = \infty$ in case $j_{k,n} = \infty$. Further, for any $k \geqq 1$, let

$$A_k = \{(i_k, i_{k+1}, \ldots) \in \prod_{n \geqq k} I_n: i_{k+n-1} = j_{k,n},\ n \geqq 1,\ or\ i_{k+n-1} = j_{k,n},$$
$$1 \leqq n \leqq m,\ and\ i_{k+m} > j_{k,m+1}\ for\ some\ m \geqq 1\}.$$

In order to characterize the set W', we need the next lemma.

Lemma 1. *Let* $(i_k, i_{k+1}, \ldots) \in \prod_{n \geqq k} I_n$. *Then* $\bar{x}_k(i_k, i_{k+1}, \ldots) = 1$ *if and only if* $(i_k, i_{k+1}, \ldots) \in A_k$.

Proof. If $i_{k+n-1} = j_{k,n}$, $n \geqq 1$, then $\bar{x}_{k,n}(i_k, \ldots, i_{k+n-1}) = 1$, $n \geqq 1$, and hence $\bar{x}_k(i_k, i_{k+1}, \ldots) = 1$. Whereas, in case $i_{k+n-1} = j_{k,n}$, $1 \leqq n \leqq m$, and $i_{k+m} > j_{k,m+1}$ for some $m \geqq 1$, we can write, using (2),

$$\begin{aligned}\bar{x}_k(i_k, i_{k+1}, \ldots) &= \underline{x}_{k,m+1}(j_{k,1}, \ldots, j_{k,m}, i_{k+m} + \bar{x}_{k+m+1}(i_{k+m+1}, i_{k+m+2}, \ldots)) \\ &\geqq \underline{x}_{k,m+1}(j_{k,1} \ldots, j_{k,m}, i_{k+m}) \\ &\geqq \bar{x}_{k,m+1}(j_{k,1}, \ldots, j_{k,m}, j_{k,m+1}) = 1,\end{aligned}$$

and so $\bar{x}_k(i_1, i_2, \ldots) = 1$.

Conversely, assume that there exists $r \geqq 1$ such that $i_{k+n-1} = j_{k,n}$, $1 \leqq n < r$, and $i_{k+r-1} < j_{k,r}$. Then, by (2), we have

$$\begin{aligned}\bar{x}_k(i_k, i_{k+1}, \ldots) &= \underline{x}_{k,r}(j_{k,1}, \ldots, j_{k,r-1}, i_{k+r-1} + \bar{x}_{k+r}(i_{k+r}, i_{k+r+1}, \ldots)) \\ &\leqq \underline{x}_{k,r}(j_{k,1}, \ldots, j_{k,r-1}, i_{k+r-1} + 1) < 1,\end{aligned}$$

and the proof is complete, q.e.d.

Theorem 1. *We have*

$$W' = \bigcup_{k \geqq 1} \{(i_1, i_2, \ldots) \in \prod_{n \geqq 1} I_n: (i_k, i_{k+1}, \ldots) \in A_k\}. \tag{11}$$

Proof. Let $(i_1, i_2, \ldots) \in W'$. If $\bar{x}(i_1, i_2, \ldots) = 1$, then $(i_1, i_2, \ldots)$ belongs to the right-hand side of (11) by Lemma 1. Whereas, in case $\bar{x}(i_1, i_2, \ldots) < 1$, choose $k = \min\{n: \bar{x}(i_1, i_2, \ldots) = \bar{x}_n(i_1, \ldots, i_n)\}$. Then, by making use of (2), we see that

$$i_n + \bar{x}_{n+1}(i_{n+1}, i_{n+2}, \ldots) < T_n, \qquad 1 \leqq n \leqq k.$$

Consequently, by the strict monotony of f_n on $[0, T_n)$, $1 \leqq n \leqq k$, and the equation

$$\underline{x}_k(i_1, \ldots, i_{k-1}, i_k + \bar{x}_{k+1}(i_{k+1}, i_{k+2}, \ldots)) = \underline{x}_k(i_1, \ldots, i_{k-1}, i_k + 1),$$

it follows that $\bar{x}_{k+1}(i_{k+1}, i_{k+2}, \ldots) = 1$. Therefore, $(i_1, i_2, \ldots)$ belongs to the right-hand side of (11) on account of Lemma 1.

Conversely, let $(i_1, i_2, \ldots)$ belong to the right-hand side of (11). Then, by

Lemma 1, there exists $k \geqq 1$ such that $\bar{x}_k(i_k, i_{k+1}, \ldots) = 1$. If $k=1$, then $\bar{x}(i_1, i_2, \ldots) = \bar{x}_n(i_1, \ldots, i_n) = 1$, $n \geqq 1$, while in case $k > 1$, $\bar{x}(i_1, i_2, \ldots) = \bar{x}_{k-1}(i_1, \ldots, i_{k-1})$ by (2). Hence $(i_1, i_2, \ldots) \in W'$, q.e.d.

1.4. **Notes.** When a is injective, Corollary 2 and Theorem 1 give a complete description of the set of realizable sequences of digits.

As for certain general algorithms, several authors acknowledge that the problem of characterizing the realizable sequences of digits is important but still open. In connection with this, see Problem 2 in [3], p. 128, and [11], p. 3. Besides our present contribution, this problem was solved in a few special cases only. Namely, Parry [8] specified the set of realizable sequences of digits in β-expansions, and Spătaru [14] characterized the same set in the more general case of f-expansions. A detailed description of this set with respect to (φ, f)-expansions is also given in [13].

2. Stochastic Independence of the Digits

2.1. Throughout the rests of this paper, we work with a fixed sequence of functions $f_n, n \geqq 1$, such that the mapping a is injective. Then, Corollary 3 ensures that the σ-algebra generated by the digits $a_n, n \geqq 1$, coincides with the σ-algebra $\mathscr{B}$ of Borel subsets of $[0,1)$. Now, for each $n \geqq 1$, consider a probability distribution $\mathbf{p}^n = (p_i^n)_{i \in I_n}$ on I_n. We next indicate necessary and sufficient conditions under which there exists a probability λ on $\mathscr{B}$ making the digits independent random variables such that, for any $n \geqq 1$, the distribution of a_n under λ is $\mathbf{p}^n$. These conditions are based on the description of W' given in Theorem 1.

Theorem 2. *If*

$$\prod_{n \geqq 1} p_{j_{k,n}}^{k+n-1} = 0, \quad k \geqq 1, \quad \textit{and} \quad \left(\prod_{n=1}^{m} p_{j_{k,n}}^{k+n-1} \right) \sum_{i > j_{k,m+1}} p_i^{k+m} = 0, \quad k \geqq 1, \ m \geqq 1, \tag{12}$$

then there exists a probability λ on $\mathscr{B}$ making the a_n independent random variables with

$$\lambda(a_n = i) = p_i^n, \qquad i \in I_n, \ n \geqq 1. \tag{13}$$

Conversely, if λ is a probability on $\mathscr{B}$ under which the digits a_n are independent and distributed according to (13) *for some $\mathbf{p}^n$, $n \geqq 1$, then* (12) *holds.*

Proof. We present a construction of λ under (12) which is adequate for our further purposes. Let $(\prod_{n \geqq 1} I_n, \mathscr{K}, \mathbf{P})$ be the product probability space formed from the probability spaces $(I_n, \mathscr{P}(I_n), \mathbf{p}^n)$, $n \geqq 1$. Then, on account of (11) and (12), it follows that $\mathbf{P}(W) = 1$. Denote $\mathscr{K}_W$ the σ-algebra of subsets of W that belong to $\mathscr{K}$, and let $\mathbf{P}_W$ be the restriction of $\mathbf{P}$ to $\mathscr{K}_W$. Since, by Corollary 2, a maps $[0,1)$ onto W, there exists a unique probability λ on $a^{-1}(\mathscr{K}_W)$ such that $\lambda a^{-1} = \mathbf{P}_W$ on $\mathscr{K}_W$. Further, as

$$a_n = \pi_n \circ a, \quad n \geqq 1, \tag{14}$$

where, for each $n \in 1$, π_n stands for the projection of $\prod_{n \geqq 1} I_n$ onto I_n, we see that $a^{-1}(\mathscr{K}_W) = \mathscr{B}$, and

$$(\lambda a^{-1})(A) = \mathbf{P}(A), \quad A \in \mathscr{K}. \tag{15}$$

Now, the independence of a_n under λ follows, on account of (14) and (15), from the independence of π_n under $\mathbf{P}$. Finally, (13) holds by (14) and (15).

Conversely, if λ is as stated in the second part of the theorem, then the left-hand sides of (12) always represent the probability of appropriate empty sets, and the theorem is proved, q.e.d.

2.2. **Notes.** The cumulative distribution function $F(x) = \lambda([0, x))$ is given by

$$F(x) = \sum_{m \geqq 1} \left(\prod_{n=1}^{m-1} p^n_{a_n(x)} \right) s^m_{a_m(x)}, \quad x \in (0, 1), \tag{16}$$

where $s^n_i = 1 - \sum_{l \geqq i} p^n_l$, $i \in I_n$, $n \geqq 1$. Indeed, by making use of Corollary 3 and the independence of a_n under λ, we have (with $\underline{x}_0 = 0$)

$$\begin{aligned}
F(x) &= \lim_{r \to \infty} \lambda([0, \underline{x}_r(a_1(x), \ldots, a_r(x))))\\
&= \lim_{r \to \infty} \sum_{m=1}^{r} \lambda([\underline{x}_{m-1}(a_1(x), \ldots, a_{m-1}(x)),\ \underline{x}_m(a_1(x), \ldots, a_{m-1}(x), a_m(x))))\\
&= \sum_{m \geqq 1} \lambda(a_n = a_n(x),\ 1 \leqq n \leqq m-1,\ a_m < a_m(x))\\
&= \sum_{m \geqq 1} \left(\prod_{n=1}^{m-1} p^n_{a_n(x)} \right) s^m_{a_m(x)}, \quad x \in (0, 1).
\end{aligned}$$

Interestingly enough, if λ is purely non-atomic, and $\lambda \perp m$, where m denotes the Lebesgue measure on $\mathscr{B}$, then (16) may provide examples of continuous, increasing, singular functions. (For $f_n = f$, $n \geqq 1$, it is shown in [14] that F is increasing if and only if all p^n_i are positive.) For instance, in case $T_n = 2$, $n \geqq 1$, and $f_n(t) = t/2$, $0 \leqq t \leqq 2$, $n \geqq 1$, Salem [10] derived the F in (16), by geometrical arguments, under the assumption $p^n_i > 0$, $i = 0, 1$, $n \geqq 1$, and $\sum_{m \geqq 1} \left(\prod_{n=1}^{m} \max_{i=0,1} p^n_i \right) < \infty$. Among Salem's functions one finds the first directly constructed, continuous, strictly monotonic, singular functions.

3. Properties of the Probability λ

3.1. In this section we present some properties of a given, but otherwise arbitrary, probability λ on $\mathscr{B}$ which makes the a_n independent and distributed according to (13). Except for the "only if" part of Theorem 5 below, these properties are consequences of the zero-one law for independent random variables. First, we consider the following easy result.

Theorem 3. *λ is either purely atomic or purely non-atomic; λ is purely non-atomic if and only if* $\prod_{n\geqq 1} \max_{i\in I_n} p_i^n = 0$.

Proof. If λ is not purely non-atomic, then there exists $x\in[0,1)$ such that $\lambda(\{x\})>0$. Therefore, by the zero-one law, $\lambda(\liminf(a_n=a_n(x)))=1$. As $\liminf(a_n=a_n(x))$ is a countable set of points in $[0,1)$, we see that λ is purely atomic. The second assertion of the theorem follows from the equation

$$\lambda(\{x\})=\prod_{n\geqq 1} p^n_{a_n(x)}, \qquad x\in[0,1),$$

thus completing the proof, q.e.d.

Now, we investigate the behaviour of λ w.r.t. m. For any $k\geqq 1$ and $i_n\in I_n$, $1\leqq n\leqq k$, define the measurable transformation $T_{i_1,\ldots,i_k}$ from $\prod_{n>k} I_n$ into $\prod_{n\geqq 1} I_n$ by

$$T_{i_1,\ldots,i_k}(i_{k+1},i_{k+2},\ldots)=(i_1,\ldots,i_k,i_{k+1},i_{k+2},\ldots), \qquad (i_{k+1},i_{k+2},\ldots)\in\prod_{n>k} I_n.$$

Whatever $B\subset\prod_{n\geqq 1} I_n$, set

$$B_k=(\pi_{k+1},\pi_{k+2},\ldots)^{-1}\Big(\bigcup_{i_1\in I_1,\ldots,i_k\in I_k} T^{-1}_{i_1,\ldots,i_k}(B)\Big), \qquad k\geqq 1.$$

The next three lemmas are needed for the proof of Theorem 4 below.

Lemma 2. *If $B\subset W$ and, for some $k\geqq 1$, $a(f_{k+1},f_{k+2},\ldots;\cdot)$ is injective, then*

$$a^{-1}(B_k)=r_k^{-1}(r_k(a^{-1}(B))). \tag{17}$$

Proof. We first verify that

$$\begin{aligned} &a^{-1}\Big(f_{k+1},f_{k+2},\ldots;\bigcup_{i_1\in I_1,\ldots,i_k\in I_k} T^{-1}_{i_1,\ldots,i_k}(B)\Big)\\ &\quad=r_k(f_1,\ldots,f_k;a^{-1}(f_1,f_2,\ldots;B)). \end{aligned} \tag{18}$$

Let x belong to the left-hand side of (18). Then there exist $i_n\in I_n$, $1\leqq n\leqq k$, and $x'\in a^{-1}(f_1,f_2,\ldots;B)$ such that

$$(i_1,\ldots,i_k,a_1(f_{k+1};x),a_2(f_{k+1},f_{k+2};x),\ldots)=a(f_1,f_2,\ldots;x'). \tag{19}$$

By making use of (5) and (19), the injectivity of $a(f_{k+1},f_{k+2},\ldots;\cdot)$ implies that $x=r_k(f_1,\ldots,f_k;x')$, and so $x\in r_k(f_1,\ldots,f_k;a^{-1}(f_1,f_2,\ldots;B))$. The converse inclusion in (18) can be proved in a similar manner. Now, on account of (5), we get

$$\begin{aligned} &a^{-1}(f_1,f_2,\ldots;B_k)\\ &\quad=r_k^{-1}\Big(f_1,\ldots,f_k;a^{-1}\Big(f_{k+1},f_{k+2},\ldots;\bigcup_{i_1\in I_1,\ldots,i_k\in I_k} T^{-1}_{i_1,\ldots,i_k}(B)\Big)\Big), \end{aligned}$$

and (18) terminates the proof, q.e.d.

Lemma 3. *There exists an increasing sequence of natural numbers $k_n, n \geqq 1$, such that $a(f_{k_n+1}, f_{k_n+2}, \ldots; \cdot)$ is injective for each $n \geqq 1$.*

Proof. We choose the sequence $k_n, n \geqq 1$, as follows. Set $k_1 = \min\{k: \bar{x}_k(0, \ldots, 0) < 1\}$, and put $k_{n+1} = \min\{k: \bar{x}_k(0, \ldots, 0) < \bar{x}_{k_n}(0, \ldots, 0)\}$, $n \geqq 1$. Then, whatever $n \geqq 1$, we have

$$\bar{x}_{r, k_n - r + 1}(0, \ldots, 0) < f_r(1), \quad 1 \leqq r < k_n, \quad f_{k_n}(1) < 1. \tag{20}$$

By making use of (20), and by repeated application of (4), we obtain

$$\begin{aligned} &r_{k_n}(f_1, \ldots, f_{k_n}; \Delta_{k_n}(0, \ldots, 0)) \\ &\quad = r_{k_n}(f_1, \ldots, f_{k_n}; [0, \bar{x}_{k_n}(0, \ldots, 0))) = [0, 1), \quad n \geqq 1. \end{aligned} \tag{21}$$

Now, assume that

$$a(f_{k_n+1}, f_{k_n+2}, \ldots; y_1) = a(f_{k_n+1}, f_{k_n+2}, \ldots; y_2). \tag{22}$$

In view of (21), $y_i = r_{k_n}(f_1, \ldots, f_{k_n}; x_i)$, where $x_i \in \Delta_{k_n}(0, \ldots, 0)$, $i = 1, 2$. Substituting y_i in (22), and using (5), we see that

$$(a_{k_n+1}(x_1), a_{k_n+2}(x_1), \ldots) = (a_{k_n+1}(x_2), a_{k_n+2}(x_2), \ldots),$$

and thus $a(x_1) = a(x_2)$. Since a is injective, we get $x_1 = x_2$, and so $y_1 = y_2$, q.e.d.

Lemma 4. *Assume that, for each $n \geqq 1$, $f_n(\{t < T_n: f_n'(t) = \infty\})$ and $\{t < T_n: f_n'(t) = 0\}$ are sets of Lebesgue measure zero. Then*

$$m(r_k^{-1}(A)) = 0, \quad m(r_k(A)) = 0, \quad k \geqq 1,$$

whenever $A \in \mathscr{B}$ is such that $m(A) = 0$.

Proof. As f_n is increasing on $[0, T_n)$, it follows by a standard argument (use, e.g., Exercise 17.25 in [4], p. 269) that $m(f_n(\{t < T_n: f_n'(t) = \infty\})) = 0$ if and only if $m(f_n(C)) = 0$ for any Borel set C of Lebesgue measure zero, and $\{t < T_n: f_n'(t) = 0\}$ is of Lebesgue measure zero if and only if $f_n^{-1}(A)$ has Lebesgue measure zero for any $A \in \mathscr{B}$ with $m(A) = 0$. Further, the lemma is established by induction, using (4) and the following equations, valid for any $A \in \mathscr{B}$,

$$r_1^{-1}(f_n; A) = \bigcup_{i \in I_n} f_n((\{i\} + A) \cap [0, T_n)), \quad n \geqq 1,$$

and

$$r_1(f_n; A) = \bigcup_{i \in I_n} (f_n^{-1}(A \cap [f_n(i), f_n(i+1))) - \{i\}). \quad n \geqq 1, \quad \text{q.e.d.}$$

Theorem 4. *Assume that, whatever $n \geqq 1$, $f_n(\{t < T_n: f_n'(t) = \infty\})$ and $\{t < T_n: f_n'(t) = 0\}$ have Lebesgue measure zero. Then either $\lambda \perp m$ or $\lambda \ll m$.*

Proof. By (15) and the equality $a^{-1}(\mathscr{K}) = \mathscr{B}$, it suffices to show that $\mathbf{P}$ is either singular or absolutely continuous w.r.t. the probability $m a^{-1}$ on $\mathscr{K}$. If it is not true that $\mathbf{P} \ll m a^{-1}$, then there exists $B \in \mathscr{K}_W$ such that $(m a^{-1})(B) = 0$ and $\mathbf{P}(B) > 0$. Since, for any $k \geqq 1$, B_k belongs to the σ-algebra generated by π_n, $n > k$, and $B \subset B_k \subset B_{k+1}$, we see that $B \cup (\bigcup_{n \geqq 1} B_{k_n})$ is a set of the tail σ-algebra of the process π_n, $n \geqq 1$, where $k_n, n \geqq 1$, is the sequence which have been chosen in Lemma 3. Hence, $\mathbf{P}(B \cup (\bigcup_{n \geqq 1} B_{k_n})) = 1$. On the other hand, by virtue of Lemmas 2 and 4, we may write

$$(m a^{-1})(B \cup (\bigcup_{n \geqq 1} B_{k_n})) \leqq (m a^{-1})(B) + \sum_{n \geqq 1} m(a^{-1}(B_{k_n}))$$
$$= \sum_{n \geqq 1} m(r_{k_n}^{-1}(r_{k_n}(a^{-1}(B)))) = 0.$$

Therefore, it follows that $\mathbf{P} \perp m a^{-1}$, q.e.d.

We now give a simple necessary and sufficient condition for any probability μ, which makes the a_n independent random variables, to be such that either $\mu \perp \lambda$ or $\mu \ll \lambda$. To do this, we use the following lemma.

Lemma 5. *Whatever $k \geqq 1$, we have $j_{k,n} > 0$ for infinitely many n.*

Proof. Assume, on the contrary, that there exist $k \geqq 1$ and $r \geqq 1$ such that $j_{k,n} = 0$, $n > r$. Then we have $j_{k,n} < \infty$, $1 \leqq n \leqq r$, and

$$\bar{x}_k(j_{k,1}, \ldots, j_{k,r}, 0, 0, \ldots) = 1.$$

Therefore, by (2) and the inequality $x_{k,r}(j_{k,1}, \ldots, j_{k,r}) < 1$, it follows that $\bar{x}_{k+r}(0, 0, \ldots) > 0$. In view of (2), this leads to $\bar{x}(0, 0, \ldots) > 0$, thus contradicting the injectivity of a, q.e.d.

Theorem 5. *The cumulative distribution function $F(x) = \lambda([0, x))$, $x \in (0, 1)$, is increasing if and only if either $\mu \perp \lambda$ or $\mu \ll \lambda$ for any probability μ on $\mathscr{B}$ which makes the a_n independent.*

Proof. Assume that F is increasing, and for each $n \geqq 1$, denote $\mathbf{q}^n = (q_i^n)_{i \in I_n}$ the distribution of a_n under μ. Then $\mathbf{q}^n \ll \mathbf{p}^n$ for all $n \geqq 1$. Indeed, whenever $q_i^n > 0$, choose $i_1 \in I_1, \ldots, i_{n-1} \in I_{n-1}$ such that

$$0 < q_{i_1}^1 \ldots q_{i_{n-1}}^{n-1} q_i^n = \mu(\Delta_n(i_1, \ldots, i_{n-1}, i)).$$

Hence $\Delta_n(i_1, \ldots, i_{n-1}, i) \neq \emptyset$, and the strict monotony of F implies that

$$p_{i_1}^1 \ldots p_{i_{n-1}}^{n-1} p_i^n = \lambda(\Delta_n(i_1, \ldots, i_{n-1}, i)) > 0.$$

Consequently, $p_i^n > 0$. Now, let $\mathbf{Q}$ be the product probability on $\mathscr{K}$ formed from $\mathbf{q}^n$, $n \geqq 1$. (Hence, $(\mu a^{-1})(A) = \mathbf{Q}(A)$, $A \in \mathscr{K}$.) Then, by a theorem of Kakutani (see, e.g., [4], p. 453), it follows that either $\mathbf{Q} \perp \mathbf{P}$ or $\mathbf{Q} \ll \mathbf{P}$. Therefore, either $\mu \perp \lambda$ or $\mu \ll \lambda$. (By the same theorem, the former case holds if and only if

$$\prod_{n \geqq 1} \left(\sum_{i \in I_n} (p_i^n q_i^n)^{1/2} \right) = 0.)$$

Conversely, assume that F is not increasing on $(0, 1)$. Then we shall construct a probability μ on $\mathscr{B}$ which makes the a_n independent, and neither $\mu \perp \lambda$ nor $\mu \ll \lambda$. As F is not increasing, the set $\{(n, i): n \geqq 1, i \in I_n, p_i^n = 0\}$ is not empty. Two cases occur: either (i) there exist $m \geqq 1$ and $l \in I_m$ such that $p_l^m = 0$ and $\sum_{i > l} p_i^m > 0$, or else (ii) $p_0^n > 0$ for all $n \geqq 1$. If (i) holds, then choose $i_n \in I_n$, $1 \leqq n \leqq m$, such that $p_{i_n}^n > 0$, $1 \leqq n \leqq m$, and $i_m > l$. Further, for each $n \geqq 1$, define a probability distribution $\mathbf{q}^n$ on I_n as follows. Set $q_{i_n}^n = 1$, $1 \leqq n < m$, $q_{i_m}^m = q_l^m = 1/2$, and put $\mathbf{q}^n = \mathbf{p}^n$ for $n > m$. Since the sequence $\mathbf{p}^n$, $n \geqq 1$, satisfies (12), notice that, whenever $r = 1, \ldots, m$, we cannot have $j_{r,n} = i_{r+n-1}$, $1 \leqq n \leqq m-r$, and $j_{r,m-r+1} \leqq l$. Consequently, (12) is also satisfied by $\mathbf{q}_n$, $n \geqq 1$. On the other hand, if (ii) holds, then choose $m \geqq 1$, $i_n \in I_n$, $1 \leqq n \leqq m$, and $l \in I_m$ such that $p_{i_n}^n > 0$, $1 \leqq n \leqq m$, $p_l^m = 0$ and $\Delta_m(i_1, \ldots, i_{m-1}, l) \neq \emptyset$. By virtue of Lemma 5, we can take $s_n = \min\{r \geqq 1 : j_{m-n+1, n+r} > 0\}$, $1 \leqq n \leqq m$. Let $s = \max_{1 \leqq n \leqq m} s_n$. For $n \geqq 1$, define a probability distribution $\mathbf{q}^n$ on I_n. Namely, let $q_{i_n}^n = 1$, $1 \leqq n < m$, $q_{i_m}^m = q_l^m = 1/2$, $q_0^n = 1$, $m < n \leqq m+s$, and set $\mathbf{q}^n = \mathbf{p}^n$ for $n > m+s$. As $\Delta_m(i_1, \ldots, i_{m-1}, l) \neq \emptyset$, we see that, for any $r = 1, \ldots, m$, we cannot have $j_{r,n} = i_{r+n-1}$, $1 \leqq n \leqq m-r$, and $j_{r,m-r+1} < l$. But, if $j_{r,n} = i_{r+n-1}$, $1 \leqq n \leqq m-r$, and $j_{r,m-r+1} = l$ for some $r \in \{1, \ldots, m\}$, then there is k, $m-r+1 < k \leqq m+s-r+1$, such that $0 < j_{r,k}$. Therefore, since $\mathbf{p}^n, n \geqq 1$, satisfies (12), we make sure the sequence $\mathbf{q}^n$, $n \geqq 1$, does so too. In both cases, denote $\mathbf{Q}$ as the product probability on $\mathscr{K}$ formed from $\mathbf{q}^n$, $n \geqq 1$. Then, according to Theorem 3, the probability μ on $\mathscr{B}$, which is associated with $\mathbf{Q}$ by $(\mu a^{-1})(A) = \mathbf{Q}(A)$, $A \in \mathscr{K}$, makes the a_n independent random variables. Now, notice that on $\prod_{n=1}^{m} I_n$ the product probability formed from $\mathbf{q}^n$, $1 \leqq n \leqq m$, is neither singular nor absolutely continuous w.r.t. the product probability formed from $\mathbf{p}^n$, $1 \leqq n \leqq m$. Hence, since on $\prod_{n>m} I_n$ the product probability formed from $\mathbf{q}^n$, $n > m$, is absolutely continuous w.r.t. the product probability formed from $\mathbf{p}^n$, $n > m$, it follows that neither $\mathbf{Q} \perp \mathbf{P}$ nor $\mathbf{Q} \ll \mathbf{P}$. Consequently, neither $\mu \perp \lambda$ nor $\mu \ll \lambda$, q.e.d.

3.2. **Notes.** Theorem 3 extends similar results due to Chatterji [1] and [2], and to Spătaru [14]. However, these results are not specific within the framework of number expansion theory, where they appeared, but originate from the general fact that the restriction of a probability to the σ-algebra generated by a countable family of discrete random variables is either purely atomic or purely non-atomic whenever the family variables are independent under this probability.

From Lemma 3 it follows actually that $a(f_{n+1}, f_{n+2}, \ldots; \cdot)$ is injective for all $n \geqq 1$. Indeed, if $a(f_{k+1}, f_{k+2}, \ldots; \cdot)$ is injective, then $a(f_k, f_{k+1}, f_{k+2}, \ldots; \cdot)$ is injective by (5), and by the injectivity of $r_1(f_k; \cdot)$ on each of the sets $(a_1(f_k; \cdot) = i)$, $i \in I_k$.

Theorem 4 is illuminating for revealing the general condition which guarantees the validity of certain very special results. Namely, in the D-adic expansion case, i.e. when $T_n = D > 1$, $n \geqq 1$, where D is integral, and $f_n(t) = t/D$, $0 \leqq t < D$, $n \geqq 1$, Chatterji [1] proved that either $\lambda \perp m$ or $\lambda \ll$m. (Without being aware of Chatterji's work, Marsaglia [7] obtained the same result, but by using a different approach.) Chatterji [2] also showed that for the continued fraction expansion case, i.e. for $f_n(t) = 1/t$, $t \geqq 1$, $n \geqq 1$, any probability making the digits independent random variables is singular w.r.t. the Lebesgue measure. Along the same line, in the Lüroth expansion case, i.e. when, for each $n \geqq 1$, the graph of f_n is the polygon joining in order the points $(i, 1/i)$, $i \geqq 1$, Jakubec [5] proved that any probability making the digits independent is either singular or absolutely continuous w.r.t. the Lebesgue measure. It should be noted that for these special cases, the condition of Theorem 4, or the corresponding one for decreasing[2] f_n, $n \geqq 1$, is trivially verified since the sets of Lebesgue measure thereof are in fact empty sets. Theorem 4 provides us with a large class of $(f_1, f_2, \ldots)$-expansions producing stochastically independent digits only under "pure" probability measures. This raises the question as to whether there exist $(f_1, f_2, \ldots)$-expansions whose digits may be independent under a probability which is neither singular nor absolutely continuous w.r.t. the Lebesgue measure. Theorem 2 in [12] answers this question affirmatively.

By the first argument in the proof of Lemma 4, and by Theorem 18.25 in [4], p. 288, it follows that the condition of Theorem 4 is equivalent to the assumption that, whatever $n \geqq 1$, f_n is absolutely continuous, and f_n^{-1} is absolutely continuous on each $[0, c]$, $c < 1$. In the next section we shall see that this condition is only sufficient for the conclusion of Theorem 4 to hold.

4. Equivalent $(f_1, f_2, \ldots)$-expansions

4.1. We say that $f_n, n \geqq 1$, is equivalent to f'_n, $n \geqq 1$, and write

$$(f_1, f_2, \ldots) \sim (f'_1, f'_2, \ldots), \quad \text{if } a(f_1, f_2, \ldots; x) = a(f'_1, f'_2, \ldots; x)$$

for all $x \in [0, 1)$. Note that any probability λ on $\mathscr{B}$ which makes the a_n independent and distributed according to (13) does so as well w.r.t. the digits a'_n. In what follows, we find the necessary and sufficient condition under which function f exists such that $(f_1, f_2, \ldots) \sim (f, f, \ldots)$, i.e. the $(f_1, f_2, \ldots)$-expansion of x is equivalent to the f-expansion of x for any $x \in [0, 1)$. The way we construct f enables us to give an example showing that the condition in Theorem 4 is not necessary for the conclusion to hold.

Theorem 6. *$(f_1, f_2, \ldots) \sim (f, f, \ldots)$ for some function f if and only if $j_{k,n} = j_{k+1,n}$, $n \geqq 1$, for each $k \geqq 1$.*

Proof. Suppose that there exists a function f on $[0, \infty)$ such that f is increasing on $[0, T)$, where $0 < T \leqq \infty$, f maps $[0, T)$ onto $[0, 1)$, $f(t) = 1$ for $t > T$, and $(f_1, f_2, \ldots) \sim (f, f, \ldots)$. For the constant sequence $f, f, \ldots$, let $j_n = j_{1,n} = j_{2,n} = \ldots$, $n \geqq 1$. We shall prove that $j_{k,n} = j_n$, $n \geqq 1$, for each $k \geqq 1$. First, since $a(f_1, f_2, \ldots; \cdot)$ is assumed injective, Corollary 2 implies that

$$W_{f_1, f_2, \ldots} = W_{f, f, \ldots}. \tag{23}$$

Now, if $T = \infty$, then $j_n = \infty$, $n \geqq 1$, and so $W_{f, f, \ldots}$ coincides with the set of all sequences of non-negative integers. Consequently, $T_k = \infty$, $k \geqq 1$, and hence $j_{k,n} = \infty$, $n \geqq 1$, for any $k \geqq 1$. If $T < \infty$, then $1 \leqq j_1 < T$, and thus $(0, \ldots, 0, j_1, 0, 0, \ldots) \in W_{f, f, \ldots}$, where here, and in the following similar sequences, j_1 (or $j_{k,1}$) is at the k-position. By

[2] We have remarked in the Introduction that each result of this paper has a correspondent when appropriate decreasing f_n, $n \geqq 1$, are considered

(23) we get $j_1 < T_k$, $k \geqq 1$, whence $1 \leqq j_1 \leqq j_{k,1}$, $k \geqq 1$. Further, using also Lemma 5, we see that

$$(0, \ldots, 0, j_{k,1}, 0, 0, \ldots) \in W_{f_1, f_2, \ldots},$$

and by (23) we find that $j_{k,1} \leqq j_1$, $k \geqq 1$. Therefore, $j_{k,1} = j_1$ for each $k \geqq 1$. This implies also that $T_k < \infty$, $k \geqq 1$. Then, whatever $k \geqq 1$ and $s \geqq 1$, we see that (*) either $j_{k+s,n} = j_{k,s+n}$, $n \geqq 1$, or there exists $r \geqq 1$ such that $j_{k+s,n} = j_{k,s+n}$, $1 \leqq n < r$, and $j_{k+s,r} > j_{k,s+r}$. Assume now that $j_{k,n} = j_n$, $1 \leqq n \leqq m$. From (11), (*) and Lemma 5, it follows that

$$(0, \ldots, 0, j_{k,1}, \ldots, j_{k,m}, j_{k,m+1}, 0, 0, \ldots) \in W_{f_1, f_2, \ldots}.$$

By virtue of (23) and (5), this amounts to $j_{k,m+1} \leqq j_{m+1}$. The converse inequality $j_{m+1} \leqq j_{k,m+1}$ is obtained similarly. Hence, the "only if" implication is established by induction.

Conversely, assume that there exist j_n, $n \geqq 1$, such that $j_{k,n} = j_n$, $n \geqq 1$, for each $k \geqq 1$. Then the following alternative holds: either (i) $j_1 = \infty$, and so $T_k = \infty$, $k \geqq 1$, or (ii) $j_1 < \infty$, and thus $T_k < \infty$, $k \geqq 1$. Now, we assume that (ii) occurs, and we construct a function f such that $(f_1, f_2, \ldots) \sim (f, f, \ldots)$. (The case when (i) occurs is similar and even simpler, so that we leave it to the reader to work out the details of the proof.) In this case, it will be helpful to keep in mind that, whatever $k \geqq 1$, we have $j_1 < T_k \leqq j_1 + 1$, and hence $I_k = \{0, 1, \ldots, j_1\} = I$ (say). Let $T = j_1 + \underline{x}(j_2, j_3, \ldots)$. Notice that, in view of Lemma 5, $j_1 < T \leqq j_1 + 1$. Define a function f from $[0, T)$ into $[0, 1]$ by

$$f(i+x) = \underline{x}(i, a_1(x), a_2(x), \ldots),$$

where $i \in I$, $x \in [0, 1)$, and $i + x < T$. We next prove that f is increasing. Consider $0 \leqq x < x' < 1$ and $i \in I$ such that $i + x < T$, $i + x' < T$. Then choose $m = \min\{n : a_n(x) \neq a_n(x')\}$. Since $a_m(x) < a_m(x')$ and, by (11) and Corollary 2, $(i, a_1(x), a_2(x), \ldots) \in W$, we may write

$$\begin{aligned} f(i+x) &\leqq \bar{x}(i, a_1(x), a_2(x), \ldots) < \bar{x}_{m+1}(i, a_1(x), \ldots, a_m(x)) \\ &\leqq \underline{x}_{m+1}(i, a_1(x'), \ldots, a_m(x')) \leqq f(i+x'). \end{aligned}$$

Further, a similar argument shows that, for $x \in [0, 1)$ and $i < j_1$, $f(i+x) < f(i+1)$, and so f being increasing is established. Our next step is to prove that f maps $[0, T)$ onto $[0, 1)$. Let D denote the set of all points $\underline{x}_n(a_1(x), \ldots, a_n(x))$, $x \in [0, 1)$, $n \geqq 1$. Then

$$\mathsf{D} \subset f((I + \mathsf{D}) \cap [0, T)).^3 \tag{24}$$

Indeed, if $\underline{x}_n(i_1, \ldots, i_n) \in \mathsf{D}$, then, in view of Corollary 2, $(i_1, \ldots, i_n, 0, 0, \ldots) \in W$, and thus $(i_2, \ldots, i_n, 0, 0, \ldots) \in W$ on account of (11). By Corollary 2, this amounts to $\underline{x}_{n-1}(i_2, \ldots, i_n) \in \mathsf{D}$. Since

$$\underline{x}_n(i_1, \ldots, i_n) = f(i_1 + \underline{x}_{n-1}(i_2, \ldots, i_n)) \tag{25}$$

(with $\underline{x}_0 = 0$) provided that $i_1 + \underline{x}_{n-1}(i_2, \ldots, i_n) < T$, to complete the proof of (24) it suffices to show that

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In fact, the two sets here are equal, but we only need this inclusion

$$\underline{x}_{n-1}(j_2,\ldots,j_n)<\underline{x}(j_2,j_3,\ldots), \qquad n>1.$$

Assume, on the contrary, that there is $m>1$ such that

$$\underline{x}_{m-1}(j_2,\ldots,j_m)=\underline{x}(j_2,j_3,\ldots). \tag{26}$$

Taking into account (*), we see that

$$j_n+\underline{x}_{n,m-n}(j_{n+1},\ldots,j_m)<T_{n-1}, \qquad 1<n\leqq m, \tag{27}$$

(with $\underline{x}_{m,0}=0$). From (1), (26) and (27), it follows that $\underline{x}_m(j_{m+1},j_{m+2},\ldots)=0$, and hence $j_n=0$, $n>m$. But this contradicts Lemma 5, and therefore (24) is proved. Now, as f is increasing and, by virtue of (6), D is dense in $[0,1)$, (24) implies that f maps $[0,T)$ onto $[0,1)$. Finally, let us show that

$$a(f_1,f_2,\ldots;x)=a(f,f,\ldots;x), \qquad x\in[0,1). \tag{28}$$

In the sequel, a prime attached to a term will indicate that the term is considered w.r.t. the constant sequence of functions $f,f,\ldots$. First, by making use of (25), it follows inductively that $\underline{x}_n(i_1,\ldots,i_n)=\underline{x}'_n(i_1,\ldots,i_n)$ whenever $\underline{x}_n(i_1,\ldots,i_n)\in\mathsf{D}$. Particularly, we have

$$\underline{x}_n(j_1,\ldots,j_n)=\underline{x}'_n(j_1,\ldots,j_n)<1, \qquad n\geqq 1.$$

Consequently, to get (28) we need only prove that

$$\bar{x}'_n(j_1,\ldots,j_n)=1, \qquad n\geqq 1. \tag{29}$$

Clearly, $\bar{x}'_1(j_1)=1$. Now assume that

$$\bar{x}'_n(j_1,\ldots,j_n)=1, \qquad 1\leqq n\leqq m. \tag{30}$$

On account of (*), we see that either (a) there is $s\in\{1,\ldots,m\}$ such that $j_n=j_{s+n}$, $1\leqq n\leqq m-s+1$, or else (b) $\underline{x}_m(j_2,\ldots,j_{m+1}+1)\in\mathsf{D}$. If (a) holds, then $\bar{x}'_{m+1}(j_1,\ldots,j_{m+1})=1$ by (30). If (b) occurs, then

$$j_1+\bar{x}'_m(j_2,\ldots,j_{m+1})=j_1+\bar{x}_m(j_2,\ldots,j_{m+1})\geqq T,$$

and again $\bar{x}'_{m+1}(j_1,\ldots,j_{m+1})=1$. Hence, (29) is obtained by induction, q.e.d.

Part i) of the following corollary is immediate; part ii) necessitates the use of (5).

Corollary 4. i) *If* $j_{k,n}=j_{k+1,n}$, $n\geqq 1$; *for each* $k>m$, *then*

$$(f_1,\ldots,f_m,f_{m+1},f_{m+2},\ldots)\sim(f_1,\ldots,f_m,f,f,\ldots)$$

for some function f.

ii) *If* $r_m([0,1))=[0,1)$ *and*

$$(f_1,\ldots,f_m,f_{m+1},f_{m+2},\ldots)\sim(f_1,\ldots,f_m,f,f,\ldots)$$

for some f, *then* $j_{k,n}=j_{k+1,n}$, $n\geqq 1$, *for all* $k>m$.

Example 1. Let $D>1$ be integral, and assume f, g, h are increasing functions on $[0, D)$ mapping $[0, D)$ onto $[0,1)$, where $f(t)=t/D$, $0 \leqq t<D$, and $g(i+x)=(i/D) + h^{-1}(x)/D^2$, $i=0, \ldots, D-1$, $x \in [0,1)$. Further, consider M a subset of natural numbers such that $n+1 \notin M$ if $n \in M$. Define a sequence of functions f_n, $n \geqq 1$. by $f_n = g$, $f_{n+1}=h$, $n \in M$, and $f_n=f$ for the other indices. Then $(f_1, f_2, \ldots) \sim (f, f, \ldots)$. Indeed, whatever $k \geqq 1$ and $i_n \in \{0, \ldots, D-1\}$, $1 \leqq n \leqq k$, we have

$$\underline{x}_k(i_1, \ldots, i_k) = \sum_{n=1}^{k} i_n/D^n = \underline{x}'_k(i_1, \ldots, i_k).$$

Now, by any of our Theorems 4 and 5, or by Corollary 1 in [1], it follows that any probability λ which makes the a'_n independent random variables is such that either $\lambda \perp m$ or $\lambda \ll m$, and hence any probability λ making the a_n independent is so too. Since we can choose either g or h not to be absolutely continuous, we see that the condition of Theorem 4 is not necessary for the conclusion of it to hold. However, for $f_n=f$, $n \geqq 1$, whether or not this condition is necessary remains an open problem.

4.2. **Notes.** The function f arising in Theorem 6 and Corollary 4 is uniquely determined. Actually, it can be proved if $a(f, f, \ldots)$ is injective, then each of $(f, f, \ldots) \sim (f', f', \ldots)$ and

$$(f_1, \ldots, f_m, f, f, \ldots) \sim (f_1, \ldots, f_m, f', f', \ldots)$$

implies that $f=f'$.

$a(f_1, f_2, \ldots; \cdot)$ being injective and $j_{k,n} = j_{k+1,n}$, $n \geqq 1$, for each $k \geqq 1$, do not imply that $T_k = T_{k+1}$, $k \geqq 1$, as it might seem likely. To show this, let $\alpha>0$ satisfy the equation $\alpha+\alpha^2=1$, and consider a sequence of functions f_n, $n \geqq 1$, such that $f_n(t) = \alpha t$, $0 \leqq t<1+\alpha$, $n>2$, $T_2=1+\alpha$, $f_2(1) \neq \alpha$, $T_1=1+f_2(1)$, and (7) holds for f_1 and f_2 with some finite γ_1 and γ_2. Then $a(f_1, f_2, \ldots; \cdot)$ is injective, and we see that $j_{k, 2n-1} = 1$, $j_{k, 2n}=0$, $n \geqq 1$ for any $k \geqq 1$. Nevertheless, $T_1 \neq T_k = 1+\alpha$, $k>1$.

If one removes the assumption $r_m([0,1))=[0,1)$, then the conclusion of Corollary 4 ii) does not generally hold. This is shown by the following example. Choose a sequence f_n, $n \geqq 1$, such that $f_n(t)=t/2$, $0 \leqq t<2$, $n>2$, $T_2>2$, $f_2(t)=t/2$, $0 \leqq t<1$, and $f_1(t)=2t$, $0 \leqq t<1/2$. Further, take $f(t)=t/2$, $0 \leqq t<2$. Then

$$r_1([0,1))=[0,1/2), \qquad (f_1, f_2, f_3, \ldots) \sim (f_1, f, f, \ldots),$$

but $j_{2,1} > j_{k,1} = 1$, $k>2$.

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