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On the Unimodality of Infinitely Divisible Distribution Functions

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1. Introduction

A distribution function $F(x)$ is said to be a distribution function of class L if there exists a sequence of independent random variables $X_1, X_2, \ldots$ such that for suitable constants A_n and $B_n > 0$ the random variables

$$Y_n = (X_1 + \cdots + X_n)/B_n - A_n \tag{1}$$

have the property that $F_{Y_n} \xrightarrow[C]{} F$ and in addition the random variables

$$X_{n,j} = X_j / B_n \quad (1 \leqq j \leqq n) \tag{2}$$

form an infinitesimal system. A distribution function $F(x)$ is stable if $F_{Y_n} \xrightarrow[C]{} F$ where the random variables $X_1, X_2, \ldots$ are identically distributed, in which case the random variables defined in (2) necessarily form an infinitesimal system. An infinitely divisible distribution function will be said to belong to class C if its Lévy spectral function is convex on $(-\infty, 0)$ and concave on $(0, \infty)$. Lévy has proved (see [3]) that an infinitely divisible distribution function belongs to class L if and only if its Lévy spectral function $M(u)$ has right and left derivatives for every $u \neq 0$ and the function $\lambda(u) = u M'(u)$ is non-increasing on $(-\infty, 0)$ and $(0, \infty)$ where $M'(u)$ denotes either the right derivative or the left derivative, perhaps different ones at different points. Thus the class of stable distribution functions is properly contained in the class of distribution functions of class L and the class of distribution functions of class L is properly contained in the class of distribution functions of class C. A distribution function $F(x)$ is said to be unimodal if there exists a constant a such that $F(x)$ is convex on $(-\infty, a)$ and concave on (a, ∞).

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The study of the unimodality of infinitely divisible distribution functions has attracted a great deal of attention during the last forty years. In 1936 A. Wintner [13] proved that the convolution of two symmetric unimodal distribution functions is unimodal. He used this theorem to prove that symmetric stable distribution functions are unimodal. In 1947 A.I. Lapin [9] claimed to prove that the convolution of any two unimodal distribution functions is unimodal. B. Gnedenko used Lapin's theorem to prove a theorem that states that every distribution function of class L is unimodal. Gnedenko's theorem appeared in Gnedenko's and Kolmogorov's book. *Limit Distributions of Sums of Independent Random Variables*, in 1949. In 1953, K.L. Chung realized, while translating Gnedenko and Kolmogorov's book into English, that Lapin's theorem was wrong. He constructed a counter-example that appeared in [1] and [3]. Thus the validity of Gnedenko's theorem was now in doubt.

In 1956, Wintner [14] generalized his theorem of twenty years earlier and showed that every symmetric distribution function of class L is unimodal. This theorem was generalized further by P. Medgyessy [10] who proved that every symmetric distribution function of class C is unimodal. In 1957, I.A. Ibragimov [5] published a paper in which he claimed to give an example of a distribution function of class L that is not unimodal. However in 1967 T.C. Sun [12] showed that the example that Ibragimov had constructed was unimodal. In 1971, Wolfe [15] proved that every distribution function of class L that has a Lévy spectral function with support on the positive axis is unimodal. It follows from this theorem that every distribution function of class L can be expressed as the convolution of at most two unimodal distribution functions of class L.

In 1959, I.A. Ibragimov and K.E. Chernin [6] published a paper in which they claimed to prove that every stable distribution function is unimodal. However M. Kanter [8] has recently shown that the proof of Ibragimov and Chernin is not valid. Thus it is not known now whether or not every stable distribution function is unimodal.

In this paper a study will be made of semigroups of infinitely divisible distribution functions. If $F(x)$ is an infinitely divisible distribution function with a characteristic function $f(u)$, then the distribution function $F_t(x)$ with characteristic function $f^t(u)$ is also infinitely divisible. Since $F_{t_1+t_2}=F_{t_1}*F_{t_2}$, the set of distribution functions $\{F_t(x): 0\leqq t<\infty\}$ forms a semigroup with respect to the operation of convolution. Semigroups of this type arise naturally in the study of stochastic processes. If $\{X_t: 0\leqq t<\infty\}$ is a centered stochastic process with independent and homogeneous increments then $\{F_{X_t}: 0\leqq t<\infty\}$ is a semigroup. A semigroup will be said to have property P if every distribution function of the semigroup has property P. A distribution function $F(x)$ can be imbedded in a semigroup if and only if the distribution function is infinitely divisible. It is obvious that every stable distribution function can be imbedded in a stable semigroup and every distribution function of class L can be imbedded in a semigroup of distribution functions of class L. In Section 2 the following theorem will be proved:

Theorem 1. *A necessary condition for a distribution function $F(x)$ to be imbedded in a unimodal semigroup is that $F(x)$ belong to class C. If $F(x)$ is symmetric then this condition is also sufficient.*

An example will be given in Section 2 to show that there exist nonsymmetric distribution functions of class C that can not be imbedded in unimodal semigroups. In Section 3 it will be shown that it is possible to construct an example of a unimodal infinitely divisible distribution function that is not contained in class C. This example has the interesting property that it can be expressed as the convolution of a normal distribution function and a non-unimodal infinitely divisible distribution function that does not have a normal component. Some remarks concerning the unimodality of infinitely divisible distribution functions will be made in Section 4.

2. Proof of Theorem 1.

Assume that $F(x)$ can be imbedded in a unimodal semigroup $\{F_t(x)\colon 0 \leqq t < \infty\}$. Without loss of generality, it can be assumed that $F(x) = F_1(x)$. It is clear that as n goes to infinity, $F_{1/n}(x)$ converges completely to the degenerate distribution function $E(x)$ defined by $E(x)=0$ for $x<0$ and $E(x)=1$ for $x>0$.

For each positive n, let $\mathcal{M}_n$ be a mode of $F_{1/n}(x)$. Lapin has proved (see [3], p. 160) that if $G_n(x)$ is a sequence of unimodal distribution functions that converge completely to $G(x)$ and if a_n is a mode of $G_n(x)$ then $G(x)$ is unimodal with a mode at $a = \limsup_{n\to\infty} a_n$. Lapin's proof can also be altered to show that $a' = \liminf_{n\to\infty} a_n$ is also a mode of $G(x)$. Since $E(x)$ has a unique mode at 0, it follows that $\lim_{n\to\infty} \mathcal{M}_n = 0$.

Let $M(u)$ be the Lévy spectral function of $F(x)$. It follows from the general limit theorem (see [3], page 124) that if x is a point of continuity of $M(u)$ less than zero then $\lim_{n\to\infty} nF_{1/n}(x) = M(x)$ and if x is a point of continuity of $M(u)$ greater than zero then $\lim_{n\to\infty} n(F_{1/n}(x)-1) = M(x)$. An argument similar to that used in the proof of Lapin's theorem can be used to show that $M(u)$ is convex on $(-\infty, 0)$ and concave on $(0, \infty)$. Thus $F(x)$ is a distribution function of class C.

If $F(x)$ is a symmetric distribution function of class C then every distribution function of the semigroup in which it can be imbedded also belongs to class C. It follows from Medgyessy's theorem that the semigroup is unimodal. □

It is possible to construct non-symmetric distribution functions of class C that can not be embedded in unimodal semigroups. Let $F(x)$ be a distribution function with characteristic function $\hat{f}(u)$ and let $G_\lambda(x)$ be the distribution function with characteristic function $\hat{g}_\lambda(u) = \exp\{\lambda(\hat{f}(u)-1)\}$. It was shown in [15] that if $F(x)$ has density $f(x)=0$ for $x<0$ and $f(x)=e^{-x}$ for $x \geqq 0$ then $G_\lambda(x)$ is unimodal for $0 \leqq \lambda \leqq 2$ and not unimodal for $\lambda > 2$. It is easy to show that if $F(x)$ has density $f(x)=0$ for $x<0$ and $x>1$ and $f(x)=1$ for $0 \leqq x \leqq 1$ then $G_\lambda(x)$ is not unimodal for any $\lambda > 0$.

The technique used in the construction of the above two examples can be generalized. Johnson and Rogers [7] have shown that if $H(x)$ is a unimodal distribution with mean m, mode $\mathcal{M}$, and variance σ^2 then $|\mathcal{M} - m|^2 \leqq 3\sigma^2$. Let $F(x)$ be a distribution function with a mean $m \neq 0$ and a variance σ^2. Then $G_\lambda(x)$ has mean λm and variance $\lambda\sigma^2$. The distribution function $G_\lambda(x)$ has a discontinuity at the origin so if it is to be unimodal then it is necessary that $(\lambda m)^2 \leqq 3\lambda\sigma^2$. It follows that $G_\lambda(x)$ can not be unimodal if $\lambda > (3\sigma^2)/m^2$.

The above examples have discontinuities at the origin. It is also possible to construct an example of an absolutely continuous distribution function of class C that is not unimodal. Let $F(x)$ be one of the non-unimodal distribution functions defined above and let $G_\eta(x)$ be the gamma distribution with density function $g_\eta(x)=0$ for $x<0$ and $g_\eta(x)=e^{-x}x^{\eta-1}/\Gamma(\eta)$ if $x>0$. It is clear that if $x \neq 0$, $\lim_{\eta\to 0} F*G_\eta(x)=F(x)$. Since $F(x)$ is not unimodal it follows that $F*G_\eta(x)$ is not unimodal for small η.

3. An Example of a Symmetric Unimodal Infinitely Divisible Distribution Function that is not Contained in Class C

Let $f(x)=0$ if $x<-2$, $f(x)=a$ if $+2\leqq x<-1$, $f(x)=b$ if $-1\leqq x<1$, $f(x)=a$ if $1\leqq x<2$, and $f(x)=0$ if $x>2$. If $2a+2b=1$ then $f(x)$ is the density function of a symmetric distribution function $F(x)$. If $b<a$ then $F(x)$ is bimodal. However it will be shown that it is possible to construct such a bimodal $F(x)$ that has the property that $F*F(x)$, $F*F*F(x)$, and $F*G(x)$ are all unimodal, where $G(x)$ is the normal distribution function with mean 0 and variance 1.

Let $f_1(x)=0$ for $x<-2$, $f_1(x)=1$ for $-2\leqq x<2$, and $f_1(x)=0$ for $x>2$. Let $f_2(x)=0$ for $x<-1$, $f_2(x)=1$ for $-1\leqq x<1$, and $f_1(x)=0$ for $x>1$. Since $f(x)=af_1(x)-(a-b)f_2(x)$, $f*f(x)=a^2 f_1*f_2(x)-2(a-b)f_1*f_2(x)+(a-b)^2 f_2*f_2(x)$. Since $(f_1*f_1)'(x)=-1$ for $0\leqq x\leqq 4$, it follows that $a-b$ can be chosen so small that $(f*f)'(x)<0$ for $0\leqq x<4$. Since $f*f(x)=0$ for $x\geqq 4$ $F*F(x)$ is then unimodal.

A similar argument can be used to show that $F*F*F(x)$ is unimodal for small $(a-b)$. Since $f_1*f_1*f_1(x)=-x^2+12$ for $0\leqq x\leqq 2$, $f_1*f_1*f_1(x)=\frac{1}{2}x^2-6x+18$ for $2<x\leqq 6$, and $f*f*f(x)=f_1*f_1*f_1(x)$ for $x>5$, it follows that if $0<\varepsilon<1$, $a-b$ can be chosen small enough to make $(f*f*f)'(x)\leqq 0$ for $x>\varepsilon$. It is then possible to compute $f*f*f(x)$ for $0<x<1$ and show that for small $a-b$ its derivative is also not positive.

If $h(x)$ is the density function of $F*G(x)$, then

$$h(x)=a[G(x+2)-G(x-2)]-(a-b)[G(x+1)-G(x-1)].$$

Thus it follows that

$$\begin{aligned} h'(x) &= a[e^{-(x+2)^2/2}-e^{-(x-2)^2/2}]-(a-b)[e^{-(x+1)^2/2}-e^{-(x-1)^2/2}] \\ &= e^{-x^2/2}[a\,e^{-2}(e^{-2x}-e^{2x})-(a-b)\,e^{-\frac{1}{2}}(e^{-x/2}-e^{x/2})]. \end{aligned}$$

If $P(y)=e^{-yx}-e^{yx}$ then $P'(y)<0$ if $x>0$. It follows that if $(a-b)\,e^{-\frac{1}{2}}<a\,e^{-2}$ then $h'(x)<0$ for $x>0$ and $F*G(x)$ is unimodal.

Let $F^{*n}(x)$ denote the convolution of $F(x)$ with itself n times and let $f^{*n}(x)$ denote the density function of $F^{*n}(x)$. Since the convolution of two symmetric unimodal distribution functions is unimodal, $F^{*n}(x)$ is unimodal for $n\geqq 2$. Let $\hat{f}(u)$ be the characteristic function of $F(x)$ and let $H_\lambda(x)$ be the infinitely divisible

distribution function with characteristic function $\hat{h}_\lambda(u)=\exp\{\lambda(\hat{f}(u)-1)\}$. Then $H_\lambda(x)=e^{-\lambda}E(x)+e^{-\lambda}\sum_{n=1}^{\infty}\lambda^n F^{*n}(x)/n!$ where $E(x)$ is the distribution function degenerate at 0.
Thus

$$H_\lambda(x)=e^{-\lambda}E(x)+e^{-\lambda}\lambda F(x)+e^{-\lambda}\lambda\sum_{n=2}^{\infty}\lambda^{n-1}F^{*n}(x)/n!.$$

Since $f(x)\leqq a$ for all x it follows that $f^{*n}(x)\leqq a$ for all x. Thus if λ is chosen so small that $\sum_{n=2}^{\infty}\lambda^{n-1}a/n!<a-b$ it follows that $H_\lambda(x)$ is not unimodal. However $G*H_\lambda(x)$ is easily seen to be unimodal. The distribution function $G*H_\lambda(x)$ is not contained in class C since $F(x)$ is not unimodal.

It should be noted that $H_\lambda(x)$, for small λ, is an example of an infinitely divisible distribution function without a normal component that has the property that its convolution with a normal distribution function is unimodal.

4. Unimodality of Infinitely Divisible Distribution Functions

It is well known that $\hat{f}(u)$ is the characteristic function of an infinitely divisible distribution function if and only if it can be expressed in the form

$$\hat{f}(u)=\exp\left\{i\gamma t-\sigma^2 t^2/2+\int_{-\infty}^{-0}+\int_{+0}^{+\infty}H(u,x)\,dM(x)\right\}$$

where

$$H(u,x)=e^{iux}-1-\frac{iux}{1+u^2},$$

γ, and $\sigma^2>0$ are constants, and $M(x)$ is a function that has the properties that it is non-decreasing on $(-\infty,0)$ and $(0,\infty)$, $M(-\infty)=M(\infty)=0$ and

$$\int_{-1}^{-0}+\int_{+0}^{+1}x^2\,dM(x)<\infty.$$

It is obvious that if $\hat{f}(u)$ is a characteristic function of a unimodal distribution function for some value of γ. Then it will be the characteristic function of a unimodal distribution function for all values of γ. The examples in this paper show that it is probably not possible to give simple conditions on σ^2 and $M(u)$ for $\hat{f}(u)$ to be the characteristic function of a unimodal distribution function $F(x)$. However it will probably be possible to improve the above results.

Examples have been constructed to show that distribution functions of class C need not be unimodal. These distribution functions fail to be unimodal because they have discontinuities at the origin and large moments or because they are close to non-unimodal distribution functions of this type. Distribution functions of class L have the property that $uM'(u)$ is non-increasing on $(-\infty, 0)$ and $(0, \infty)$. Apparently this property not only forces a distribution function to be absolutely continuous but keeps it from being close to a distribution function with a discontinuity at the origin.

If $M(u)$ has support on $(0, \infty)$, then it follows from the author's theorem that $F(x)$ is unimodal if $uM'(u)$ is non-increasing. It may be possible to generalize this theorem and show that $F(x)$ is unimodal under some broader condition, such as the requirement that $u^\alpha M'(u)$ be non-increasing where $0<\alpha<1$.

If $M(u)$ has support on $(-\infty, 0)$ and $(0, \infty)$ the situation is more complicated. At the present time, it is not even known whether or not all stable distribution functions are unimodal. It is the feeling of the author that a large class of infinitely divisible distribution functions, that includes the class of distribution functions of class L, are unimodal.

Results similar to those in this paper can be obtained for discrete distributions on the lattice of integers. A discrete distribution $\{q_k\}$ is said to be unimodal with a mode at N if $q_i \leqq q_j$ for $i<j \leqq N$ and $q_i \geqq q_j$ for $N \leqq i<j$. A study of the unimodality of discrete distributions has been made by Medgyessy in [11]. An infinitely divisible distribution $\{P_k\}$ will be said to belong to class C if it has a characteristic function of the form $f(u)=\exp\{\lambda(g(u)-1)\}$ where $g(u)$ is the characteristic function of a distribution $\{q_k\}$ with the property that $q_i \leqq q_j$ for $i<j<0$ and $q_i \geqq q_j$ for $0<i<j$. Dharmadhikari and Jogdeo [2] have shown that the convolution of two symmetric unimodal distributions is unimodal. Once this is known, the following analogue of Theorem 1 can be proved.

Theorem 2. *A necessary condition for a distribution to be imbedded in a unimodal semigroup is that $F(x)$ belong to class C. If $F(x)$ is symmetric then this condition is also sufficient.*

Let $\{q_k\}^{*n}$ denote the convolution of $\{q_k\}$ with itself n times. If $q_0=3/5, q_1$, $=1/5$, and $q_2=1/5$, then $\{q_k\}^{*2}$ is not unimodal. It is possible to show that if λ is small, $P_1<P_0<P_2$. Thus $\{q_k\}$ yields an infinitely divisible distribution function of class C that is not unimodal. If $q_{-2}=4/17$, $q_{-1}=3/17$, $q_0=3/17$, $q_1=3/17$, and $q_2=4/17$, then $\{q_k\}^{*n}$ is unimodal for $n>1$. Thus $\{q_k\}$ yields, for large λ, a symmetric unimodal distribution that is not contained in class C. The easy computations are left to the reader.

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Note Added in Proof

Yamazato [Unimodality of infinitely divisible distribution functions of class L, Annals of Probability **6**, 523–531 (1978)] has recently shown that all distribution functions of class L are unimodal.