Z. Wahrscheinlichkeitstheorie verw. Gebiete
45, 297–313 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Random Invariant Measures for Markov Chains, and Independent Particle Systems

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Summary. Let P be the transition operator for a discrete time Markov chain on a space S. The object of the paper is to study the class of random measures on S which have the property that $MP=M$ in distribution. These will be called random invariant measures for P. In particular, it is shown that $MP=M$ in distribution implies $MP=M$ a.s. for various classes of chains, including aperiodic Harris recurrent chains and aperiodic irreducible random walks. Some of this is done by exploiting the relationship between random invariant measures and entrance laws. These results are then applied to study the invariant probability measures for particle systems in which particles move independently in discrete time according to P. Finally, it is conjectured that every Markov chain which has a random invariant measure also has a deterministic invariant measure.

1. Introduction

One of the main problems in the theory of interacting particle systems is to determine the structure of the set of invariant measures for the process. The solution to this problem is now well understood in certain contexts, but in most cases, very little is known about it. [8] contains a survey of results and open problems in this area, and a list of references. One interesting example of a process for which little is known concerning the set of invariant measures is Spitzer's zero range interaction process [12]. Spitzer gives a class of measures which are invariant for it, but it is not known in general whether these exhaust the class of all invariant measures.

Since many interacting particle systems are obtained by superimposing some type of interaction on an independent particle system, it seems reasonable to study the invariant measures for the basic independent system without interactions, even though the techniques will of course not carry over to the

* Research supported in part by N.S.F. Grant No. MCS 77-02121

0044-3719/78/0045/0297/$03.40

interacting case. The fact that independent systems often have invariant measures which are Poisson point processes has been known for a long time [4]. We are interested in finding all the invariant measures for the system, and in particular, in determining conditions under which all extremal invariant measures are Poisson. It turns out that under a weak assumption, this problem reduces quickly to a question involving only the underlying one particle Markov chain, and most of this paper will be devoted to that question. I am grateful to Claude Kipnis for pointing out that even in the independent case, the structure of the set of invariant measures for an infinite particle system had not been adequately treated.

Let S be a locally compact, second countable Hausdorff space, and let $P(x, dy)$ be the transition probabilities for a discrete time Markov chain on S. Thus we assume that for each $x \in S$, $P(x, \cdot)$ is a Borel probability measure on S, and that for each Borel set $E \subset S$, $P(\cdot, E)$ is a Borel measurable function of x. Let $\{\eta_n\}$ be the discrete time particle system on S determined by the requirement that particles move independently on S at integer times according to the transition law $P(x, dy)$. η_n is regarded as a random integer valued measure on S, so we let $\eta_n(E)$ denote the number of particles in E at time n. Let $\mathscr{M}$ be the set of integer valued measures η on S which are finite on compact subsets of S. $\mathscr{M}$ is endowed with the smallest σ-algebra with respect to which $\eta(A)$ is a measurable function of η for each Borel set $A \subset S$. This is the same as the Borel σ-algebra corresponding to the vague topology on $\mathscr{M}$. Of course, $\eta_0 \in \mathscr{M}$ does not necessarily imply that $\eta_n \in \mathscr{M}$ a.s. for each n. Still, we will say that a probability measure μ on $\mathscr{M}$ is invariant for $\{\eta_n\}$ provided that

$$\mu(A) = \int P^\eta[\eta_1 \in A]\, \mu(d\eta)$$

for all measurable subsets A of $\mathscr{M}$.

If m is a measure on S which is finite on compact sets, let μ_m be the Poisson probability measure on $\mathscr{M}$ with mean m, which is determined by

$$\mu_m\{\eta(E_i) = k_i \quad \text{for } 1 \leqq i \leqq n\} = \prod_{i=1}^{n} e^{-m(E_i)} \frac{[m(E_i)]^{k_i}}{k_i!}$$

for disjoint sets E_i with $m(E_i) < \infty$ for each i. Unless otherwise stated, M will always denote a random measure on S which satisfies

$$M(C) < \infty \qquad \text{a.s. for each compact } C \subset S. \tag{1.1}$$

For such an M, define the mixed Poisson probability measure μ_M on $\mathscr{M}$ governed by M by

$$\mu_M(A) = \int \mu_m(A)\, P\{M \in dm\}.$$

Of course, μ_M depends only on the distribution of M.

In Section 4, the following result will be proved, together with applications obtained by combining it with the theorems in Sections 2 and 3. Let $P^n(x, dy)$ be the n'th iterate of $P(x, dy)$, and let $\mathscr{I}$ be the set of all measures m on S which are finite on compact sets and satisfy $mP = m$.

Theorem 1.2. (a) μ_M *is invariant for* $\{\eta_n\}$ *if and only if* $MP=M$ *in distribution.*
Suppose now that for every compact set $C \subset S$,

$$\lim_{n\to\infty} \sup_{x\in S} P^n(x, C)=0. \tag{1.3}$$

Then

(b) *every invariant probability measure on* $\mathcal{M}$ *for* $\{\eta_n\}$ *is of the form* μ_M *for some* M, *and*

(c) *the extremal invariant probability measures on* $\mathcal{M}$ *for* $\{\eta_n\}$ *are exactly* $\{\mu_m: m\in\mathcal{I}\}$ *if and only if* $MP=M$ *in distribution implies* $MP=M$ *a.s.*

Parts (b) and (c) of the above theorem are false without assumption (1.3). They fail, for example, for any positive recurrent chain, since then there are extremal invariant probability measures for $\{\eta_n\}$ which concentrate on $\{\eta:\eta(S)=k\}$. Another example is given by the uniform motion process on the integers, since then the pointmass on the configuration which has one particle on each site is invariant.

It should be noticed that Theorem 1.2 often allows one to determine the invariant measures for an independent particle system even if the motion of the particles is in continuous time and is non-Markovian. To see this, suppose that $\{X^x(t), x\in S\}$ is a collection of stochastic processes on S with $X^x(0)=x$, and let η_t be the corresponding independent particle system. Then one can apply the above theorem to the discrete time Markovian system with transition probabilities $P(x, dy)=P[X^x(t)\in dy]$ for some fixed $t>0$.

Theorem 1.2 suggests the following problems for the Markov chain P, which are of interest even when condition (1.3) is not satisfied:

(a) What are all the random measures M on S for which $MP=M$ in distribution?

(b) Under what assumptions on P is it the case that $MP=M$ in distribution implies $MP=M$ a.s., i.e., $M\in\mathcal{I}$ a.s.?

In Section 2, we will prove that for aperiodic Harris recurrent chains, $MP=M$ in distribution implies $MP=M$ a.s. In this case, P has a unique (up to constant multiples) σ-finite invariant measure, so that the answers to (a) and (b) are complete.

The transient case is more complex, and will be studied in Section 3. Only partial answers to these questions will be obtained. One of the interesting results is that there is a close relationship between the solutions to $MP=M$ in distribution and the entrance laws for P. An entrance law for P is a collection $\{\pi_n, -\infty<n<\infty\}$ of measures on S such that $\pi_n(C)<\infty$ for all compact sets C and $\pi_n P=\pi_{n+1}$ for each n. As an example of this relationship, suppose P has an entrance law $\{\pi_n\}$ which satisfies

$$\sum_{n=-\infty}^{\infty} \pi_n(C)<\infty \tag{1.4}$$

for each compact set C, and let $\{W_n, -\infty<n<\infty\}$ be any positive stationary process with finite mean. Then $M=\sum_{n=-\infty}^{\infty} W_n \pi_n$ is a solution of $MP=M$ in distribution. To see this, write

$$MP=\sum_n W_n \pi_n P=\sum_n W_n \pi_{n+1}=\sum_n W_{n-1} \pi_n,$$

which has the same distribution as M, since $\{W_n\}$ is stationary. If $\{W_n\}$ is an independent and identically distributed nonconstant sequence, for example, then M will not satisfy $MP=M$ a.s. An irreducible, aperiodic chain for which there exists an entrance law satisfying (1.4) is given by:

S = integers, $P(x,x+1)=1$ if $x<0$, $P(x,x+1)=\alpha$ if $x\geqq 0$, $P(x,x-1)=1-\alpha$ if $x\geqq 1$, and $P(0,x)=(1-\alpha)2^x$ if $x\leqq -1$, where $\frac{1}{2}<\alpha<1$. An entrance law which satisfies (1.4) is given by: π_n = pointmass on n for $n\leqq 0$, and $\pi_n=\pi_0 P^n$ for $n\geqq 0$. Similarly, if $\{\pi_n\}$ is an entrance law which satisfies $\pi_{n+d}=\pi_n$ for some d and all n, a solution to $MP=M$ in distribution can be obtained by letting $M=\sum_{k=1}^{d} \pi_k W_k$, where $W_k\geqq 0$ and the distribution of $(W_1,\ldots,W_d)$ is invariant under cyclic permutations. Many periodic chains have periodic entrance laws.

A more complete relationship to entrance laws is given by the fact that there is a natural one-to-one correspondence between (the distribution of) solutions to $MP=M$ in distribution and (the distribution of) stationary processes of entrance laws. By a stationary process of entrance laws, we mean a sequence $\{\Pi_n, -\infty<n<\infty\}$ of random measures on S which satisfies $\Pi_n(C)<\infty$ a.s. for compact sets $C\subset S$,

(a) $\Pi_n P=\Pi_{n+1}$ a.s. for each n, and

(b) $\{\Pi_n, -\infty<n<\infty\}$ and $\{\Pi_{n+1}, -\infty<n<\infty\}$ have the same distribution.

Therefore, if one knows enough about the structure of the set of entrance laws, one can determine all the solutions to $MP=M$ in distribution. For example, it will be proved in Section 3 that if every extremal entrance law is of the form

$$\pi_n=\lambda^n \pi_0 \tag{1.5}$$

for some $\lambda>0$, then $MP=M$ in distribution implies $MP=M$ a.s. As an application, we will show that if S is an Abelian group and P is the transition law for an irreducible aperiodic random walk, then $MP=M$ in distribution implies $MP=M$ a.s. In Section 4, this will be combined with Theorem 1.2 to show that in this case, the extremal invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ are exactly $\{\mu_m: m\in\mathscr{I}\}$.

In general, not much appears to be known about the structure of the set of entrance laws, except that when properly normalized, it is a Choquet simplex [5]. Cox [2] has studied entrance laws which satisfy $\pi_n(S)=1$, and has given conditions under which there exist such entrance laws which are not stationary distributions. In our context, however, one must deal with infinite entrance laws as well. One interesting problem is to determine reasonable conditions on P under which all extremal entrance laws satisfy (1.5). Of course this is a natural extension of Cox's problem, since whenever $\{\pi_n\}$ is an entrance law which satisfies (1.5) and $\pi_n(S)<\infty$, it follows that $\lambda=1$ and hence π_n is a stationary measure. A somewhat related question is whether it is possible for an extremal entrance law to satisfy both $\sup_{n\geqq 0}\pi_n(C)=\infty$ and $\sup_{n\leqq 0}\pi_n(C)=\infty$ for some compact $C\subset S$.

Finally, we will state a specific unsolved problem relating to random invariant measures:

Conjecture 1.6. *If there exists a nonzero solution of $MP=M$ in distribution, then $\mathscr{I}$ contains a nonzero measure.*

Of course, if $MP=M$ in distribution and $EM(C)<\infty$ for all compact sets C, then $m=EM\in\mathscr{I}$. When M has infinite mean, however, the problem of using M to construct an element of $\mathscr{I}$ is open. An example which bears on this point is given in Section 3. Of course, by part (a) of Theorem 1.2, in order to prove Conjecture 1.6, it suffices to show that if $\{\eta_n\}$ has a nontrivial invariant probability measure on $\mathscr{M}$, then $\mathscr{I}$ contains a nonzero measure.

2. Recurrent Chains

Throughout this section, we will assume that P is Harris recurrent. For the definition and properties of Harris recurrent chains, see [11]. P has a unique (up to constant multiples) σ-finite invariant measure, which will be denoted by m. In studying the random invariant measures for P, we will need to consider separately the cases in which M is singular with respect to m a.s. and in which M is absolutely continuous with respect to m a.s. We begin with the singular case.

Lemma 2.1. *Suppose M is a random measure on S such that $MP=M$ in distribution and M is singular with respect to m a.s. Then $M=0$ a.s.*

Proof. By iterating, $MP^n=M$ in distribution for all $n\geqq 1$. This, together with the singularity of M, implies that MP^n is singular with respect to m a.s. for each $n\geqq 1$. Therefore there is a subset Ω_0 of the probability space on which M is defined such that Ω_0 has probability one and MP^n is singular with respect to m for all $n\geqq 1$ and all $\omega\in\Omega_0$. Since P is Harris recurrent, $\sum_{n=1}^{\infty} P^n(x,A)=\infty$ for all $x\in S$ if $m(A)>0$ (page 75 of [11]). Therefore

$$\sum_{n=1}^{\infty} MP^n(A)=\int M(dx)\sum_{n=1}^{\infty} P^n(x,A)=\infty \tag{2.2}$$

for all A such that $m(A)>0$ and all ω such that M is not the zero measure. Suppose now that $M(S)>0$ with positive probability, and choose $\omega\in\Omega_0$ so that $M(S)>0$ for that ω. For this ω, MP^n is singular with respect to m for all $n\geqq 1$, so there exists a Borel set A so that $MP^n(A)=0$ for all $n\geqq 1$ and $m(A)>0$. This contradicts (2.2), so we conclude that $M=0$ a.s.

Lemma 2.3. *Suppose $M(\omega,dx)$ is a random measure on S. Then there exists a jointly measurable function $F(\omega,x)$ and a random measure $M^1(\omega,dx)$ on S such that*

$$M(\omega,dx)=F(\omega,x)\,m(dx)+M^1(\omega,dx)$$

and $M^1(\omega,dx)$ is singular with respect to m a.s.

Proof. The proof of Lemma 5.3 of Chapter 1 of [11] can be used to obtain this result in case $M(\omega, S)$ is bounded in ω. To extend this to the general case, note that by (1.1), there exists a strictly positive Borel function h on S for which $Z(\omega) = \int_S h(x)\, M(\omega, dx) < \infty$ a.s. Let

$$M_n(\omega, dx) = \begin{cases} M(\omega, dx) & \text{if } n \leqq Z(\omega) < n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_S h(x)\, M_n(\omega, dx) \leqq n+1$, so that by the earlier case, there exist jointly measurable functions $F_n(\omega, x)$ and random measures $M_n^1(\omega, dx)$ which are singular with respect to m a.s. so that

$$h(x)\, M_n(\omega, dx) = F_n(\omega, x)\, m(dx) + M_n^1(\omega, dx).$$

The required F and M^1 are then given by $F(\omega, x) = h^{-1}(x)\, F_n(\omega, x)$ and $M^1(\omega, dx) = h^{-1}(x)\, M_n^1(\omega, dx)$, where for each ω, n is chosen so that $n \leqq Z(\omega) < n+1$.

The proof of the next lemma is most transparent in case S is countable, so we will carry it out under that assumption before treating the general case. We will say that $MP \leqq M$ in distribution if there exists a random measure N so that $MP + N = M$ in distribution. For the definition and discussion of periodicities for Harris recurrent chains, the reader is referred to the first part of Chapter 6 of [11].

Lemma 2.4. *Suppose that P is aperiodic. If M is a random measure on S such that $MP \leqq M$ in distribution and M is absolutely continuous with respect to m* a.s., *then $MP = M$ a.s.*

Proof. *The countable case.* We assume here that S is countable and P is an irreducible, aperiodic, recurrent Markov chain on S. Let φ be a bounded, increasing, strictly concave continuous function on $[0, \infty)$. By Jensen's inequality and the fact that $mP^n = m$,

$$\varphi\left[\frac{MP^n(y)}{m(y)}\right] m(y) \geqq \sum_{x \in S} \varphi\left[\frac{M(x)}{m(x)}\right] m(x)\, P^n(x, y) \tag{2.5}$$

for every $y \in S$ and positive integer n. Since $MP^n(y) \leqq M(y)$ in distribution and φ is increasing,

$$E\varphi\left[\frac{MP^n(y)}{m(y)}\right] \leqq E\varphi\left[\frac{M(y)}{m(y)}\right].$$

Together with (2.5), this implies that $E\varphi\left[\frac{M(y)}{m(y)}\right] m(y)$ is an excessive measure for P^n. But P^n is irreducible and recurrent, since P is irreducible, recurrent and aperiodic. Therefore $E\varphi\left[\frac{M(y)}{m(y)}\right]$ is independent of y by Proposition 2.10 of Chapter 3 of [11], so equality holds a.s. in (2.5) since both sides have the same expected value. Since φ is strictly concave, this implies that for each $y \in S$, $M(x)/m(x)$ is independent of x a.s. for all x for which $P^n(x, y) > 0$. By the aperiodicity and irreducibility of P, for any $y\in S$, the sets $\{x\colon P^n(x,y)>0$ for all $n\geqq N\}$ increase to S. Hence $M(x)/m(x)$ is independent of x a.s., which gives the desired result.

Proof. *The general case.* Since $mP=m$, $m(E)=0$ implies that $P(x,E)=0$ for a.e. x with respect to m. Therefore

$$\gamma \ll m \quad \text{implies } \gamma P \ll m, \tag{2.6}$$

and hence MP^n is absolutely continuous with respect to m a.s. By Lemma 2.3, there exist jointly measurable functions $F_n(\omega,x)$ such that $MP^n(\omega,dx)=F_n(\omega,x)\,m(dx)$ a.s. We will usually suppress the dependence of these quantities on ω. By the definition of MP^n, for each Borel set $A\subset S$,

$$\int_A F_n(x)\,m(dx)=\int F_0(x)\,m(dx)\,P^n(x,A) \quad \text{a.s.}, \tag{2.7}$$

where the exceptional set does not depend on A. For fixed $u>0$, define $\varphi_1(t)=t$, $\varphi_2(t)=u$, and $\varphi(t)=\min\{\varphi_1(t),\varphi_2(t)\}$. Note that φ is bounded, increasing, and concave on $[0,\infty)$. Given any Borel set $A\subset S$ for which $m(A)<\infty$, define random sets $A_n^1=\{y\in A\colon F_n(y)\leqq u\}$ and $A_n^2=\{y\in A\colon F_n(y)>u\}$. By (2.7) and $mP=m$, since φ_1 and φ_2 are linear,

$$\int_{A_n^i} \varphi_i[F_n(y)]\,m(dy)=\int \varphi_i[F_0(x)]\,m(dx)\,P^n(x,A_n^i)$$

for each $i=1,2$. Therefore since A is the disjoint union of A_n^1 and A_n^2,

$$\begin{aligned}&\int_A \varphi[F_n(y)]\,m(dy)-\int \varphi[F_0(x)]\,m(dx)\,P^n(x,A)\\ &\quad=\sum_{i=1}^{2}\int \{\varphi_i[F_0(x)]-\varphi[F_0(x)]\}\,m(dx)\,P^n(x,A_n^i)\geqq 0.\end{aligned} \tag{2.8}$$

Since $MP^n\leqq M$ in distribution and φ is increasing,

$$\int_A \varphi[F_n(y)]\,m(dy)\leqq \int_A \varphi[F_0(y)]\,m(dy)$$

in distribution, so that $\int_A E\varphi[F_n(y)]\,m(dy)\leqq \int_A E\varphi[F_0(y)]\,m(dy)$. By taking expected values in (2.8), it then follows that the measure $E\varphi[F_0(y)]\,m(dy)$ is excessive for P^n. Since P is aperiodic, P^n is Harris recurrent for each $n\geqq 1$. Therefore by Proposition 2.10 of Chapter 3 of [11], $E\varphi[F_0(y)]$ is constant a.e. with respect to m. It follows that the expected value of the left side of (2.8) is zero, so

$$\int \{\varphi_i[F_0(x)]-\varphi[F_0(x)]\}\,m(dx)\,P^n(x,A_n^i)=0 \quad \text{a.s.}$$

for $i=1,2$. Therefore for a.e. ω, and for each $n\geqq 1$,

$$\begin{aligned}&F_0(x)\leqq u \quad \text{for a.e. } x \text{ with respect to } m(dx)\,P^n(x,A_n^1),\\ &\text{and}\\ &F_0(x)\geqq u \quad \text{for a.e. } x \text{ with respect to } m(dx)\,P^n(x,A_n^2).\end{aligned} \tag{2.9}$$

Fix an ω for which (2.9) holds for all $n \geqq 1$ and all rational u. We will show that for such an $\omega, F_0(x)$ is constant a.e. with respect to m. By Lemma 1.1 and Proposition 1.2 of Chapter 6 of [11], there exists a set $C \subset S$ with $0 < m(C) < \infty$ such that for every $x \in S$ there exists an $n(x)$ so that $n \geqq n(x)$, $B \subset C$ and $m(B) > 0$ imply $P^n(x, B) > 0$. Suppose that $F_0(x)$ is not constant a.e. with respect to m. Then there is a rational u and a $k \geqq 1$ so that $m\{x: F_0(x) < u,\ n(x) \leqq k\} > 0$ and $m\{x: F_0(x) > u,\ n(x) \leqq k\} > 0$. Let $B = \{y \in C: F_k(y) \leqq u\}$. If $m(B) > 0$, then $P^k(x, B) > 0$ for all x such that $n(x) \leqq k$. Therefore by (2.9), $F_0(x) \leqq u$ a.e. with respect to m on the set $\{x: n(x) \leqq k\}$, which is a contradiction. A similar contradiction is obtained if $m(C \setminus B) > 0$. Therefore with probability one, $F_0(x)$ is independent of x a.e. with respect to m, which proves the lemma.

Theorem 2.10. *Suppose that P is aperiodic. If M is a random measure on S such that $MP = M$ in distribution then $MP = M$ a.s.*

Proof. By Lemma 2.3, there exist random measures M_1 and M_2 so that $M = M_1 + M_2$, M_1 is absolutely continuous with respect to m a.s., and M_2 is singular with respect to m a.s. Let $MP = \tilde{M}_1 + \tilde{M}_2$ be the corresponding decomposition of MP. Then

$$M_1 P + M_2 P = \tilde{M}_1 + \tilde{M}_2. \tag{2.11}$$

$M_1 P \ll m$ a.s. by (2.6), so the uniqueness of the Lebesgue decomposition gives

$$M_1 P \leqq \tilde{M}_1 \quad \text{a.s.} \tag{2.12}$$

Since $MP = M$ in distribution, it follows that

$$\tilde{M}_1 = M_1 \quad \text{in distribution.} \tag{2.13}$$

Thus $M_1 P \leqq M_1$ in distribution by (2.12). Lemma 2.4 then gives $M_1 P = M_1$ a.s. This, together with (2.12) and (2.13) imply that $M_1 = \tilde{M}_1 = M_1 P$ a.s., so that $M_2 P = \tilde{M}_2$ by (2.11). Since $M_2 = \tilde{M}_2$ in distribution, it follows from Lemma 2.1 that $M_2 = 0$ a.s. Thus $M = M_1$ a.s., and finally $MP = M$ a.s.

The requirement that $M(C) < \infty$ a.s. for compact $C \subset S$ in our definition of a random measure was made primarily because of the connection with independent particle systems which is studied in Section 4. Since the invariant measure for a Harris recurrent chain is merely σ-finite, and is not necessarily finite on compact sets, it would be more natural in this section to assume instead that there is a sequence $S_n \uparrow S$ such that $m(S_n) < \infty$ and $M(S_n) < \infty$ a.s. for each n. The results of this section carry through under this assumption with no change. The conclusion of Theorem 2.10 could then be restated in the following way: all solutions of $MP = M$ in distribution are given by $M(dx) = Wm(dx)$, where W is a nonnegative random variable.

3. Entrance Laws

We will begin by exhibiting the relationship between the random invariant measures for P and the stationary processes of entrance laws which are defined in the introduction.

Lemma 3.1. *If $\{\Pi_n, -\infty<n<\infty\}$ is a stationary process of entrance laws for P, then Π_0 satisfies $\Pi_0 P=\Pi_0$ in distribution. Conversely, if M is a random measure on S which satisfies $MP=M$ in distribution, then there exists a stationary process of entrance laws $\{\Pi_n, -\infty<n<\infty\}$ such that $M=\Pi_0$ in distribution.*

Proof. The first statement is immediate. For the converse, suppose that $MP=M$ in distribution, and let $\Pi_n=MP^n$ for $n\geqq 0$. Then $\{\Pi_n, 0\leqq n<\infty\}$ is a one sided stationary process of entrance laws. The space of all Radon measures on S with the topology of vague convergence is a Polish space (Theorem A 7.7 of [6]), so the Kolmogorov extension theorem (Corollary on page 83 of [10]) may be applied to extend $\{\Pi_n, 0\leqq n<\infty\}$ to a two sided stationary process. Since $\Pi_n P=\Pi_{n+1}$ a.s. for $n\geqq 0$, the same will be true for all n for the extended process, and therefore this is a stationary process of entrance laws.

Suppose $l_n(x)$ is a strictly positive Borel function on $Z\times S$, and let $\mathscr{E}$ be the collection of all entrance laws $\pi=\{\pi_n\}$ on S such that

$$\sum_{n=-\infty}^{\infty} \int l_n(x)\,\pi_n(dx)=1. \tag{3.2}$$

Then $\mathscr{E}$ is a convex set. Let $\mathscr{E}_e$ be the extreme points of $\mathscr{E}$. A general entrance law $\pi=\{\pi_n\}$ is called extremal if $\pi_n=\pi_n^1+\pi_n^2$ for entrance laws $\{\pi_n^1\}$ and $\{\pi_n^2\}$ implies that π_n^1 and π_n^2 are constant multiples of π_n. Of course $\mathscr{E}_e$ is the set of all extremal entrance laws which are in $\mathscr{E}$. Noting that an entrance law is simply a stationary measure for the associated space-time process on $Z\times S$, Theorem 12.2 of [5] can be applied to conclude that for each $\pi=\{\pi_n\}\in\mathscr{E}$, there is a unique probability measure ν on $\mathscr{E}_e$ such that

$$\pi_n=\int \tilde{\pi}_n\,\nu(d\tilde{\pi}). \tag{3.3}$$

For the next result, let $\{C_k\}$ be a sequence of compact sets which increase to S.

Lemma 3.4. *Assume that every extremal nonzero entrance law $\tilde{\pi}=\{\tilde{\pi}_n\}$ is of the form $\tilde{\pi}_n=\lambda^n\tilde{\pi}_0$ for some $\lambda=\lambda(\tilde{\pi})>0$. Let π be any entrance law in $\mathscr{E}$, and let ν be the probability measure on $\mathscr{E}_e$ which corresponds to it via* (3.3).

(a) *If $\nu\{\tilde{\pi}:\lambda(\tilde{\pi})>1\}>0$, then $\lim_{n\to\infty}\pi_n(C_k)=\infty$ for some k.*

(b) *If $\nu\{\tilde{\pi}:\lambda(\tilde{\pi})<1\}>0$, then $\lim_{n\to-\infty}\pi_n(C_k)=\infty$ for some k.*

Proof. The proofs of the two parts are similar, so we will prove only (a). Write

$$\begin{aligned}\pi_n(C_k)&=\int \tilde{\pi}_n(C_k)\,\nu(d\tilde{\pi})\\ &=\int [\lambda(\tilde{\pi})]^n\,\tilde{\pi}_0(C_k)\,\nu(d\tilde{\pi}).\end{aligned} \tag{3.5}$$

It follows from (3.2) and the assumption of the lemma that $\tilde{\pi}_0(S)>0$ for all $\tilde{\pi}\in\mathscr{E}$. Therefore $\lim_{k\to\infty}\nu\{\tilde{\pi}:\tilde{\pi}_0(C_k)>0\}=1$, and hence under the assumption of (a), there is a k for which

$$\nu\{\tilde{\pi}:\lambda(\tilde{\pi})>1, \tilde{\pi}_0(C_k)>0\}>0.$$

Applying Fatou's lemma to (3.5) gives $\lim_{n\to\infty}\pi_n(C_k)=\infty$ for that k.

Theorem 3.6. *Assume that every extremal nonzero entrance law $\pi=\{\pi_n\}$ is of the form $\pi_n=\lambda^n\pi_0$ for some $\lambda=\lambda(\pi)>0$. Then $MP=M$ in distribution implies that $MP=M$ a.s.*

Proof. Suppose that M is a random measure on S which satisfies $MP=M$ in distribution. Without loss of generality, we may assume that $M(S)>0$ a.s. Let $\{\Pi_n, -\infty<n<\infty\}$ be the stationary process of entrance laws which is associated with M via Lemma 3.1. By (1.1), there is a strictly positive Borel function $l(x)$ on S such that $\int l(x)M(dx)<\infty$ a.s. Therefore $\int l(x)\Pi_n(dx)<\infty$ a.s. for each n, so there is a positive sequence $\{\alpha_n\}$ for which $W=\sum_{n=-\infty}^{\infty}\alpha_n\int l(x)\Pi_n(dx)<\infty$ a.s. Let $l_n(x)=\alpha_n l(x)$, and consider the corresponding set $\mathscr{E}$ of entrance laws. Then $\{W^{-1}\Pi_n, -\infty<n<\infty\}\in\mathscr{E}$ a.s. Let v_ω be the probability measure on $\mathscr{E}_e$ which corresponds to it via (3.3). If $C\subset S$ is compact, $\{\Pi_n(C)\}$ is a stationary sequence of finite random variables, and therefore the probability that $\lim_{n\to\infty}\Pi_n(C)=\infty$ or that $\lim_{n\to-\infty}\Pi_n(C)=\infty$ is zero. Hence by Lemma 3.4, for a.e. ω, v_ω concentrates on $\{\pi\in\mathscr{E}_e:\lambda(\pi)=1\}$, which is the same as $\{\pi\in\mathscr{E}_e:\pi_n$ is independent of $n\}$. Therefore $\Pi_0=\Pi_1$ a.s., and finally $MP=M$ a.s. since (Π_0,Π_1) and (M,MP) have the same joint distributions.

We will next verify the assumption of Theorem 3.6 in several cases. Ted Cox showed me the proof of the first corollary.

Corollary 3.7. *Suppose that $P=\alpha I+\beta Q$ where $\alpha+\beta=1$, $0<\alpha<1$, and $Q\geqq 0$. Then $MP=M$ in distribution implies that $MP=M$ a.s.*

Proof. Suppose $\{\pi_n\}$ is a nonzero extremal entrance law for P. Then $\pi_{n+1}=\pi_n P=\alpha\pi_n+\beta\pi_n Q\geqq\alpha\pi_n$. Let $\{\tilde{\pi}_n\}$ be the extremal entrance law defined by $\tilde{\pi}_n=\pi_{n+1}$ for each n. Then $\tilde{\pi}_n\geqq\alpha\pi_n$, so that $\tilde{\pi}_n=\alpha\pi_n+(\tilde{\pi}_n-\alpha\pi_n)$ is a representation of $\tilde{\pi}_n$ as a sum of two entrance laws. Since $\tilde{\pi}_n$ is extremal, it follows that $\tilde{\pi}_n=\lambda\pi_n$ for some $\lambda>0$ and all n. Thus $\pi_n=\lambda^n\pi_0$, and the result follows from Theorem 3.6.

Corollary 3.8. *Suppose S is an Abelian group, and $P(x,E)=P(0,E-x)$. Assume that $P(0,dy)$ is not supported by a translate of a proper closed subgroup of S. Then $MP=M$ in distribution implies $MP=M$ a.s.*

Proof. Since entrance laws for P are stationary measures for the associated space-time chain, and since the space-time chain is a random walk on $Z\times S$, we will verify the assumption of Theorem 3.6 by applying Theorem 3 of [1]. In order to apply that theorem, it is necessary to verify that the transition measure for the space-time chain is not supported by a proper closed subgroup of $Z\times S$. To do this, let T be a closed subgroup of $Z\times S$ which contains $\{1\}\times$support $P(0,dy)$. Put $H_n=\{x\in S:(n,x)\in T\}$. Then H_0 is a closed subgroup of S, and $H_1\neq\emptyset$. Let y be any point in H_1. Then $H_n=\{ny+x:x\in H_0\}$ for each $n\in Z$, so that $T=\{(n,ny+x):x\in H_0, n\in Z\}$. Therefore $P(0,dy)$ is supported by $\{y+x, x\in H_0\}$, and hence $H_0=S$ by assumption, and $T=Z\times S$. Using the theorem of Choquet and Deny [1], it follows that if $\{\pi_n\}$ is a nonzero extremal entrance law for P, then $\pi_n(dx)=f(n,x)\,m(dx)$, where m is Haar measure on S and $f(n,x)$ is a continuous function on $Z\times S$ which satisfies $f(k+l,x+y)=f(k,x)f(l,y)$. Putting $x=y=0$ in this gives $f(k,0)=\lambda^k$ for some $\lambda>0$, and then putting $k=y=0$ gives $f(l,x)=\lambda^l f(0,x)$. Therefore $\pi_n=\lambda^n\pi_0$, so Theorem 3.6 applies.

As a final application of Theorem 3.6, consider the renewal chain on $S=\{0,1,2,\ldots\}$ with transition probabilities $P(x,x+1)=1-\varepsilon(x)$ and $P(x,0)=\varepsilon(x)$, where $0<\varepsilon(x)<1$. If $\sum_x \varepsilon(x)=\infty$, then the chain is recurrent, and the results of Section 2 apply. Therefore, we will assume that $\sum_x \varepsilon(x)<\infty$. In this case the chain is transient, and it is easy to check that it has no nonzero invariant measure. It is therefore clear that there is no nonzero random measure M on S such that $MP=M$ in distribution and $EM(x)<\infty$ for each x. It is not so obvious that there is no random invariant measure with infinite mean. However, we will deduce this from Theorem 3.6. Cox [2] has proved that this chain has no entrance laws $\{\pi_n\}$ for which $\sum_x \pi_n(x)<\infty$. We will need to consider infinite entrance laws as well.

Corollary 3.9. *There is no nonzero random measure M such that $MP=M$ in distribution for the above chain.*

Proof. Let $u(0)=1$ and $u(x)=\prod_{y=0}^{x-1}[1-\varepsilon(y)]$ for $x\geqq 1$. Note that $u(x)\downarrow c>0$ since $\sum_y \varepsilon(y)<\infty$. Let $\{\pi_n\}$ be any entrance law for P and define $\gamma_n(x)=\pi_n(x)/u(x)$. Then

$$\gamma_{n+1}(x)=\gamma_n(x-1) \tag{3.10}$$

for $x\geqq 1$, and

$$\gamma_{n+1}(0)=\sum_{x=0}^{\infty}[u(x)-u(x+1)]\,\gamma_n(x). \tag{3.11}$$

Let $v(n)=\gamma_n(0)$. By (3.10), $\gamma_n(x)=v(n-x)$ for $x\in S$ and $n\in Z$. Therefore by (3.11),

$$v(n+1)=\sum_{x=0}^{\infty}[u(x)-u(x+1)]\,v(n-x). \tag{3.12}$$

By the theorem of Choquet and Deny [1], the extremal positive solutions of (3.12) are of the form $v(n)=\alpha\lambda^n$ for some $\alpha>0$ and some $\lambda>0$ for which

$$\lambda=\sum_{x=0}^{\infty}[u(x)-u(x+1)]\,\lambda^{-x}. \tag{3.13}$$

There is at most one solution of (3.13), and if there is a solution, it is not 1, since $\sum_{x=0}^{\infty}[u(x)-u(x+1)]=1-c<1$. Therefore either P has no entrance laws, or all entrance laws for P are of the form $\pi_n(x)=\alpha\lambda^{n-x}u(x)$ where λ is the solution to (3.13). By Theorem 3.6, $MP=M$ in distribution implies that $MP=M$ a.s. Since P has no invariant measure, it follows that there is no nonzero solution of $MP=M$ in distribution.

The above example gives some additional evidence for Conjecture 1.6. A related observation, which indicates some of the difficulties involved in proving the conjecture is the following: Suppose in the above example, the $\varepsilon(x)$ are chosen so that $\sum_{x=0}^{\infty}[u(x)-u(x+1)]^{\alpha}=1$ for some $\alpha\in(0,1)$, and let $\{W(x), x\in S\}$ be independent and identically distributed positive random variables with the one-sided stable distribution of exponent α. Then it is not hard to check that $M(x) = u(x)\,W(x)$ satisfies $MP(x)=M(x)$ in distribution for each x, although of course the joint distributions are not the same. Thus while proving the conjecture in the trivial case when M has finite mean requires only $MP(x)=M(x)$ in distribution for each x, proving it when M has infinite mean will require that the joint distributions be equal as well.

One case in which the conjecture can be easily verified is that in which the random measure M satisfies $M(S)<\infty$ a.s. In this case, when $M(dx)$ is conditioned on $\{M(S)\leqq\alpha\}$ for some large α, one obtains a new random invariant measure $\tilde{M}$ for which $\tilde{M}(S)\leqq\alpha$ a.s. The mean is then finite, so one can let $m = E\tilde{M}$. Of course if the chain is irreducible, it then follows that it is positive recurrent since $m(S)<\infty$, so the results of Section 2 apply to give all solutions of $MP=M$ in distribution.

We will next give a general sufficient condition under which all random invariant measures M for P with finite mean satisfy $MP=M$ a.s. Note that each element m of $\mathscr{I}$ can be thought of as an entrance law by letting $\pi_n(dx)=m(dx)$ for each n. However, it is not necessarily the case that every extremal element of $\mathscr{I}$ is an extremal entrance law.

Theorem 3.14. *Suppose that*

$$\textit{every extremal element of } \mathscr{I} \textit{ is an extremal entrance law.} \tag{3.15}$$

If M is a random measure on S such that $EM(C)<\infty$ for all compact $C\subset S$, and $MP=M$ in distribution, then $MP=M$ a.s.

Proof. Assume that $M\neq 0$ a.s. Let $\{\Pi_n, -\infty<n<\infty\}$ be the stationary process of entrance laws which is associated with M via Lemma 3.1, and let $m=EM = E\Pi_n$ for each n. Writing the expectation as an integral over the probability space on which $\{\Pi_n\}$ is defined, exhibits m as an average of the entrance laws $\{\Pi_n\}$. Let $l(x)$ be a strictly positive function on S for which $\int l(x)\,m(dx)=1$, let α_n be positive numbers such that $\sum_{n=-\infty}^{\infty}\alpha_n=1$, and put $l_n(x)=\alpha_n\,l(x)$. Then

$$E\sum_{n=-\infty}^{\infty}\alpha_n\int l(x)\,\Pi_n(dx)=\int l(x)\,m(dx)=1,$$

so that with probability one, the entrance law Π_n is a multiple of an entrance law in $\mathscr{E}$. By Theorem 12.2 of [5], both $\mathscr{E}$ and $\{m\in\mathscr{I}:\int l(x)\,m(dx)=1\}$ are Choquet simplexes, so that by (3.15), Π_n is independent of n a.s., which gives the desired conclusion.

We conjecture that Theorem 3.14 is true without the assumption that $EM(C)<\infty$ for compact C. Removing that condition, however, seems to involve the same difficulties as the proof of Conjecture 1.6. We further conjecture that (3.15) is both a necessary and a sufficient condition for $MP=M$ in distribution to imply $MP=M$ a.s.

4. Independent Particle Systems

This section is devoted to the relationship between the random invariant measures for P and the invariant probability measures for $\{\eta_n\}$. We will use the following two identities, which are valid for nonnegative Borel functions f on S. They are proved first for simple functions f, and then the general case is obtained by passing to the limit using the monotone and dominated convergence theorems. For (4.2), see for example page 8 of [6].

$$E^\eta \exp[-\int f(x)\,\eta_n(dx)] = \exp\{\int \log[\int e^{-f(y)} P^n(x,dy)]\,\eta(dx)\} \tag{4.1}$$

$$\int \exp[-\int f(x)\,\eta(dx)]\,\mu_M(d\eta) = E\exp[-\int (1-e^{-f(x)})\,M(dx)]. \tag{4.2}$$

The following result is Corollary 3.2 of [6]. Lemma 4.4 has appeared elsewhere (see Satz 5.4.4 of [16], for example), but we include a proof for the sake of completeness.

Lemma 4.3. *If M_1 and M_2 are random measures on S, then $\mu_{M_1}=\mu_{M_2}$ if and only if $M_1=M_2$ in distribution.*

Lemma 4.4. *If η_0 has distribution μ_M, then η_1 has distribution μ_{MP}.*

Proof. Suppose η_0 has distribution μ_M, let f be a nonnegative Borel function on S, and let

$$g(x) = -\log \int e^{-f(y)} P(x,dy) \geqq 0.$$

Then

$$\begin{aligned} E^{\mu_M} \exp[-\int f(x)\,\eta_1(dx)] &= \int E^\eta\{\exp[-\int f(x)\,\eta_1(dx)]\}\,\mu_M(d\eta) \\ &= \int \exp[-\int g(x)\,\eta(dx)]\,\mu_M(d\eta) \\ &= E\exp[-\int (1-e^{-g(x)})\,M(dx)], \end{aligned}$$

where the last two equalities follow from (4.1) and (4.2) respectively. On the other hand, by (4.2) and Fubini's theorem,

$$\begin{aligned} \int \exp[-\int f(x)\,\eta(dx)]\,\mu_{MP}(d\eta) &= E\exp[-\int (1-e^{-f(y)})\,MP(dy)] \\ &= E\exp[-\int\int (1-e^{-f(y)})\,M(dx)\,P(x,dy)] \\ &= E\exp[-\int (1-e^{-g(x)})\,M(dx)]. \end{aligned}$$

Therefore

$$E^{\mu_M} \exp[-\int f(x)\,\eta_1(dx)] = \int \exp[-\int f(x)\,\eta(dx)]\,\mu_{MP}(d\eta)$$

for all $f \geqq 0$, and hence η_1 has distribution μ_{MP} by Theorem 3.1 of [6].

Corollary 4.5. μ_M is invariant for $\{\eta_n\}$ if and only if $MP=M$ in distribution.

The following theorem may be known, but I have not been able to find an explicit statement of it in the literature. Of course, both the result and its proof are very close to the classical theorem which states that the only possible limit distributions for the partial sums from infinitesimal triangular arrays of independent Bernoulli random variables are the Poisson distributions. Assumption 4.7 is the infinitesimal condition.

Theorem 4.6. *Suppose*

$$\lim_{n\to\infty} \sup_{x\in S} P^n(x; C)=0 \tag{4.7}$$

for every compact $C\subset S$. Then every invariant probability measure on $\mathscr{M}$ for $\{\eta_n\}$ is of the form μ_M for some random measure M on S.

Proof. Suppose μ is an invariant probability measure on $\mathscr{M}$ for $\{\eta_n\}$. Let f be a nonnegative continuous function on S with compact support, and let

$$g_n(x) = -\log \int e^{-f(y)} P^n(x, dy) \geqq 0.$$

By (4.1) and the invariance of μ,

$$\begin{aligned}\int \exp[-\int f(x)\eta(dx)]\,\mu(d\eta) &= \int E^\eta\{\exp[-\int f(x)\eta_n(dx)]\}\,\mu(d\eta)\\ &= \int \exp\{-\int g_n(x)\eta(dx)\}\,\mu(d\eta).\end{aligned}$$

By (4.7) and the fact that f has compact support, $\int e^{-f(y)} P^n(x,dy)\to 1$ as $n\to\infty$ uniformly in x. Therefore there are $\varepsilon_n \to 0$ so that

$$(1-\varepsilon_n)\int [1-e^{-f(y)}]\,P^n(x,dy) \leqq g_n(x) \leqq (1+\varepsilon_n)\int [1-e^{-f(y)}]\,P^n(x,dy).$$

Thinking of $(\mathscr{M},\mu)$ as a probability space, we may define a sequence M_n of random measures on S by

$$M_n(dy) = \int P^n(x,dy)\,\eta(dx).$$

Since μ is invariant and concentrates on $\mathscr{M}$, $\eta_n(C)<\infty$ a.s. for a.e. η_0 with respect to μ and for every compact set $C\subset S$. Noting that $M_n(C) = \int P^n(x, C)\,\eta(dx) = E^\eta \eta_n(C)$, it follows that M_n satisfies (1.1), since for each η, $\eta_n(C)$ is a sum of independent Bernoulli random variables. Therefore

$$\int \exp[-\int f(x)\eta(dx)]\,\mu(d\eta) = \lim_{n\to\infty} E\exp\{-\int [1-e^{-f(y)}]\,M_n(dy)\}. \tag{4.8}$$

Making the transformation $h(y)=1-e^{-f(y)}$, we obtain

$$\int \exp[\int \log[1-h(x)]\,\eta(dx)]\,\mu(d\eta) = \lim_{n\to\infty} E\exp\{-\int h(y)\,M_n(dy)\}$$

for all continuous functions h with compact support which satisfy $0\leqq h<1$. Replacing h by εh yields

$$\int \exp[\int \log[1-\varepsilon h(x)]\,\eta(dx)]\,\mu(d\eta) = \lim_{n\to\infty} E\exp[-\varepsilon \int h(y)\,M_n(dy)] \tag{4.9}$$

for all nonnegative continuous functions h with compact support and sufficiently small $\varepsilon>0$. Since μ concentrates on $\mathscr{M}$, the left side of (4.9) tends to one as $\varepsilon\to 0$. Therefore, by the convergence theorem for Laplace transforms, $\int h(y)\,M_n(dy)$ converges in distribution as $n\to\infty$ for each such function h. By Lemma 5.1 of

[6], it follows that there is a random measure M such that $\int h(y) M_n(dy)$ converges in distribution to $\int h(y) M(dy)$ for each such h. Taking limits in (4.8) gives

$$\int \exp[-\int f(x)\,\eta(dx)]\,\mu(d\eta) = E\exp\{-\int [1-e^{-f(y)}]\,M(dy)\},$$

and therefore $\mu = \mu_M$ by (4.2) and Theorem 3.1 of [6].

Corollary 4.10. *Assume that P satisfies* (4.7). *Then the invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ are given exactly by the class $\{\mu_M : MP = M$ in distribution$\}$. Therefore the extremal invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ are exactly $\{\mu_m : m \in \mathscr{I}\}$ if and only if $MP = M$ in distribution implies $MP = M$* a.s.

As was pointed out in the introduction, condition (4.7) is not always satisfied, and is required for the conclusions of Theorem 4.6 and Corollary 4.10 to hold. It does appear, however, to be a rather weak assumption. We will now verify it for some of the chains considered in Sections 2 and 3, so that those results can be combined with Corollary 4.10 to find all the invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ in these cases.

One situation in which (4.7) is easy to verify is that in which S is countably infinite, and $P(x,y)$ is symmetric and irreducible, since then

$$[\sup_x P^n(x,y)]^2 \leqq \sum_x [P^n(x,y)]^2 = P^{2n}(y,y) \to 0$$

for each $y \in S$. Thus if in addition, P is aperiodic and recurrent, it follows from Theorem 2.10 and Corollary 4.10 that the extremal invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ are exactly $\{\mu_{cm}, c \geqq 0\}$ where m is the counting measure on S.

Theorem 4.11. *Suppose S is a noncompact Abelian group, and $P(x,E) = P(0, E-x)$. Assume that $P(0,dy)$ is not supported by a translate of a proper closed subgroup of S. Then the extremal invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ are exactly $\{\mu_m : m \in \mathscr{I}\}$.*

Proof. Let C be a compact subset of S, and $P_s(x,dy)$ be the transition probabilities for the symmetrized random walk: $P_s(0,dy)$ is the convolution of $P(0,dy)$ and $P(0,-dy)$. Then

$$\sup_x [P^n(x,C)]^2 \leqq P_s^n(0, C-C).$$

By the assumption, $P_s(0,dy)$ is not supported by a compact subgroup of S, and therefore $\lim_{n\to\infty} P_s^n(0, C-C) = 0$. This last fact follows, for example, from Corollary 1 and Proposition 2 of [14]. Hence (4.7) holds, and Corollaries 3.8 and 4.10 give the desired conclusion.

$\mathscr{I}$ is known completely in the situation covered by Theorem 4.11 ([1]), so in this case, the invariant probability measures on $\mathscr{M}$ for $\{\eta_n\}$ are completely determined. In particular, it should be noted there are often invariant probability measures for $\{\eta_n\}$ which are not translation invariant. This should be kept in mind in comparing Theorem 4.11 with other results in this situation ([3, 9, 13]) which essentially consider only invariant probability measures which are translation invariant.

The next result deals with a case which is very natural from the point of view of infinite interacting particle systems. In that context, the basic one-particle motion is often a continuous time Markov chain X_t in which the parameters of the exponential holding times are uniformly bounded on S. The $P(x, dy)$ which would occur when our results are applied in that context is then $P^x[X_t \in dy]$ for some fixed t. Thus, for example, Theorem 4.12 solves the problem of finding all the invariant probability measures for Spitzer's zero range interaction process [12] in case his speed function $c_k(x)$ is identically one, which corresponds to the absence of interaction.

Theorem 4.12. *Suppose that* (a) $P = \alpha I + \beta Q$ *where* $\alpha + \beta = 1$, $0 < \alpha < 1$, *and* $Q \geqq 0$, (b) *P has the Feller property in the sense that Pf is continuous whenever f is continuous with compact support, and* (c) *P has no finite invariant measure. Then the extremal invariant probability measures on* $\mathscr{M}$ *for* $\{\eta_n\}$ *are exactly* $\{\mu_m : m \in \mathscr{I}\}$.

Proof. Let $m_n(dy)$ be the pointmass at x_n, where $\{x_n\}$ is any sequence of points in S. Then

$$m_n P^n = \sum_{k=0}^{n} \binom{n}{k} \beta^k \alpha^{n-k} m_n Q^k,$$

and

$$m_n P^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \beta^k \alpha^{n+1-k} m_n Q^k,$$

so that $m_n P^n - m_n P^{n+1} \to 0$ in total variation. Therefore by (b), $mP \leqq m$ for any vague limit m of a subsequence $\{m_{n_k}\}$. Since m is a finite measure, it follows that $mP = m$, and hence by (c) that $m = 0$. Thus $m_n P^n \to 0$ vaguely, so that $P^n(x_n, C) \to 0$ for each compact $C \subset S$. This gives (4.7), so that the desired conclusion follows from Corollaries 3.7 and 4.10.

Finally, combining Theorem 2.10 and Corollary 4.10 gives the following result.

Theorem 4.13. *Suppose P is an aperiodic Harris recurrent chain which satisfies* (4.7). *Then the extremal invariant probability measures on* $\mathscr{M}$ *for* $\{\eta_n\}$ *are exactly* $\{\mu_m : m \in \mathscr{I}\}$.

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Received February 21, 1978; in revised form June 8, 1978