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Changes of Filtrations and of Probability Measures

Pierre Brémaud[1] and Marc Yor[2]

[1] IRIA/Laboria, Domaine de Voluceau, 78150-Le Chesnay, (France)
[2] Laboratoire de Probabilités associé au C.N.R.S. n° 224, Université Pierre et Marie Curier, 75230 Paris Cedex 05, (France)

Notation

If $(E, \mathscr{E})$ is a measurable space, $b(\mathscr{E})$ denotes the space of bounded real $\mathscr{E}$ measurable functions.

Let (Ω, F, P) be a complete probability space. It will generally be endowed with P-complete and right-continuous filtrations $\mathscr{F} = (\mathscr{F}_t, t \geqq 0)$ or $\mathscr{G} = (\mathscr{G}_t, t \geqq 0)$.

• $\mathscr{P}(\mathscr{F})$ (resp: $\mathscr{O}(\mathscr{F})$) is the predictable (resp: optional) σ-field on $\Omega \times \mathbb{R}_+$ associated with $\mathscr{F}$.

• To a real valued, measurable process X on (Ω, F), we associate the natural filtration $\mathscr{F}^X$, once P-completed and made right continuous.

• We do not distinguish between two measurable processes that differ only on a P-evanescent set.

We use the notations of Dellacherie [2] for projections relative to a filtration: for instance, if there is no possible confusion about the filtration or the probability with respect to which such projections are taken, we denote by ${}^1X$(resp: ${}^3X$) the optional (resp: predictable) projection of X; if A is an increasing process, A^3 is the dual predictable projection of A. If the filtration $\mathscr{F}$ (or the probability P, or both) needs to be made precise, we write for instance: ${}^{1\mathscr{F}}X$, ${}^{3\mathscr{F}}X$; ${}^{1P/\mathscr{F}}X$, ${}^{3P/\mathscr{F}}X$; A^{P3}.

If X is a $\mathscr{F}$-semi-martingale, $\mathscr{F}.\int H\,dX$ is the stochastic integral of H relative to X, via the filtration $\mathscr{F}$. If this integral does not depend on $\mathscr{F}$, we may suppress the letter $\mathscr{F}$.

If $\mathscr{C}$ is a class of processes, we note $\mathscr{C}^0 = \{C \in \mathscr{C} \mid C_0 = 0\}$ and $\mathscr{C}_{\text{loc}}$ consists of all processes C such that there is a sequence T_n of stopping times, increasing P a.e. to ∞, and $C^{T_n} = C_{\cdot \wedge T_n}$ (if $C \in \mathscr{C}_{\text{loc}}$ we also write: C is locally in $\mathscr{C}$).

$\mathscr{M}_{\text{loc}}$ (or: $\mathscr{M}_{\text{loc}}(P, \mathscr{F})$) is the space of P-local martingales relative to $\mathscr{F}$, and $\mathscr{M}^2$ (or: $\mathscr{M}^2(P, \mathscr{F})$) the subspace of square integrable $(P, \mathscr{F})$ martingales, H^1 (or: $H^1(P, \mathscr{F})$) the space of martingales X such that $X^*_\infty = \sup_{t \geqq 0} |X_t| \in L^1$. Let us recall ([10]) that $\mathscr{M}_{\text{loc}} = H^1_{\text{loc}}$.

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Finally, increasing processes and stochastic integrals are always supposed to vanish at the origin.

0. Outline

The original aim of this work is to generalize a result of Communications theory known as the theorem of *separation of detection and filtering*, and to set out minimal assumptions under which a recursive filtering formula can be obtained. This project has presented several natural questions on stochastic calculus. As a result the present article is constructed as follows:

In Section 1, we consider the unique solution $D(\varphi)$ of Doléans-Dade's equation ([3]):

$$D(\varphi)=1+\int D(\varphi)_-\,\varphi\,dX$$

when X is a $(P,\mathscr{F})$ martingale and $\varphi\in\mathscr{P}(\mathscr{F})$. Moreover, we suppose that $D(\varphi)$ is a non-negative uniformly integrable martingale so that we can define a new probability Q by: $dQ=D(\varphi)_\infty\,dP$. Under certain conditions, mainly:

a) the existence of $\langle X,X\rangle^{P,\mathscr{F}}$ such that $\langle X,X\rangle^{P,\mathscr{F}}\in\mathscr{P}(\mathscr{F}^X)$

b) X has the predictable representation property, e.g.: all $(P,\mathscr{F}^X)$ local martingales can be represented as: $a+\int H\,dX$, with $a\in\mathbb{R}$, $H\in\mathscr{P}(\mathscr{F}^X)$,

we obtain:

$$^{1P}(D(\varphi))=D(^{3Q}\varphi), \qquad (*)$$

or equivalently:

$$^{1P}(D(\varphi))=1+\int {}^{1P}(D(\varphi))_-\,{}^{3Q}\varphi\,dX,$$

all projections being taken with respect to $\mathscr{G}=\mathscr{F}^X$.

This result is a separation result in that we obtain the likelihood ratio of Q with respect to P relative to the filtration $\mathscr{F}^X$ of observations (that is to say $^{1P}(D(\varphi))$) in two steps: first, we estimate or "filter" φ relatively to $\mathscr{F}^X$ and Q, and then we enter the predictable version $^{3Q}\varphi$ in the operator D. For communications engineers, φ is the signal and X the observation corrupted additively by a noise B defined by:

$$X=\int\varphi\,d\langle X,X\rangle^{P,\mathscr{F}}+B$$

(from Van Schuppen and Wong [12], B is a $(Q,\mathscr{F})$ local martingale).

In the course of obtaining (*), some natural questions concerning the optional projections of semi-martingales and stochastic integrals arise. They are answered in Section 2, using the Hilbert space theory of square integrable martingales and stable subspaces. More precisely, for two filtrations $\mathscr{F}$ and $\mathscr{G}$ such that $\mathscr{G}\subset\mathscr{F}$, we consider the $\mathscr{G}$-stable subspace $\mathscr{L}$ of square integrable martingales with respect to $\mathscr{F}$ *and* $\mathscr{G}$. For $M\in\mathscr{L}$, and $H\in\mathscr{O}(\mathscr{F})$ such that

$E\int_0^\infty H_s^2\,d[M,M]_s<\infty$, we have:

$$P_{\mathscr{L}}{}^1(\mathscr{F}.\int H\,dM)=P_{\mathscr{L}}(\mathscr{G}.\int {}^1H\,dM) \tag{**}$$

where $P_{\mathscr{L}}$ is the Hilbert projector on $\mathscr{L}$ and optional projections are relative to $\mathscr{G}$. Moreover, we consider the subset $\mathscr{L}'$ of $\mathscr{L}$, consisting of martingales M such that their $\mathscr{F}$-predictable increasing process is adapted to $\mathscr{G}$, or, equivalently, such that their $\mathscr{F}$ and $\mathscr{G}$-predictable increasing processes coincide.

If $\mathscr{L}'$ is a (stable) subspace of $\mathscr{L}$, and $H\in\mathscr{P}(\mathscr{F})$ is such that $E\int_0^\infty H_s^2\,d[M,M]_s<\infty$, we have:

$$P_{\mathscr{L}'}{}^1(\mathscr{F}.\int H\,dM)=\mathscr{F}.\int {}^3H\,dM=\mathscr{G}.\int {}^3H\,dM. \tag{***}$$

The results of Section 1 are a particular case of those of Section 2, if we remark that, under the hypothesis of Section 1, for $\mathscr{G}=\mathscr{F}^X$, $M=X$, $H=D(\varphi)_-\,\varphi$, we have $\mathscr{L}=\mathscr{L}'=\mathscr{M}^2(P,\mathscr{G})$, and we use the formula:

$$^{1P}(D(\varphi))_-\;{}^{3Q}\varphi={}^{3P}(D(\varphi)_-\,\varphi). \tag{****}$$

However, the computational method of Section 1, where predictable and optional projections are defined as Radon-Nikodym derivatives, has an independent interest, it may be carried out in situations not covered in the present paper, (see [1] for the case of an observation which contains continuous martingales and marked point processes; we also have in mind a possible extension to the two parameter filtering problem [13], for which projection theorems similar to those in Dellacherie [2] are not (yet) available.)

Section 3 is devoted to the filtering problem. It is a generalization, on one hand, of results of Duncan [5], and Zakaï [17], where an approach to the filtering problem is made via the reference probability method. On the other hand, we obtain a unified recursive filtering equation extending Kunita's equation to the case where X is only supposed to have the predictable representation property (in Kunita [8], or [1], X is a Wiener process, which is well known to have this property).

Section 2 is technically independent of Sections 1 and 3.

1. Projections of Martingale Exponentials

1.1. Preliminaries

If $(E,\mathscr{E})$ is a measurable space, and μ a positive finite measure on $(E,\mathscr{E})$, the conditional expectation $\mu(h|\mathscr{B})=\left(\frac{\mu}{\mu(1)}\right)(h|\mathscr{B})$ is well defined and known, for h in $L^1(E,\mathscr{E},\mu)$ or $L^2(E,\mathscr{E},\mu)$, and $\mathscr{B}$ any sub-σ-field of $\mathscr{E}$. If μ is positive, and $\mathscr{B}$ σ-finite (e.g.: there exists an increasing sequence (B_n), $B_n\in\mathscr{B}$, such that: $E=\bigcup_n B_n$

and $\mu(B_n)<\infty$ for every n), and if $h\in L^1(E,\mathscr{E},\mu)$, the formulas $1_{B_n}\mu(h|\mathscr{B})=1_{B_n}$ $\mu_n(h|B)$ where $\mu_n(A)=\mu(A\cap B_n)(A\in\mathscr{E})$, are consistent and define $(\mu(h|\mathscr{B})$. A similar argument allows us to define $\mu(h|\mathscr{B})$ if h is $\mathscr{B}$-locally in $L^1(E,\mathscr{E},\mu)$, e.g.: there exists $B_n\in\mathscr{B}$, increasing to E, such that $1_{B_n}h\in L^1(E,\mathscr{E},\mu)$ for every n. If $h\in L^2(E,\mathscr{E},\mu)$, $\mu(h|\mathscr{B})$ is simply defined as the L^2-projection of h on $L^2(E,\mathscr{B},\mu)$. This extends as well if h is $\mathscr{B}$ locally in $L^2(E,\mathscr{E},\mu)$.

We shall apply the above remarks in the following setting: (Ω,F,P) is a probability space, with $\mathscr{F}=(\mathscr{F}_t,t\geqq 0)$ a filtration of sub σ-fields of F, satisfying the usual conditions. Let X be an element of $\mathscr{M}^2_{\text{loc}}(P,\mathscr{F})$, with the associated increasing $\mathscr{F}$-predictable process $\langle X,X\rangle$ $(=\langle X,X\rangle^{P,\mathscr{F}})$. We suppose that the following hypothesis is verified:

$$\langle X,X\rangle \quad \text{is } \mathscr{F}^X \text{ predictable.} \tag{H.1}$$

Remarks. 1. E. Lenglart has pointed out to us that (H.1) implies:

$$X\in\mathscr{M}^2_{\text{loc}}(P,\mathscr{F}^X).$$

Indeed, let S_m be a sequence of $\mathscr{F}^X$ stopping times increasing P a.e. to ∞, such that $\langle X,X\rangle_{\cdot\wedge S_m}$ is bounded. Existence of such a sequence follows from (H.1). By Doob's inequality ([11], VI, 1), we get:

$$E[\sup_t X^2_{t\wedge S_m}]\leqq 4E[\langle X,X\rangle_{S_m}]$$

and this terminates the proof.

2. (H.1) is automatically verified if X is continuous, since then:

$$\langle X,X\rangle_t=P\text{-}\lim_{(n\to\infty)}\sum_{\tau_n}(X_{t_{i+1}}-X_{t_i})^2$$

where τ_n is a sequence of refining subdivisions of $[0,t]$, the mesh of which decreases to 0. The validity of H.1 will be more thoroughly investigated in Subsection 2.3. ■

Let $\varphi\in\mathscr{P}(\mathscr{F})$ be such that:

$$E\int_0^\infty \varphi_s^2\,d\langle X,X\rangle_s<\infty, \tag{H.2}$$

and define L as the unique ([3]) solution of:

$$L_t=1+\int_0^t L_{s-}\,\varphi_s\,dX_s \tag{1.1}$$

From [3], it is known that:

$$L_t=\exp\left(\int_0^t\varphi_s\,dX_s-\tfrac{1}{2}\int_0^t\varphi_s^2\,d\langle X^c,X^c\rangle_s\right)\prod_{s\leqq t}(1+\varphi_s\Delta X_s)\exp(-\varphi_s\Delta X_s)$$

From (H.2), $L\in\mathscr{M}^2_{\text{loc}}(P,\mathscr{F})$, but we shall need the additional assumptions:

$$L\in\mathscr{M}^2(P,\mathscr{F}) \quad\text{and:}\quad \varphi\,\Delta X+1\geqq 0 \tag{H.3}$$

L is now a non-negative martingale, with mean 1, so that we can define a probability Q on (Ω, F) by:

$$dQ = L_\infty dP. \tag{1.2}$$

Moreover, it follows from Van Schuppen-Wong [12] that:

$$M = X - \int \varphi d\langle X, X\rangle \in \mathcal{M}_{\mathrm{loc}}(Q, \mathcal{F}) \tag{1.3}$$

We now make use of the first part of the preliminaries with $E = \Omega \times \mathbb{R}_+$, $\mathscr{E} = F \otimes \mathscr{B}(\mathbb{R}_+)$, $d\mu(s,\omega) = d\langle X, X\rangle_s(\omega) dP(\omega)$, and $\mathscr{B} = \mathscr{P}(\mathscr{F}^X)$. We first remark that $L_- \varphi$ belongs to $L^2(E, \mathscr{E}, \mu)$ since

$$E \int_0^\infty (L_{s-} \varphi_s)^2 d\langle X, X\rangle_s = E(L_\infty - 1)^2 < \infty.$$

To show that φ is $\mathscr{B}$-locally in $L^1(E, \mathscr{E}, \mu^Q)(\mu^Q = d\langle X, X\rangle dQ)$, we use the stopping times S_m of Remark 1:

$$E_Q \int_0^{S_m} |\varphi_s| d\langle X, X\rangle_s = E_P \left[L_\infty \int_0^{S_m} |\varphi_s| d\langle X, X\rangle_s \right]$$
$$\leqq (E_P(L_\infty^2))^{\frac{1}{2}} \left\{ E_P \left(\int_0^{S_m} |\varphi_s| d\langle X, X\rangle_s \right)^2 \right\}^{\frac{1}{2}}$$

and

$$E_P \left(\int_0^{S_m} |\varphi_s| d\langle X, X\rangle_s \right)^2 \leqq E_P \left[\left(\int_0^{S_m} (\varphi_s)^2 d\langle X, X\rangle_s \right) \langle X, X\rangle_{S_m} \right] < \infty.$$

We shall note $\mu(L_- \varphi | \mathscr{B}) = {}^{3P}(L_- \varphi)$ and $\mu^Q(\varphi | \mathscr{B}) = {}^{3Q}\varphi$, thereby indicating the obvious links with Dellacherie's predictable projections.

Remark. In [1] and [21], such a definition of the predictable projection is used. It is interesting to show how these previous predictable projections are extensions of Dellacherie's. Suppose for simplicity that $X \in \mathcal{M}^2(P, \mathscr{F})$. If we use the notations of [2] for $(P, \mathscr{F}^X)$-projections, we have, from the predictable section theorem: $\forall H \in b(F \otimes \mathscr{B}(\mathbb{R}_+))$, $|{}^3H| \leqq {}^3|H|$ and $({}^3H)^2 \leqq {}^3(H^2)$ outside a P evanescent set; thus, the application $H \to {}^3H$ defined on $b(F \otimes \mathscr{B}(\mathbb{R}_+))$ extends uniquely in a linear contraction from $L^{1(2)}(F \otimes \mathscr{B}(\mathbb{R}_+), \mu)$ to $L^{1(2)}(\mathscr{P}(\mathscr{F}^X), \mu)$.

1.2. The Separation Theorem

The following representation hypothesis is crucial for our purpose:

$$L^2(\mathscr{F}_\infty^X, P) = \left\{ a + \int_0^\infty H_s dX_s \,|\, a \in \mathbb{R}, H \in \mathscr{P}(\mathscr{F}^X);\ E_P \int_0^\infty H_s^2 d\langle X, X\rangle_s < \infty \right\} \tag{H.4}$$

This representation property is the subject of [14, 15 and 7 (see in particular Theorem 1.5)]; when it is verified, we say that X has the predictable representation property. Let us point out that (H.4) immediately implies a representation of square integrable martingales as stochastic integrals. Moreover, we know from [15 and 7] that (H.4) also implies the following representation of *local* martingales:

Proposition 1. (H.4) *is equivalent to: every local* $(P, \mathscr{F}^X)$ *martingale* M *can be written as:*

$$M_t = a + \int_0^t H_s \, dX_s$$

with $a \in \mathbb{R}_+$, *and* $H \in \mathscr{P}(\mathscr{F}^X)$ *such that* $\left(\int_0^{\cdot} H_s^2 \, d[X,X]_s\right)^{\frac{1}{2}}$ *is* $(P, \mathscr{F}^X)$ *locally integrable.*

We now state the main result of this section. *All notions involved in the next theorem are relative to the filtration* $\mathscr{F}^X$.

Theorem 1. *Let* ${}^{1P}L$ *be a right continuous version of the* $(P, \mathscr{F}^X)$ *martingale* $E_P\left(\frac{dQ}{dP}\middle|\mathscr{F}_t^X\right)$. *Then, under* H.1, H.2, H.3, H.4:

$$ {}^{1P}L = 1 + \int {}^{3P}(L_- \varphi) \, dX \tag{1.4}$$

$$ = 1 + \int ({}^{1P}L)_- \, {}^{3Q}\varphi \, dX \tag{1.5}$$

(equivalently: ${}^{1P}D(\varphi) = D({}^{3Q}\varphi)$*)*

Moreover, $\tilde{X} = X - \int {}^{3Q}\varphi \, d\langle X, X\rangle \in \mathscr{M}_{\text{loc}}(Q, \mathscr{F}^X)$

Remarks. 1. The notations of Theorem 1 are consistent since ${}^{1P}L$ is indeed the optional projection of L onto $\mathscr{F}^X$ in the sense of [2].

2. In the next section, formula (1.4) will appear as a particular case of a general projection formula of stochastic integrals.

3. From [24] (Theorem 3.1), we know that the hypothesis (H.4) implies:

$$L^2(\mathscr{F}_\infty^X, Q) = \left\{ \int_0^\infty H_s \, d\tilde{X}_s \,\middle|\, H \in \mathscr{P}(\mathscr{F}^X), E \int_0^\infty H_s^2 \, d\langle \tilde{X}, \tilde{X}\rangle^{Q, \mathscr{F}^X} < \infty \right\} \tag{H.$\tilde{4}$}$$

This result will only be used in the sequel as a remark concerning the Equation (3.7) (Theorem 4); it appears in Kunita [18], when X is continuous, and is also known when $X = N - A$ is the martingale process obtained from the point process N_t with stochastic intensity A_t.

Proof of Theorem 1. From the end of Subsection 1.1, the two sides of (1.4) are well defined. In order to prove this formula, it is sufficient to verify:

$$E_P({}^{1P}L_t \, V_t) = E_P\left[\left(1 + \int_0^t {}^{3P}(L_- \varphi) \, dX\right) V_t\right]$$

for all square integrable $(P, \mathscr{F}^X)$ martingales V, with $V_0 = 0$. By H.4, V can be written as $\int H\,dX$, with $H \in \mathscr{P}(\mathscr{F}^X)$. Therefore:

$$\begin{aligned} E_P[{}^{1P}L_t V_t] &= E_P[L_t V_t] \\ &= E_P\left[\left(\int_0^t L_- \varphi\, dX\right)\left(\int_0^t H\, dX\right)\right] \\ &= E_P\left[\int_0^t H L_- \varphi\, d\langle X, X\rangle\right] \\ &= E_P\left[\int_0^t H\, {}^{3P}(L_- \varphi)\, d\langle X, X\rangle\right] \end{aligned}$$

and this last expression is equal to $E_P\left[\left(1+\int_0^t {}^{3P}(L_-\varphi)\,dX\right) V_t\right]$ thus proving (1.4).

Again using the $\mathscr{F}^X$ stopping times S_m of the remark in Subsection 1.1, we can suppose $\langle X, X\rangle$ bounded.

Then, ${}^{3P}(L_-\varphi)$ and $({}^{1P}L)_-\,{}^{3Q}\varphi$ belong to $L^1(d\langle X,X\rangle\, dP)$ as we show, for example, for the second term:

$$\begin{aligned} E_P&\left[\int_0^\infty {}^{1P}L_{s-} |{}^{3Q}\varphi_s|\, d\langle X, X\rangle_s\right] \\ &= E_P\left[{}^{1P}L_\infty \int_0^\infty |{}^{3Q}\varphi_s|\, d\langle X, X\rangle_s\right] \\ &= E_P\left[L_\infty \int_0^\infty |{}^{3Q}\varphi_s|\, d\langle X, X\rangle_s\right] \\ &= E_Q\left(\int_0^\infty |{}^{3Q}\varphi_s|\, d\langle X, X\rangle_s\right) \leqq E_Q \int_0^\infty {}^{3Q}|\varphi|_s\, d\langle X, X\rangle_s \\ &\qquad\qquad = E_Q \int_0^\infty |\varphi|\, d\langle X, X\rangle_s < \infty \end{aligned}$$

from the end of Subsection 1.1.

To show that ${}^{3P}(L_-\varphi) = ({}^{1P}L)_-\,{}^{3Q}\varphi$, $d\langle X,X\rangle\, dP$ a.e. and thus obtain (1.5), it is now sufficient to verify that:

$$E_P\left[\int_0^\infty {}^{3P}(L_-\varphi)_s H_s\, d\langle X, X\rangle_s\right] = E_P\left[\int_0^\infty ({}^{1P}L)_{s-}\, {}^{3Q}\varphi_s H_s\, d\langle X, X\rangle_s\right]$$

for every bounded, $\mathscr{P}(\mathscr{F}^X)$ measurable process H.

The right-hand side is equal to:

$$\begin{aligned} E_P&\left[{}^{1P}L_\infty \int_0^\infty {}^{3Q}\varphi_s H_s\, d\langle X, X\rangle_s\right] \\ &= E_Q\left[\int_0^\infty {}^{3Q}\varphi_s H_s\, d\langle X, X\rangle_s\right] = E_Q\left[\int_0^\infty \varphi_s H_s\, d\langle X, X\rangle_s\right] \end{aligned}$$

which, in turn, is equal to:

$$E_P \int_0^\infty L_{s-} \varphi_s H_s \, d\langle X, X\rangle_s = E_P \int_0^\infty {}^{3P}(L_- \varphi)_s H_s \, d\langle X, X\rangle_s. \quad \blacksquare$$

Examples. i) If X is a continuous martingale of $\mathcal{M}_{\text{loc}}(P, \mathscr{F})$, we have:

$$L = \exp \int_0^{\bullet} \varphi \, dX - \tfrac{1}{2} \int_0^{\bullet} \varphi^2 \, d\langle X, X\rangle.$$

Theorem 1 reads:

$${}^{1P}L = \exp \int_0^{\bullet} {}^{3Q}\varphi \, dX - \tfrac{1}{2} \int_0^{\bullet} ({}^{3Q}\varphi)^2 \, d\langle X, X\rangle$$

and

$$\tilde{X} = X - \int_0^{\bullet} {}^{3Q}\varphi \, d\langle X, X\rangle \in \mathcal{M}_{\text{loc}}(Q, \mathscr{F}^X).$$

The equality $\langle \tilde{X}, \tilde{X}\rangle^{Q, \mathscr{F}^X} = \langle X, X\rangle$ proceeds from the quadratic approximation of these processes which has already been mentioned several times. Consequently, if X is a $(P, \mathscr{F})$ Wiener process (e.g.: $\langle X, X\rangle_t = t$) it follows from Doob's characterization theorem that $\tilde{X}$ is a $(Q, \mathscr{F}^X)$ Wiener.

ii) Let $X = N - A$ be the compensated martingale of N, a counting process whose jumps are totally inaccessible (thus, A is continuous). Then:

$$L = \exp\left(-\int_0^{\bullet} \varphi_s \, dA_s\right) \prod_{s \leqq \cdot} (1 + \varphi_s \Delta N_s)$$

and, from Theorem 1:

$${}^{1P}L = \exp\left(-\int_0^{\bullet} {}^{3Q}\varphi_s \, dA_s\right) \prod_{s \leqq \cdot} (1 + {}^{3Q}\varphi_s \Delta N_s)$$

Finally, let us remark, using the end of Theorem 1, that $\int_0^{\bullet} (1 + {}^{3Q}\varphi_s) \, dA_s$ is the $(Q, \mathscr{F})$ dual predictable projection of N. $\blacksquare$

2. Projections of Semi-Martingales

Let (Ω, F, P) be a complete probability space, endowed with a filtration $\mathscr{F} = (\mathscr{F}_t, t \geqq 0)$ satisfying the usual conditions.

In the sequel, we shall need the following extension of the optional projections of bounded measurable processes: if X is a $\mathscr{B}(\mathbb{R}_+) \otimes F$ measurable process, such that:

$$\forall\, T \in \mathscr{T}(\mathscr{F}), \; E[|X_T| \, 1_{(T < \infty)}] < \infty,$$

then, there exists a unique optional process Y valued in $\mathbb{R}$, verifying: for every $\mathscr{F}$ stopping time T, $E[X_T 1_{(T<\infty)} | \mathscr{F}_T] = Y_T 1_{(T<\infty)}$, P a.e. and, we shall write: $Y = {}^1X$.

Also, we recall the definition of a quasi-martingale: a process $X=(X_t, t\geqq 0)$ is a $\mathscr{F}$-quasi-martingale if, and only if, it is adapted, right continuous, left-hand limited, and such that

$$V_{\mathscr{F}}(X)=\sup_\tau E\left[\sum_{i=0}^{n-1} |E(X_{t_{i+1}}-X_{t_i}|\mathscr{F}_{t_i})|+|X_{t_n}|\right]<\infty,$$

where the supremum is taken over the finite sequences $\tau=(t_0, t_1, \dots, t_n)$, with $0\leqq t_0<t_1<\dots<t_n<\infty$, and $n\in\mathbb{N}$.

It is well known that X is a quasi-martingale if, and only if, it is the difference of two positive, right-continuous, and integrable supermartingales.

2.1. Canonical Projections of Semi-Martingales

In this paper, a $\mathscr{F}$ semi-martingale is a process $X=M+A$ such that M is a uniformly integrable $\mathscr{F}$ martingale and A a right continuous $\mathscr{F}$ adapted process with bounded integrable variation, i.e.: $E\int_0^\infty |dA_s|<\infty$. Note that this definition is different from Meyer's [10]; in particular, a semi-martingale X — in our sense — is a quasi-martingale, as:

$$V_{\mathscr{F}}(X)\leqq E\left[\int_0^\infty |dA_s|\right]<\infty.$$

The following lemma shows the existence and uniqueness of a canonical decomposition for our semi-martingale.

Lemma 1. *Let $X=M+A$ be a $\mathscr{F}$ semi-martingale. Then, X can be written as: $X=N+B$, where N is a uniformly integrable $\mathscr{F}$ martingale, and B a $\mathscr{F}$ predictable process with bounded integrable variation $\left(e.g.: E\int_0^\infty |dB_s|<\infty\right)$. Such a decomposition is unique.*

Proof: routine.

The decomposition obtained in the lemma is called the canonical decomposition of X. Using the notations of the lemma, we write $B=X^3$. This is consistent with Dellacherie's system of notations, since if $X=A$, then indeed $X^3=A^3$.

With the complete probability space (Ω, F, P), we also suppose that two filtrations $\mathscr{F}=(\mathscr{F}_t, t\geqq 0)$ and $\mathscr{G}=(\mathscr{G}_t, t\geqq 0)$ are given. They satisfy the usual conditions, as well as:

$$\forall t\geqq 0, \quad \mathscr{G}_t\subseteq\mathscr{F}_t, \quad \text{and} \quad \vee_t \mathscr{F}_t=F.$$

All projections considered in this subsection will be relative to $\mathscr{G}$, and this will no longer be mentioned.

Proposition 2. *The optional projection ${}^1X$ of a $\mathscr{F}$-quasi-martingale is a $\mathscr{G}$ quasi-martingale.*

Moreover, we have the inequality:

$$V_{\mathscr{G}}({}^1X) \leqq V_{\mathscr{F}}(X) \tag{2.1}$$

Proof. Let $0 \leqq t_0 < t_1 < \dots < t_n < \infty$.

The inequality (2.1) is a consequence of:

$$E\left[\sum_{i=0}^{n-1} |E({}^1X_{t_{i+1}} - {}^1X_{t_i} | \mathscr{G}_{t_i})| + |{}^1X_{t_n}|\right]$$
$$\leqq E\left[\sum_{i=0}^{n-1} |E(X_{t_{i+1}} - X_{t_i} | \mathscr{F}_{t_i})| + |X_{t_n}|\right]$$

It remains to show that if X is a $\mathscr{F}$-quasi-martingale, ${}^1X$ is right-continuous, with left-hand limits. By difference, we may suppose that X is a positive, integrable, right-continuous (and thus, left-hand limited) $\mathscr{F}$-super-martingale. For every $n \in \mathbb{N}$, $X \wedge n$ has the same properties: therefore, its $\mathscr{G}$-optional projection ${}^1(X \wedge n)$ – which is a $\mathscr{G}$-supermartingale is also right continuous ([2], T 20, p. 101). Finally, by the optional section theorem, the sequence $({}^1(X \wedge n), n \in \mathbb{N})$ is increasing, and its limit is ${}^1X$. From ([11], T16, p. 135), ${}^1X$ is therefore a right-continuous $\mathscr{G}$-supermartingale. ■

We are now interested in the $\mathscr{G}$-optional projection of a $\mathscr{F}$ semi-martingale; the proof of the next proposition – which is a reforment of proposition 2 – is straightforward.

Proposition 3. *The optional projection ${}^1X$ of a $\mathscr{F}$-semi-martingale is a $\mathscr{G}$-semi-martingale. Moreover, if $X = N + B$ is the canonical $\mathscr{F}$ decomposition of X, we have:*

$$({}^1X)^3 = B^3. \tag{2.2}$$

Here is an immediate consequence of Proposition 3: let X be a $\mathscr{F}$-semi-martingale, with $\mathscr{F}$-canonical decomposition $X = N + B$. If, moreover, X is $\mathscr{G}$-adapted, then it is a $\mathscr{G}$-semi-martingale with $\mathscr{G}$-canonical decomposition $X = \{N + (B - B^3)\} + B^3$. The aim of the end of Section 2 is to study the optional projections of $\mathscr{F}$-semi-martingales defined by stochastic integrals, for instance: $U = \mathscr{F}. \int H\,dX$, with H a $\mathscr{F}$-predictable (or optional) process, when X is a $\mathscr{G}$-semi-martingale, as well as a $\mathscr{F}$ one. More precisely, we want to compare ${}^1U$ and $\mathscr{G}. \int ({}^{1 \text{ or } 3}H)\,dX$, in order to extend to semi-martingales the following result: if A is a $\mathscr{G}$-predictable process with integrable bounded variation $\left(E\int_0^\infty |dA_s| < \infty\right)$ and H a bounded $F \otimes \mathscr{B}(\mathbb{R}_+)$ measurable process (for instance, a bounded $\mathscr{F}$ predictable process), then $(\int H\,dA)^3 = \int {}^3H\,dA$, and so, from proposition 1, we get: $[{}^1(\int H\,dA)]^3 = \int {}^3H\,dA$.

2.2. The Spaces $\mathscr{L}$, $\mathscr{L}'$, and the Associated Hilbert Projections of Stochastic Integrals

The space $\mathscr{L}$ consisting of the square integrable martingales M (e.g.: $E(M_\infty^2)<\infty$, with $M_0=0$, which are simultaneously $\mathscr{F}$ and $\mathscr{G}$-martingales is particularly relevant to the previously presented projection problem.

M belonging to $\mathscr{L}$, we examine how the different increasing processes associated with M, via $\mathscr{F}$ or $\mathscr{G}$, and in general via a filtration with respect to which M is a martingale, depend, or do not depend on such a filtration.

– First, $[M,M]$ is independent of $\mathscr{F}$ or $\mathscr{G}$, since

$$[M,M]_t = P.\lim_{(n\to\infty)} \sum_{\tau_n} (M_{t_{i+1}}-M_{t_i})^2,$$

where τ_n is a sequence of refining subdivisions of $[0,t]$, the mesh of which decreases to 0. Let now $M\in\mathscr{L}$, be written as $M=M^c+M^d=\tilde{M}^c+\tilde{M}^d$, where M^c+M^d (resp: $\tilde{M}^c+\tilde{M}^d$) is its $\mathscr{F}$ (resp: $\mathscr{G}$) decomposition as a sum of a continuous and a "purely discontinuous" (e.g.: compensated sum of jumps) martingale. Then, the increasing processes $\langle M^c, M^c\rangle$ (relative to $\mathscr{F}$) and $\langle \tilde{M}^c, \tilde{M}^c\rangle$ (relative to $\mathscr{G}$) are equal, as both are the continuous part of $[M,M]$.

– On the contrary, $\langle M,M\rangle$ may depend on the filtration, and we shall write: $\langle\!\langle M,M\rangle\!\rangle=\langle M,M\rangle^{\mathscr{F}}$ and $\langle M,M\rangle=\langle M,M\rangle^{\mathscr{G}}$. As in Subsection 2.1, all projections are now taken with respect to the smaller filtration $\mathscr{G}$.

Next lemma ensures that no confusion is possible when one deals with stochastic integration of $\mathscr{G}$-predictable processes (which are also $\mathscr{F}$-predictable):

Lemma 2. *Let $M\in\mathscr{L}$*

(a) *Then, $\langle\!\langle M,M\rangle\!\rangle^3=\langle M,M\rangle$*

(b) *Let $H\in\mathscr{P}(\mathscr{G})$ such that $E\int_0^\infty H_s^2\,d\langle M,M\rangle_s<\infty$. Then, $\mathscr{F}.\int H\,dM$ is well defined and:*

$$\mathscr{F}.\int H\,dM=\mathscr{G}.\int H\,dM. \tag{2.3}$$

Proof. (a) is a consequence of the following equalities: for $s<t$,

$$E[\langle\!\langle M,M\rangle\!\rangle_t-\langle\!\langle M,M\rangle\!\rangle_s|\mathscr{G}_s]=E[M_t^2-M_s^2|\mathscr{G}_s]=E[\langle M,M\rangle_t-\langle M,M\rangle_s|\mathscr{G}_s]$$

Now, from (a), we obtain the equality:

$$E\left(\int_0^\infty H_s^2\,d\langle\!\langle M,M\rangle\!\rangle_s\right)=E\left(\int_0^\infty H_s^2\,d\langle M,M\rangle_s\right)<\infty, \tag{2.4}$$

and therefore, $\mathscr{F}.\int H\,dM$ and $\mathscr{G}.\int H\,dM$ are defined. Moreover, if H is an elementary process of $\mathscr{P}(\mathscr{G})$, $H=\sum H_{t_i}1_{]t_i,t_{i+1}]}$ satisfying the integrability condition, the equality (2.3) is immediate. Using the isometry formula (2.4), (2.3) extends to all $H\in\mathscr{P}(\mathscr{G})$ such that $E\int_0^\infty H_s^2\,d\langle M,M\rangle_s<\infty$. ■

In fact, we have just proved that $\mathscr{L}$ is a $\mathscr{G}$-stable subspace of $\mathscr{M}^2(\mathscr{G})$, as defined by Kunita-Watanabe in [19], i.e.: $\mathscr{L}$ is a closed subspace of $\mathscr{M}^2(\mathscr{G})$, stable by integration of $\mathscr{G}$-predictable integrands H with respect to $M \in \mathscr{L}$ $\left(\text{under the condition: } E\left(\int_0^\infty H_s^2 \, d\langle M, M\rangle_s\right) < \infty\right)$.

The set $\mathscr{L}'$ of martingales $M \in \mathscr{L}$ such that $\langle\langle M, M\rangle\rangle$ is $\mathscr{G}$-predictable plays a fundamental part in the projection problem. The following lemma gives a better understanding of $\mathscr{L}'$:

Lemma 3. *If $M \in \mathscr{L}$, the following assertions are equivalent:*

1) $M \in \mathscr{L}'$
2) $\langle\langle M, M\rangle\rangle = \langle M, M\rangle$
3) $M^2 - \langle M, M\rangle$ *is a $\mathscr{F}$-martingale.*

Proof. 1) $\Rightarrow$ 2): $\langle\langle M, M\rangle\rangle - \langle M, M\rangle$ is a $\mathscr{G}$-predictable martingale, with bounded variation, which is null at 0, and so identically null. 2) $\Rightarrow$ 1) or 3) is obvious.

The proof of: 3) $\Rightarrow$ 2) is similar to that of 1) $\Rightarrow$ 2). ∎

There is no guarantee in general that $\mathscr{L}'$ be a vector space. However, let us study this case.

Lemma 4. *The following assertions are equivalent:*

1) $\mathscr{L}'$ *is a vector space.*
2) $\forall M \in \mathscr{L}'$, $\forall N \in \mathscr{L}'$, $\langle\langle M, N\rangle\rangle = \langle M, N\rangle$
3) $\forall M \in \mathscr{L}'$, $\forall N \in \mathscr{L}'$, $MN - \langle M, N\rangle$ is a $\mathscr{F}$-martingale.

If either one of the above assumptions is satisfied, $\mathscr{L}'$ is a $\mathscr{G}$ stable subspace of $\mathscr{M}^2(\mathscr{G})$.

Proof. 1) $\Leftrightarrow$ 2) is an immediate consequence of the identity:

$$\langle\langle M+N, M+N\rangle\rangle = \langle\langle M, N\rangle\rangle + 2\langle\langle M, N\rangle\rangle + \langle\langle N, N\rangle\rangle$$

and the analogous one for $\langle M+N, M+N\rangle$. The proof of 2) $\Leftrightarrow$ 3) is the same as in Lemma 3.

Suppose now that $\mathscr{L}'$ is a vector space. It is closed in $\mathscr{M}^2(\mathscr{G})$, as if $M^{(n)} \in \mathscr{L}'$, $M^{(n)} \xrightarrow[\mathscr{M}^2(\mathscr{G})]{} M$, then $\langle M^{(n)}, M^{(n)}\rangle_t$ converges to $\langle M, M\rangle_t$ in L^1 uniformly in t. Moreover, if $H \in \mathscr{P}(\mathscr{G})$ and $M \in \mathscr{L}'$ are such that $E \int_0^\infty H_s^2 \, d\langle M, M\rangle_s < \infty$, then $U = \mathscr{F}.\int H\, dM = \mathscr{G}.\int H\, dM \in \mathscr{L}$ and $\langle\langle U, U\rangle\rangle = \int H^2 \, d\langle\langle M, M\rangle\rangle = \int H^2 \, d\langle M, M\rangle = \langle U, U\rangle$ thus showing: $U \in \mathscr{L}'$. $\mathscr{L}'$ is therefore a $\mathscr{G}$ stable subspace of $\mathscr{M}^2(\mathscr{G})$. ∎

At this point, we may ask the question: is $\mathscr{L}'$ a strict subset of $\mathscr{L}$? It is not so easy to find a quite general counterexample. However, here is one: let T be a $\mathscr{F}$-predictable stopping time and $H \in L^2(\mathscr{F}_T) \ominus L^2(\mathscr{F}_{T-})$. The process M defined by $M_t = H\, 1_{(T \leqq t)}$ is a square integrable $\mathscr{F}$-martingale. We have: $\langle\langle M, M\rangle\rangle_t = E[H^2 | \mathscr{F}_{T-}]\, 1_{(T \leqq t)}$. Take $\mathscr{G} = \mathscr{F}^M$. In general, $\langle\langle M, M\rangle\rangle$ is not $\mathscr{G}$-predictable, as the following particular case shows: suppose $\mathscr{F}_t = \mathscr{F}_0$ for every $t < 1$, and $\mathscr{F}_t = \mathscr{F}_1$

for every $t \geqq 1$; take $T=1$, $H \in L^2(\mathscr{F}_1) \ominus L^2(\mathscr{F}_0)$. Then, $\langle\langle M, M \rangle\rangle \in \mathscr{P}(\mathscr{F}^M)$ if, and only if $E[H^2 | \mathscr{F}_0] \in \sigma(H)$ and it is now easy to give an explicit counterexample: let $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$ be given with the probability $P = \mu \otimes \nu$, μ and ν already being probabilities on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, having finite second moments, and such that $\int y \, d\nu(y) = 0$. X and Y denote the coordinate variables on $\mathbb{R}^2 (X(x,y)=x, Y(x,y)=y)$, $\mathscr{F}_0 = \sigma\{X\} \vee \mathscr{N}_P$, $\mathscr{F}_1 = \sigma\{X, Y\} \vee \mathscr{N}_P$, where $\mathscr{N}_P$ is the class of P negligible sets of R^2. Then, $H = XY \in L^2(\mathscr{F}_1) \ominus L^2(\mathscr{F}_0)$, and $E[H^2 | \mathscr{F}_0] = X^2 E(Y^2) \notin \sigma\{H\} \vee \mathscr{N}_P$ for "general" μ and ν.

Let us now slightly change our point of view. The filtration $\mathscr{F}$ being fixed, it is natural to consider the square-integrable $\mathscr{F}$ martingales M which belong to $\mathscr{L}'(\mathscr{F}, \mathscr{G})$ (using an obvious notation) for all $\mathscr{G}$ such that $\mathscr{F}^M \subseteq \mathscr{G} \subseteq \mathscr{F}$.

In other words, we consider the class

$$\mathscr{I} = \{M \in \mathscr{M}^2(\mathscr{F}) / \langle\langle M, M \rangle\rangle \in \mathscr{P}(\mathscr{F}^M)\}$$

(the letter $\mathscr{I}$ is used for intrinsic). Here are two remarkable sub-classes of $\mathscr{I}$:

– the square integrable continuous $\mathscr{F}$-martingales: indeed, for such M's, $\langle\langle M, M \rangle\rangle$ is adapted to $\mathscr{F}^M$ (from the second remark in Subsection 1.1), and continuous; therefore, it is in $\mathscr{P}(\mathscr{F}^M)$.

– the compensated sums of $\mathscr{F}$ totally inaccessible stopping times. Let (T_n) be a sequence of $\mathscr{F}$ totally inaccessible stopping times, which are strictly ordered, e.g.: $T_n < T_{n+1}$ P a.e. and increasing P a.e. to infinity. Let M be defined by $M_t = \sum_n 1_{(T_n \leqq t)} - A_t$, where A_t is the $\mathscr{F}$ dual predictable projection of $\sum_n 1_{(T_n \leqq t)}$. A is continuous; therefore, the T_n's are the successive jumping times of M, and so, are $\mathscr{F}^M$ stopping times. This implies that A is $\mathscr{F}^M$ adapted, and so, $A \in \mathscr{P}(\mathscr{F}^M)$. Moreover, $\langle\langle M, M \rangle\rangle = A$, so that: $M \in \mathscr{I}$.

2.3. $\mathscr{G}$ Projections of $\mathscr{F}$ Stochastic Integrals

In the present subsection, we consider measurable integrands (not necessarily optional). This provides a natural setting for our results, although the extension of stochastic integration to optional integrands, made by P.A. Meyer in [10], is maximal in the following sense: indeed, let $M \in \mathscr{M}_{\text{loc}}(P, \mathscr{F})$ and H be a $F \otimes \mathscr{B}(\mathbb{R}_+)$ measurable process such that $E \int_0^\infty H_s^2 \, d[M, M]_s < \infty$; then if $\mathscr{F}. \int H \, dM$ is defined by the same method as P.A. Meyer's in [10], it is shown in [16] (Proposition 1, page 483) that $\mathscr{F}. \int H \, dM = \mathscr{F}. \int {}^1H \, dM$ (${}^1H$ is here the $L^2(\mathscr{O}(\mathscr{F}), d[M, M] \, dP)$ projection of H (see Subsection 1.1).

Proposition 4. *Let $M \in \mathscr{L}$, and H be a $F \otimes \mathscr{B}(\mathbb{R}_+)$ measurable process such that $E \int_0^\infty H_s^2 \, d[M, M]_s < \infty$.*

Then, $P^1_{\mathscr{G}}(\mathscr{F}. \int H \, dM) = P_{\mathscr{G}}(\mathscr{G}. \int H \, dM) = P_{\mathscr{G}}(\mathscr{G}. \int {}^1H \, dM)$ where(1) *${}^1H$ is here the $L^2(\mathscr{O}(\mathscr{G}), d[M, M] \, dP)$ projection of H.*

(1) See the last remark of Subsection 1.1.

Proof. The last equality comes from $\mathscr{G}.\int H\,dM=\mathscr{G}.\int {}^1H\,dM$. Let N be another martingale of $\mathscr{L}$. Then:

$$\begin{aligned} E\int_0^\infty H_s\,d[M,N]_s &= E[(\mathscr{F}.\int H\,dM)_\infty N_\infty] \\ &= E[{}^1(\mathscr{F}.\int H\,dM)_\infty N_\infty] \\ &= E[\{P_{\mathscr{L}}{}^1(\mathscr{F}.\int H\,dM)\}_\infty N_\infty]. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} E\int_0^\infty H_s\,d[M,N]_s &= E[(\mathscr{G}.\int H\,dM)_\infty N_\infty] \\ &= E[\{P_{\mathscr{L}}(\mathscr{G}.\int H\,dM)\}_\infty N_\infty]. \end{aligned}$$

As the equality $E[\{P_{\mathscr{L}}{}^1(\mathscr{F}.\int H\,dM)\}_\infty N_\infty]=E[\{P_{\mathscr{L}}(\mathscr{G}.\int H\,dM)\}_\infty N_\infty]$ takes place for every $N\in\mathscr{L}$, we obtain:

$$P_{\mathscr{L}}{}^1(\mathscr{F}.\int H\,dM)=P_{\mathscr{L}}(\mathscr{G}.\int H\,dM).$$

Let us remark that no hypothesis on the $\mathscr{G}$-martingales, or even on the space $\mathscr{L}$, has been made in Proposition 4, but the result does not give an explicit formula.

However, we now give a sufficient condition to obtain such a formula:

Corollary. *Let the hypotheses of Proposition 4 be verified. Note* $U={}^1(\mathscr{F}.\int H\,dM)$ *and* $V=\mathscr{G}.\int {}^1H\,dM$.

If $E(U_\infty^2)=E(V_\infty^2)$, *$U$ belongs to $\mathscr{L}$ iff V belongs to $\mathscr{L}$, and then* $U=V$.

Proof. From Proposition 4, we have: $P_{\mathscr{L}}(U)=P_{\mathscr{L}}(V)$. Then, U belongs to $\mathscr{L}$ iff $U=P_{\mathscr{L}}(V)$.

Writing $V=P_{\mathscr{L}}(V)+P_{\mathscr{L}^\perp}(V)$, we get:

$$E(V_\infty^2)=E[(P_{\mathscr{L}}(V))_\infty^2]+E(P_{\mathscr{L}^\perp}(V))_\infty^2).$$

If $U=P_{\mathscr{L}}(V)$, the hypothesis $E(U_\infty^2)=E(V_\infty^2)$, implies then:

$$P_{\mathscr{L}^\perp}(V)=0, \quad \text{and so} \quad V=P_{\mathscr{L}}(V)=U.$$

Conversely, if V belongs to $\mathscr{L}$, we have: $V=P_{\mathscr{L}}(U)$, and the equality:

$$E[U_\infty^2]=E[P_{\mathscr{L}}(U)_\infty^2]+E[P_{\mathscr{L}^\perp}(U)_\infty^2]$$

then gives $P_{\mathscr{L}^\perp}(U)=0$, and $U=P_{\mathscr{L}}(U)=V$.

Remark. If M is quasi-left continuous (for $\mathscr{F}$, and then for $\mathscr{G}$), $H\in\mathcal{O}(\mathscr{F})$, and $U\in\mathscr{L}$, the inequality $E(V_\infty^2)\leqq E(U_\infty^2)$ is always verified.

Indeed, we have:

$$\begin{aligned} E(V_\infty^2) &= E\int_0^\infty ({}^1H)_s^2\,d[M,M]_s \\ &\leqq E\int_0^\infty H_s^2\,d[M,M]_s=E(U_\infty^2). \end{aligned}$$

Let us now study the particular case when $M \in \mathscr{L}$, and $H = 1_{(\Delta M = 0)}$. Then, we have:

$$U = \mathscr{F}. \int H\, dM = M^{c/\mathscr{F}} \quad \text{and} \quad V = \mathscr{G}. \int H\, dM = M^{c/\mathscr{G}}$$

and the equality $E(U_\infty^2) = E(V_\infty^2) (= E([M, M]_\infty^c))$ from the beginning of Subsection 2.3.

Then, from the corollary, if $M^{c/\mathscr{F}} \in \mathscr{M}^2(\mathscr{G})$ (or: $M^{c/\mathscr{G}} \in \mathscr{M}^2(\mathscr{F})$) we have: $M^{c/\mathscr{F}} = M^{c/\mathscr{G}}$. ■

When looking at projections on $\mathscr{L}'$ (when $\mathscr{L}'$ is a stable subspace), we get the following theorem, the proof of which is similar to that of Proposition 4.

Theorem 2. *Suppose that $\mathscr{L}'$ is a vector space. Then, it is a $\mathscr{G}$ stable subspace of $\mathscr{M}^2(P, \mathscr{G})$.*

Let $M \in \mathscr{L}'$, and H be a $\mathscr{F}$ predictable process such that $E \int_0^\infty H_s^2\, d\langle\langle M, M\rangle\rangle_s < \infty$.

Then,

$$P_{\mathscr{L}'}\, {}^1(\mathscr{F}. \int H\, dM) = \int {}^3H\, dM.$$

We now look at the dual situation, i.e.: we study the processes of the form ${}^1\left(\int_0^\bullet H_s\, dM_s\right)$, where H is $\mathscr{G}$ predictable, and M is a $\mathscr{F}$-martingale.

Proposition 5. *Let $M \in \mathscr{M}^2(\mathscr{F})$, and H be a bounded $\mathscr{G}$-predictable process. Then:*

$$P_{\mathscr{L}}\, {}^1(\mathscr{F}. \int H\, dM) = \int H\, dP_{\mathscr{L}}({}^1M).$$

Proof. $\mathscr{L}$ being a $\mathscr{G}$-stable space, we only have to show the equality:

$$E[(\mathscr{F}. \int H\, dM)_\infty Z_\infty] = E[(\mathscr{G}. \int H\, d^1M)_\infty Z_\infty], \tag{2.5}$$

for every martingale $Z \in \mathscr{L}$.

By a density argument, we may suppose that H can be written as:

$$H = \sum_{i=0}^{n-1} H_{t_i} 1_{]t_i, t_{i+1}]}, \quad \text{where } 0 \leqq t_0 < t_1 < \cdots < t_n < \infty,$$

and $H_{t_i} \in b(\mathscr{G}_{t_i})$, for every $i \leqq n-1$.

The right member of (2.5) is then equal to:

$$E[\sum_i H_{t_i}({}^1M_{t_{i+1}} - {}^1M_{t_i}) Z_\infty] = E[(\mathscr{F}. \int H\, dM)_\infty Z_\infty],$$

and therefore, (2.5) is proved. ■

The corollary of Proposition 4 and its proof are still valid if one takes $\mathscr{U} = {}^1(\mathscr{F}. \int H\, dM)$ and $V = \int H\, d^1M$, with the hypotheses in force in Proposition 5.

2.4. A Simplifying Hypothesis

Let us now express our previous results under the following fundamental hypothesis:

($\mathscr{H}$) *Every square integrable $\mathscr{G}$-martingale is a $\mathscr{F}$-martingale.*

We first remark, by density arguments, that ($\mathscr{H}$) holds if, and only if, every bounded $\mathscr{G}$-martingale (resp: every $\mathscr{G}$ local martingale) is a $\mathscr{F}$-martingale, (resp. a $\mathscr{F}$ local martingale).

Hypothesis ($\mathscr{H}$) implies that $\mathscr{G}$ has a nice structure, relatively to $\mathscr{F}$, as is shown – among other properties – in the following theorem:

Theorem 3. *Let $\mathscr{G}=(\mathscr{G}_t, t\geqq 0)$ be a sub-filtration of $\mathscr{F}=(\mathscr{F}_t, t\geqq 0)$ e.g. for every t, $\mathscr{G}_t \subseteq \mathscr{F}_t$.*

The following assertions are equivalent:

(1) *Hypothesis ($\mathscr{H}$) is verified.*

(2) *For every t, $\mathscr{F}_t$ and $\mathscr{G}_\infty$ are conditionally independent with respect to $\mathscr{G}_t$.*

(3) *For every t, $\mathscr{G}_t$ is the (F,P) complete σ-field generated by the variables $E[G|\mathscr{F}_t]$ $(G\in L^2(\mathscr{G}_\infty))$.*

(4) *For every process A, $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{G}_\infty$ measurable with bounded integrable variation, the equality $A^{\mathscr{G}3}=A^{\mathscr{F}3}$ holds.*

(5) *For every $X\in\mathscr{M}^2(\mathscr{F})$, and every deterministic bounded function h, the equality: ${}^1(\int h\,dX)=\int h\,d\,{}^1X$ holds.*

Moreover, if one of the properties (1)…(5) is verified, the equality $\mathscr{G}_t=\mathscr{F}_t\cap\mathscr{G}_\infty$ holds, for every $t\geqq 0$.

Remark. In [23] (Lemma 4), Sekiguchi already remarked that hypothesis ($\mathscr{H}$) implies: $\forall t\geqq 0$, $\mathscr{G}_t=\mathscr{F}_t\cap\mathscr{G}_\infty$.

Moreover, in Theorem 1 of the same paper, the equivalences of (1) and of different assertions similar to (3) are shown, for $\mathscr{G}$, the filtration generated by the continuous $\mathscr{F}$-martingales.

Proof of Theorem 3. It is easily seen that ($\mathscr{H}$) is verified iff:

$$\forall t\geqq 0, \quad \forall X\in L^1(\mathscr{G}_\infty), \quad E[X\,|\,\mathscr{G}_t]=E[X\,|\,\mathscr{F}_t]$$

which is an expression-among many – of (2). Thus, the equivalence between (1) and (2) is obtained.

Let us now remark that (2) implies: $\mathscr{G}_t=\mathscr{F}_t\cap\mathscr{G}_\infty$, for every $t\geqq 0$. Indeed, we only have to show $\mathscr{F}_t\cap\mathscr{G}_\infty\subseteq\mathscr{G}_t$. But if $A\in\mathscr{F}_t\cap\mathscr{G}_\infty$, we have from (2):

$$1_A=E[1_A\,|\,\mathscr{G}_\infty]=E[1_A\,|\,\mathscr{G}_t] \qquad P \text{ a.e., and thus, } A\in\mathscr{G}_t.$$

(2)⇒(3): $\mathscr{G}_t$ is the (F,P) complete σ-field generated by the variables $E[X\,|\,\mathscr{G}_t]$ $(X\in L^2(\mathscr{G}_\infty))$. But, from (2), $E[X\,|\,\mathscr{G}_t]=E[X\,|\,\mathscr{F}_t]$ and so, we have (3).

(3)⇒(2): If $G\in L^2(\mathscr{G}_\infty)$, one has: $E[G\,|\,\mathscr{G}_t]=E[E(G\,|\,\mathscr{F}_t)\,|\,\mathscr{G}_t]=E[G\,|\,\mathscr{F}_t]$, from the definition of $\mathscr{G}_t$.

(2)⇒(4): If A is a $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{G}_\infty$ measurable process, with integrable bounded variation, the following equality follows from (2): $\forall(s,t)$, such that $0\leqq s\leqq t$: $E[A_t-A_s|\mathscr{G}_s]=E[A_t-A_s|\mathscr{F}_s]$. This may also be written as:

$$E[A_t^{\mathscr{G}3}-A_s^{\mathscr{G}3}\,|\,\mathscr{G}_s]=E[A_t^{\mathscr{F}3}-A_s^{\mathscr{F}3}\,|\,\mathscr{F}_s].$$

Again using (2), we obtain:

$$E[A_t^{\mathscr{G}3}-A_s^{\mathscr{G}3}\,|\,\mathscr{F}_s]=E[A_t^{\mathscr{F}3}-A_s^{\mathscr{F}3}\,|\,\mathscr{F}_s].$$

Thus, $A^{\mathscr{F}3}-A^{\mathscr{G}3}$ is a $\mathscr{F}$-predictable martingale, null at $t=0$, and with bounded variation: therefore, it is null for all t, and $A^{\mathscr{F}3}=A^{\mathscr{G}3}$.

(4)⇒(2): If $X\in b(\mathscr{G}_\infty)$, and $t\geqq 0$, let us note: $A_u=X1_{(t\leqq u)}$. Then, we have:

$$A_u^{\mathscr{G}3}=E[X\,|\,\mathscr{G}_{t-}]\,1_{(t\leqq u)},$$
$$A_u^{\mathscr{F}3}=E[X\,|\,\mathscr{F}_{t-}]\,1_{(t\leqq u)}.$$

Therefore, if (4) is true, we have for every t:

$$E[X\,|\,\mathscr{F}_{t-}]=E[X\,|\,\mathscr{G}_{t-}]\;P\text{ a.e.}$$

Replacing t by $(t+h)$, and letting h decrease to 0, we obtain (2).

(2)⇔(5): By a density argument, we may only make use of the functions $h(u)=1_{]s,\infty]}(u)\,(s\in\mathbb{R}_+)$.

If $X\in\mathscr{M}^2(\mathscr{F})$, the equality:

$$E[X_\infty-X_s\,|\,\mathscr{G}_s]={}^1X_\infty-{}^1X_s$$

is equivalent to:

$$E[X_s\,|\,\mathscr{G}_\infty]=E[X_s\,|\,\mathscr{G}_s]$$

and this verified for all $X\in\mathscr{M}^2(\mathscr{F})$ iff (2) is true. ■

The different equivalences obtained in Theorem 3 obviously imply many consequences, of which we shall only give a few important ones: if $(\mathscr{H})$ is verified,

(c.1) T is a $\mathscr{G}$-stopping time iff it is a $\mathscr{F}$-stopping time, which is also $\mathscr{G}_\infty$ measurable; moreover, $E^{\mathscr{G}_T}=E^{\mathscr{F}_T}E^{\mathscr{G}_\infty}=E^{\mathscr{G}_\infty}E^{\mathscr{F}_T}$ (e.g.: $\mathscr{F}_T$ and $\mathscr{G}_\infty$ are independent, conditionally to $\mathscr{G}_T=\mathscr{F}_T\cap\mathscr{G}_\infty$).

(c.2) H is a $\mathscr{G}$-optional (resp: predictable) process iff it is a $\mathscr{F}$ optional (resp: predictable) one, such that, for all t, H_t is $\mathscr{G}_\infty$ measurable. Moreover, if H is bounded, and $\mathscr{F}$-optional, its $\mathscr{G}$-optional projection cannot be distinguished from its "optional" projection on the constant filtration $(\mathscr{G}_\infty)$ (we may also replace optional by predictable).

(c.3) This last result may be extended as follows: let $M=(M_t,t\geqq 0)$ be a uniformly integrable $\mathscr{F}$-martingale. Then, for every random variable $S\colon\Omega\to\mathbb{R}_+$,

$\mathscr{G}_\infty$-measurable, one has:

$$^1M_S = E[M_S|\mathscr{G}_\infty] \tag{2.6}$$

and

$$^1M_{S-} = E[M_{S-}|\mathscr{G}_\infty] \quad (\text{if } P(S>0)=1)$$

These formulas are obvious, from (2), when S takes a countable number of values, and then, formulas (2.6) follow from the right-continuity (resp: left-continuity) of M and ^1M (resp: M_- and $^1M_-$).

(c.4) $\mathscr{L}=\mathscr{L}'=\mathscr{M}^2(P,\mathscr{G})$. This is proved as follows: if $M\in\mathscr{M}^2(P,\mathscr{G})$, then $A=[M,M]$ is an increasing process satisfying the conditions listed in the assertion (4) of Theorem 3. Thus, $A^{\mathscr{G}3}=\langle M,M\rangle$, and $A^{\mathscr{F}3}=\langle\langle M,M\rangle\rangle$ are equal, therefore, M belongs to $\mathscr{L}'$.

Remark. Incidentally, using this last consequence of hypothesis ($\mathscr{H}$), it is easy to show that in subsection 1, under the representation property (H.4), hypothesis (H.1) is equivalent to the apparently (but not really) weaker hypothesis (H.1)': $X\in\mathscr{M}^2_{\text{loc}}(P,\mathscr{F}^X)$.

We now compare the $\mathscr{F}$ and $\mathscr{G}$ stochastic integrations of $\mathscr{G}$ optional processes relatively to $M\in\mathscr{M}_{\text{loc}}(P,\mathscr{G})$, and we show that ($\mathscr{H}$) allows us to extend the result of Lemma 2 from $\mathscr{G}$ predictable to $\mathscr{G}$ optional processes.

Proposition 6. *Let ($\mathscr{H}$) be verified.*

If $H\in\mathcal{O}(\mathscr{G})$ and $M\in\mathscr{M}_{\text{loc}}(P,\mathscr{G})$ are such that $\left(\int_0^\cdot H_s^2\, d[M,M]_s\right)^{\frac{1}{2}}$ is $\mathscr{G}$ locally integrable, then:

$$\mathscr{F}.\int H\,dM=\mathscr{G}.\int H\,dM.$$

Proof. 1) Let us remark that the space $W(P,\mathscr{G})=\{M\in\mathscr{M}(P,\mathscr{G})/E\int_0^\infty |dM_s|<\infty\}$ is dense in: $H^{1,d}(P,\mathscr{G})=H^1(P,\mathscr{G})\cap\mathscr{M}^d(P,\mathscr{G})$. This is a consequence of the density of $\mathscr{M}^{2,d}(P,\mathscr{G})$ in $H^{1,d}(P,\mathscr{G})$ and of the orthogonal decomposition of a martingale belonging to $\mathscr{M}^{2,d}(P,\mathscr{G})$ as a (generally infinite) sum of discontinuous martingales, having only one jump.

2) If $H\in b(\mathcal{O}(\mathscr{G}))$, and $M\in W(P,\mathscr{G})$, we get:

$$\mathscr{F}.\int_0^t H_s\,dM_s=\int_0^t H_s\,dM_s-\left(\int_0^\cdot H_s\,dM_s\right)_t^{\mathscr{F}3}$$

([16], Proposition 3), where $\int_0^t H_s\,dM_s$ is a Stieltjes integral. The same equality is true, when $\mathscr{F}$ is replaced by $\mathscr{G}$. So, from Theorem 3, (4), we obtain: $\mathscr{F}.\int H\,dM=\mathscr{G}.\int H\,dM$.

3) We still suppose $H\in b(\mathcal{O}(\mathscr{G}))$, but M is now a $\mathscr{G}$ (and $\mathscr{F}$) local martingale, which we can suppose to be in $H^1(P,\mathscr{G})$. Let $M=M_0+M^c+M^d$ be the $\mathscr{G}$ decomposition of M as a sum of a $\mathscr{G}$ continuous local martingale and a purely

discontinuous one. Then, since the set $\{H \neq {}^{3\mathscr{G}}H\}$ is a denumberable union of $\mathscr{G}$-stopping time graphs ([2], page 101, T 19), we have: $\mathscr{G}. \int H\, dM^c = \mathscr{F}. \int H\, dM^c$. So, we only have to take $M \in H^{1,d}(P, \mathscr{G})$ in the following. The applications $M \to \{(\mathscr{F}$ or $\mathscr{G}). \int H\, dM\}_\infty$ are continuous from $H^1(P, \mathscr{G})$ to $L^1(\mathscr{F}_\infty, P)$ (and their norms are smaller than or equal to $c\|H\|_\infty$, where c is a universal constant appearing in Davis' inequality [10]). From 1) and 2), these applications are equal on $W(P, \mathscr{G})$, which is dense in $H^{1,d}(P, \mathscr{G})$, and so, are equal on all of $H^{1,d}(P, \mathscr{G})$. Then, if $H \in b(\mathscr{O}(\mathscr{G}))$, and M is a $\mathscr{G}$ local martingale, we have: $\mathscr{F}. \int H\, dM = \mathscr{G}. \int H\, dM$.

4) If $M \in \mathscr{M}_{\mathrm{loc}}(P, \mathscr{G})$, the equality $\mathscr{F}. \int H\, dM = \mathscr{G}. \int H\, dM$ now extends (by density and continuity) to all $H \in \mathscr{O}(\mathscr{G})$ such that $\left(\int_0^{\cdot} H_s^2\, d[M, M]_s\right)^{\frac{1}{2}}$ is $\mathscr{G}$-locally integrable. ■

We now give the simplified form under which our results in Propositions 4, 5 and Theorem 2, appear, when the hypothesis $(\mathscr{H})$ is verified.

Proposition 7. *Let $(\mathscr{H})$ be true and M be a $\mathscr{G}$ local martingale.*

1) *If H is a $F \otimes \mathscr{B}(\mathbb{R}_+)$ measurable (or $\mathscr{F}$. optional) process such that, either $E \int_0^\infty H_s^2\, d[M, M]_s < \infty$, or H is bounded, then:*

$${}^1(\mathscr{F}. \textstyle\int H\, dM) = \mathscr{G}. \int {}^1H\, dM = \mathscr{F}. \int {}^1H\, dM$$

2) *Under one of the following hypotheses:*

(i) *M is locally square integrable*[2], *and H is a $\mathscr{F}$-predictable process such that $E \int_0^\infty H_s^2\, d\langle M, M\rangle_s < \infty$*

(ii) *H is a $\mathscr{F}$ predictable bounded process, then:*

$${}^1(\mathscr{F}. \textstyle\int H\, dM) = \int {}^3H\, dM.$$

Proof. The equalities, under square integrability conditions, come directly from Proposition 4 and Theorem 2, as the hypothesis $(\mathscr{H})$ implies $\mathscr{L} = \mathscr{L}' = \mathscr{M}^2(P, \mathscr{G})$ (consequence (c.4) of Theorem 3).

If H is bounded, the previous equalities extend to all $\mathscr{G}$ local martingales M, as $\mathscr{M}^2(P, \mathscr{G})$ is dense in $H^1(P, \mathscr{G})$, and the applications

$$M \to {}^1(\mathscr{F}. \textstyle\int H\, dM)_\infty \text{ (or: } \{\mathscr{G}. \int {}^1H\, dM\}_\infty, \text{ or: } \{\int {}^3H\, dM\}_\infty)$$

are continuous from $H^1(P, \mathscr{G})$ to $L^1(\mathscr{F}_\infty, P)$. ■

Proposition 8. *Let $(\mathscr{H})$ be verified and M be a square integrable $\mathscr{F}$-martingale. If H is a $\mathscr{G}$ predictable process such that $E \int_0^\infty H_s^2\, d[M, M]_s < \infty$,* then the integral $E\left(\int_0^\infty H_s^2\, d[{}^1M, {}^1M]_s\right)$ is also finite, and the equality ${}^1(\mathscr{F}. \int H\, dM) = \int H\, d^1M$ holds.

---

[2] From the remark made in Subsection 1.1, there is no need to specify whether it is with respect to $\mathscr{F}$ or $\mathscr{G}$.

Proof. First, remark that ${}^1(M^2)-({}^1M)^2$ is a $\mathscr{G}$-submartingale. Indeed, for all couples (s,t) such that $0 \leqq s \leqq t$, one has:

$$\begin{aligned} E[({}^1M)_t^2-({}^1M)_s^2 \mid \mathscr{G}_s] &= E[({}^1M_t-{}^1M_s)^2 \mid \mathscr{G}_s] \\ &= E[(E(M_t-M_s \mid \mathscr{G}_\infty))^2 \mid \mathscr{G}_s] \\ &\leqq E[(M_t-M_s)^2 \mid \mathscr{G}_s] \\ &\leqq E[M_t^2-M_s^2 \mid \mathscr{G}_s] = E[{}^1(M^2)_t-{}^1(M_s^2) \mid \mathscr{G}_s] \end{aligned}$$

Now, from the inequality:

$$E[({}^1M)_t^2-({}^1M)_s^2 \mid \mathscr{G}_s] \leqq E[M_t^2-M_s^2 \mid \mathscr{G}_s],$$

it follows that:

$$E[[{}^1M,{}^1M]_t-[{}^1M,{}^1M]_s \mid \mathscr{G}_s] \leqq E[[M,M]_t-[M,M]_s \mid \mathscr{G}_s]$$

This implies that for every $\mathscr{G}$ predictable process H, one has:

$$E\left(\int_0^\infty H_s^2\, d[{}^1M,{}^1M]_s\right) \leqq E\left(\int_0^\infty H_s^2\, d[M,M]_s\right)$$

The equality ${}^1(\mathscr{F}\!\int H\,dM)=\int H\,d^1M$ now comes from Proposition 5, using the density of bounded $\mathscr{G}$-predictable processes in $L^2(\mathscr{P}(\mathscr{G}), d[M,M]_s\,dP)$. ■

We now give two simple examples where $(\mathscr{H})$ is verified (see Sekiguchi [23] for another):

– the simplest of all is most certainly given by the filtrations $\mathscr{G}_t=\mathscr{F}_{t\wedge T}$, with T a $\mathscr{F}$-stopping time;

– let $\mathscr{F}$ (resp: $\mathscr{G}$) be the usually completed filtration of a n-dimensional Brownian motion $(B_t, t \geqq 0)$ (resp: of $R_t=|B_t|$).

Then, $(\mathscr{H})$ is verified, which can be seen from either of the following points:

(i) $R_t=|B_t|$ is a $\mathscr{F}$-Markov process, which implies property 2) of theorem 3 and thus $(\mathscr{H})$.

More generally let $\mathscr{F}$ (resp: $\mathscr{G}$) be the filtration of a Hunt process X, valued in a Polish space E (resp: of $Y=\varphi(X)$, valued in K, another Polish space), where $\varphi: E \to K$ is continuous, and lets the semi-group (P_t) of X invariant (see [25] for more details). This last condition implies that Y is a $\mathscr{F}$-Markov process, e.g. for every $t \geqq 0$, the future of Y and the past of X at time t are independent, conditionally to Y_t. From this, it follows easily that assertion (2) of Theorem 3, and therefore $(\mathscr{H})$ are verified.

(ii) From [25], $\mathscr{G}_t$ is equal to the usually completed filtration of

$$\begin{aligned} Y_t &= \int_0^t \operatorname{sgn}(B_s)\, dB_s \qquad \text{(if } n=1) \\ &= \int_0^t \sum_{i=1}^n \frac{B_s^i\, dB_s^i}{R_s} \qquad \text{(if } n>1) \end{aligned}$$

Y is a standard real Brownian motion, which implies that every $\mathscr{G}$-martingale is a stochastic integral for Y, and so, a $\mathscr{F}$-martingale.

Apart from using the equivalences of Theorem 3, a practical means of verifying hypothesis $(\mathscr{H})$ consists in establishing that the space of square integrable $\mathscr{G}$-martingales is generated (as a stable space) by a finite (or infinite) family of martingales M which are also $\mathscr{F}$ martingales. (for instance see point (ii) of the last example).

Thus we are naturally concerned with the characterization of martingales which have the predictable representation property (see the outline, subsection 1.2, and [14, 15, 7]). As the reader may have noticed, this question is underlying the whole present work. But, it is important here to know whether this representation property is relative to $\mathscr{F}$ or $\mathscr{G}$. The following proposition fully answers this question:

Proposition 9. *Let X be a $\mathscr{F}$- and $\mathscr{G}$-local martingale such that every $\mathscr{F}$ local martingale M can be represented as:*

$$M = a + \mathscr{F}.\int H\,dX \qquad (a\in\mathbb{R}, H\in\mathscr{P}(\mathscr{F})).$$

Then the two following assertions are equivalent:

a) *every $\mathscr{G}$ local martingale N can be represented as:*

$$N = b + \mathscr{G}.\int K\,dX \qquad (b\in\mathbb{R}, K\in\mathscr{P}(\mathscr{G}))$$

b) *all $\mathscr{G}$ local martingales are $\mathscr{F}$ local martingales.*

Proof. a) $\Rightarrow$ b) is obvious, as $\mathscr{G}.\int K\,dX = \mathscr{F}.\int K\,dX$ for all $\mathscr{G}$ predictable processes such that $(\int_0^{\cdot} K_s^2\,d[X,X]_s)^{\frac{1}{2}}$ is $\mathscr{G}$ locally integrable (this is a slight generalization of Lemma 2).

b) $\Rightarrow$ a) From Proposition 1, it is sufficient to show that every square-integrable $\mathscr{G}$ martingale N may be represented as:

$$N = b + \mathscr{G}.\int K\,dX \qquad (b\in\mathbb{R}, K\in\mathscr{P}(\mathscr{G})).$$

From b), we have:

$$N_t = E[N_\infty \mid \mathscr{F}_t] = a + \mathscr{F}.\int H\,dX,$$

with $a\in\mathbb{R}$, $H\in\mathscr{P}(\mathscr{F})$ such that $E\left(\int_0^\infty H_s^2\,d[X,X]_s\right) < \infty$.

By a density argument, we can suppose that H is bounded. Then, from Proposition 8, 2), ii), we have:

$$N = a + {}^1(\mathscr{F}.\int H\,dX) = a + \int {}^3H\,dX. \quad \blacksquare$$

Remark. Let X be a $\mathscr{F}$ local martingale, verifying the hypothesis of proposition 9. This hypothesis is also obviously verified by $Y = \mathscr{F}.\int H\,dX$ where H is a $\mathscr{F}$ predictable process, which is null at most on an evanescent set (and is, for simplicity, bounded). Then, we can apply Proposition 9 to Y, with $\mathscr{G} = \mathscr{F}^Y$.

3. On a Filtering Equation

3.1. Preliminaries

Going back to and using the notations of section 1, we now consider the problem of calculating $E_Q[U|\mathscr{F}_t^X]$, for bounded and $\mathscr{F}_t$ measurable random variables U.

For all such U's,

$$E_Q[U|\mathscr{F}_t^X]=\frac{E_P[L_t U|\mathscr{F}_t^X]}{E_P(L_t|\mathscr{F}_t^X)} \qquad Q \text{ a.s.}$$

(let us remark that the set $\{\omega/E_P(L_t|\mathscr{F}_t^X)(\omega)=0\}$ is negligible for Q, and more generally the processes L, L_-, ${}^{1P}L$, ${}^{1P}L_-$ are strictly positive outside a Q evanescent set) and so, the problem is to express $E_P[L_t U|\mathscr{F}_t^X\mathrm{I}$ or, more precisely, ${}^{1P}(LU)$ (once again, all projections are relative to the filtration $\mathscr{F}^X$, and are defined from Dellacherie's book [2], and Section 1 if necessary).

We work in the following particular situation, already mainly considered in [1, 5, 17]:

In addition to H.1, H.2, H.3, H.4, we suppose:

$$\mathscr{F}_t=\mathscr{F}_t^X\vee\mathscr{U}, \qquad \text{with } \mathscr{F}_\infty^X \text{ and } \mathscr{U} \qquad P \text{ independent.} \tag{H.5}$$

Note that, for any $U\in b(\mathscr{U})$, $L_t U=U+\int_0^t U L_{s-}\,\varphi_s dX_s$ is a $(P,\mathscr{F})$ martingale.

Moreover, we have:

Lemma 6. *Let $U\in b(\mathscr{U})$. Then,*

$${}^{1P}(LU)=E_Q[U]+\int_0^t {}^{3P}(L_-\,U\varphi)\,dX. \tag{3.1}$$

The proof is obtained by putting together the proof of Theorem 1 and the additional equalities (deduced from H.5):

$$E_P(U|\mathscr{F}_t^X)=E_P(U)=E_P(L_0 U)=E_Q(U).$$

3.2. The Fundamental Filtering Equation

Let us specialize the above situation by assuming:

There is a Fellerian Markov process Z, valued in E (polish space, with Borel σ-field $\mathscr{E}$) such that $\mathscr{U}=\mathscr{F}_\infty^Z$. (H.6)

Let us recall that therefore the semi-group $P_t(x,dy)$ on $(E,\mathscr{E})$ attached to Z verifies:

$$\forall f\in C_b(E), \qquad (t,x)\to P_t(x,f) \text{ is bicontinuous.}$$

There is a bounded, $\mathscr{P}(\mathscr{F}^X)\otimes\mathscr{E}$ measurable function (H.7)

$\psi: (s,\omega,z)\to\psi_s(\omega,z)$ such that $\varphi_s(\omega)=\psi_s(\omega,Z_s(\omega))$

Taking $U=g(Z_{t_1})$, with $g\in b(\mathscr{E})$, we deduce from (3.1):

$$E_P[L_{t_1}g(Z_{t_1})|\mathscr{F}^X_{t_1}]=E_Q[g(Z_{t_1})]+\int_0^{t_1} {}^{3P}(L_-\,g(Z_{t_1})\,\psi(Z))\,dX \tag{3.2}$$

Using mainly (H.5) and (H.6), we obtain the:

Theorem 4. *Let* $g\in b(\mathscr{E})$, *and* $t_1\geqq 0$.
Under the hypotheses (H.1) ... (H.7), *we have:*

$$E_P[L_{t_1}g(Z_{t_1})|\mathscr{F}^X_{t_1}]=E_Q[g(Z_{t_1})]+\int_0^{t_1} dX_s\,{}^{3P}(L_{s-}\,\psi_s(Z_s)\,P_{t_1-s}\,g(Z_s)). \tag{3.3}$$

Proof. Let $\mathscr{G}$ be the right continuous P complete filtration obtained from $\mathscr{G}^0_t=\mathscr{F}^X_t\vee\mathscr{F}^Z_t$. Then:

$$\begin{aligned}{}^{3P}(L_-\,g(Z_{t_1})\,\psi(Z)) &= {}^{3P}({}^{1P/\mathscr{G}}(L_-\,g(Z_{t_1})\,\psi(Z)))\\ &= {}^{3P}(L_-\,\psi(Z)\,{}^{1P/\mathscr{G}}\,g(Z_{t_1}))\\ &= {}^{3P}(L_{t-}\,\psi_t(Z_t)\,P_{t_1-t}\,g(Z_t))\end{aligned}$$

(in the second equality, we consider $g(Z_{t_1})$ as a stochastic process, which is constant in $t\in R_+$; moreover, the time set here is $[0,t_1]$). Now, (3.3) follows from (3.2). ∎

We shall now obtain a generalization of Kunita's recursive filtering equation [8], the generalization being that we only work with a local martingale X having the predictable representation property described in Subsection 1.2, instead of taking a Wiener process for X (also, see [1]).

From now on, the time set is $[0,t_1]$, for a fixed $t_1>0$.

We need more notations: – for $g\in b(\mathscr{E})$, we set $\bar g_{t_1}=E_Q[g(Z_{t_1})]$

- $P^{t_1}g$ is the process: $(t,\omega)\to P_{t_1-t}\,g(Z_t(\omega))$
- we still write ψ for the process: $(t,\omega)\to\psi(t,\omega,Z_t(\omega))$
- if H is a bounded measurable process, we note $\hat H={}^{3Q}H$
- Finally, let $U_t=\bar g_{t_1}+\int_0^t {}^{3P}(L_-\,\psi\,P^{t_1}g)_s\,dX_s$ and $V_t={}^{1P}L_t$.

Before writing the filtering equation, we need to identify some projections.

Lemma 7. *If* $g\in b(\mathscr{E})$, *we have under* (H.1) ... (H.7):

$$\frac{U}{V}={}^{1Q}(g(Z_{t_1}))={}^{1Q}(P^{t_1}g) \tag{3.4}$$

and

$$\frac{U_-}{V_-}=\widehat{P^{t_1}g}. \tag{3.5}$$

Proof. From (H.6), if $g \in C_b(E)$, the process $\overset{t_1}{P}g$ is right continuous, and so, has a right continuous $\mathscr{F}^X$ optional projection. So, to show (3.4) for such a g, it is sufficient to show it for every t, and the result will follow by right continuity. By the monotone class theorem, (3.4) will then be valid for every $g \in b(\mathscr{E})$ so, we now look at $E_Q[P_{t_1-t}\, g(Z_t)|\mathscr{F}_t^X]$ which is equal (Subsection (3.1)) to:

$$\frac{E_P[L_t P_{t_1-t}\, g(Z_t)|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]} = \frac{E_P[L_t E_P[g(Z_{t_1})|\mathscr{G}_t]\,|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]}$$

(the filtration $\mathscr{G}$ has already been used in Theorem 3).

The last expression is obviously equal to:

$$\frac{E_P[L_t\, g(Z_{t_1})|\mathscr{F}_t^X]}{E_P(L_t|\mathscr{F}_t^X)}$$

which is $E_Q(g(Z_{t_1})|\mathscr{F}_t^X)$ thereby proving the right hand side of (3.4).

Moreover,

$$\frac{E_P[L_t\, g(Z_{t_1})|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]} = \frac{E_P[L_{t_1}\, g(Z_{t_1})|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]} + \Delta \tag{3.6}$$

with

$$\Delta = -\frac{E_P[(L_{t_1} - L_t)\, g(Z_{t_1})|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]}$$

L is a $\mathscr{F}$ martingale, which implies, as $g(Z_{t_1}) \in b(\mathscr{F}_0)$, that $g(Z_{t_1})L$ is also a $\mathscr{F}$ martingale. Then,

$$\Delta = -\frac{E_P[E_P[(L_{t_1} - L_t)\, g(Z_{t_1})|\mathscr{F}_t]|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]} = 0$$

So, from (3.6), we get:

$$E_Q[g(Z_{t_1})\,|\mathscr{F}_t^X] = \frac{E_P[L_{t_1}\, g(Z_{t_1})|\mathscr{F}_t^X]}{E_P[L_t|\mathscr{F}_t^X]}$$

and, from (3.3) (Theorem 3), we get the left hand side of (3.4). Now, (3.5) proceeds from (3.4) as the predictable projection of a martingale M is M_-.

For simplicity we now assume:

X is quasi-left continuous (relatively to $\mathscr{F}^X$). (H.8)

From [15] or [7], we know that (H.8) and the predictable representation property assumed for X by (H.4) (Subsection 1.2) imply the existence of a $\mathscr{F}^X$ predictable set A (resp: process f) such that:

$$E_P \int 1_{A^c}\, d\langle X^c, X^c\rangle = E_P \int 1_A\, d\langle X^d, X^d\rangle = 0$$

resp: $\Delta X = f\, I_{\Delta X \neq 0}$

Theorem 5. *Let $g \in b(\mathscr{E})$, and $t_1 > 0$.*

Under the hypotheses (H.1), ..., (H.8) *and supposing that the processes $(1 + \hat{\psi} f)^{-1}$ is $(\mathscr{F}^X, Q)$ locally bounded, we have:*

$$E_Q[g(Z_{t_1})|\mathscr{F}_{t_1}^X] = E_Q[g(Z_{t_1})] + \int_0^{t_1} [\widehat{\psi \times P\overset{t_1}{g}} - \hat{\psi}\widehat{P\overset{t_1}{g}}] \left(1_A + \frac{1_{A^c}}{1+\hat{\psi}f}\right) d\tilde{X} \tag{3.7}$$

with $\tilde{X} = X - \int_0^{\cdot} \hat{\psi}\, d\langle X, X\rangle \in \mathscr{M}_{\text{loc}}(Q, \mathscr{F}^X)$

Remarks. 1) The appearence of $d\tilde{X}$ in the right-member of (3.7) is a priori expected from Remark 3) in Subsection 1.2.

2) The technical hypothesis made on $(1+\hat{\psi}f)^{-1}$ does not seem too stringent, as this process is equal to V_-/V, on $\Delta X \neq 0$, and Q almost every trajectory of V_-/V is bounded on every compact set of $\mathbb{R}_+$.

Proof of the Theorem. Applying Ito's formula under P, to U_t/V'_t, with $V' = (V \vee \varepsilon)$, (which is again, from [10], and with the definition there of, a semimartingale), we get:

$$\begin{aligned} \frac{U_t}{V'_t} = \frac{U_0}{V'_0} &+ \int_0^t \left[\frac{dU_s}{V'_{s-}} - \frac{U_{s-}}{(V'_{s-})^2} dV'_s\right] + \int_0^t \frac{U_{s-}}{(V'_{s-})^3} d\langle (V')^c, (V')^c\rangle_s \\ &- \int_0^t \frac{1}{(V')^2_{s-}} d\langle U^c, (V')^c\rangle_s + \sum_{0<s\leq t} \frac{U_s}{V'_s} - \frac{U_{s-}}{V'_{s-}} - \left\{\frac{\Delta U_s}{V'_{s-}} - \frac{U_{s-}\Delta V_s}{(V'_{s-})^2}\right\} \end{aligned} \tag{3.8}$$

This equality is also valid Q as. Also, from [10] (Chapt. 6), we have:

$$(V')^c_t = \int_0^t 1_{(V_s > \varepsilon)}\, dV^c_s. \tag{3.9}$$

We note $T_\varepsilon = \inf\{t > 0;\ V_t < \varepsilon\}$.

As we have already remarked, V and V_- are strictly positive outside a Q evanescent set. So, the stopping times T_ε increase to ∞, Q a.s., as ε decreases to 0.

Moreover, on $\{\omega \mid T_\varepsilon(\omega) > t_1\}$, we can replace V' by V in Equation (3.8), using (3.9), and the local character of stochastic integrals [10]. Finally, Equation (3.8) is true under Q, with V replacing V'.

Let us now recall that from the definition of U, we have:

$$dU = V_- \times \widehat{\psi \times P\overset{t_1}{g}}\, dX \qquad P \text{ a.s. and } Q \text{ a.s.}$$

From Theorem 1, we get:

$$dV = V_-\, \hat{\psi}\, dX \qquad P \text{ a.s. and } Q \text{ a.s.}$$

Finally, from Lemma 7:

$$\frac{U_-}{V_-} = \widehat{P\overset{t_1}{g}} \qquad Q \text{ a.s.}$$

So, after some calculus, the expression of U_t/V_t becomes:

$$\begin{aligned} \frac{U_t}{V_t} = \bar{g}_{t_1} &+ \int_0^t [\widehat{\psi \times P\overset{t_1}{g}} - \hat{\psi} \times \widehat{P\overset{t_1}{g}}](dX - \hat{\psi}\, d\langle X^c, X^c\rangle) \\ &- \sum_{(0<s\leq t)} [\widehat{\psi \times P\overset{t_1}{g}} - \widehat{P\overset{t_1}{g}} \times \hat{\psi}]_s \frac{\hat{\psi}_s (\Delta X_s)^2}{(1+\hat{\psi}_s \Delta X_s)} \qquad Q \text{ a.s.} \end{aligned}$$

Now remembering that $\Delta X = f I_{\Delta X \neq 0}$, with $f \in \mathscr{P}(\mathscr{F}^X)$, we get:

$$\frac{U_t}{V_t} = \bar{g}_{t_1} + \int_0^t [\widehat{\psi \times P^{t_1} g} - \hat{\psi} \times \widehat{P^{t_1} g}] (dX - \hat{\psi} d\langle X^c, X^c \rangle)$$

$$- \int_0^t [\widehat{\psi \times P^{t_1} g} - \widehat{P^{t_1} g} \times \hat{\psi}] \left(\frac{\hat{\psi}}{1 + \hat{\psi} f}\right) (d\langle X^d, X^d \rangle + \Delta X \, dX) \qquad Q \text{ a.s.}$$

Here, we have used Pratelli-Yoeurp's formula (see [10]): P a.s. and Q a.s.,

$$\sum_{0 < s \leqq t} (\Delta X_s)^2 - \langle X^d, X^d \rangle_t = [X, X]_t - \langle X, X \rangle_t = \int_0^t (\Delta X_s) \, dX_s$$

In our case, $\Delta X \, dX = f I_{\Delta X \neq 0} \, dX = f \, dX^d$.
Again transforming the expression of U_t/V_t, we get, under Q:

$$\frac{U_t}{V_t} = \bar{g}_{t_1} + \int_0^t [\widehat{\psi \times P^{t_1} g} - \hat{\psi} \times \widehat{P^{t_1} g}] \left\{ dX^c + \frac{1}{1 + \hat{\psi} f} dX^d \right\}$$

$$- \int_0^t [\widehat{\psi \times P^{t_1} g} - \hat{\psi} \times \widehat{P^{t_1} g}] \hat{\psi} \left\{ d\langle X^c, X^c \rangle + \frac{1}{1 + \hat{\psi} f} d\langle X^d, X^d \rangle \right\}$$

Now, we make use of the $\mathscr{F}^X$ predictable set A, whose existence was recalled before the theorem, by remarking:

$$\left(1_A + \frac{1}{1 + \hat{\psi} f} 1_{A^c}\right) d\langle X, X \rangle = d\langle X^c, X^c \rangle + \frac{1}{1 + \hat{\psi} f} d\langle X^d, X^d \rangle \qquad Q \text{ a.s.}$$

and:

$$\left(1_A + \frac{1_{A^c}}{1 + \hat{\psi} f}\right) dX = dX^c + \frac{1}{1 + \hat{\psi} f} dX^d \qquad Q \text{ a.s.}$$

Then, (3.7) proceeds from the last expression we obtained for U_t/V_t.

Remarks. 1) From the end of the proof, Equation (3.7) can also be written as:

$$E_Q[g(Z_{t_1}) | \mathscr{F}_{t_1}^X]$$
$$= E_Q[g(Z_{t_1})] + \int_0^{t_1} [\widehat{\psi \times P^{t_1} g} - \hat{\psi} \times \widehat{P^{t_1} g}] \left\{ d\tilde{X}^c + \frac{1}{1 + \hat{\psi} f} d\tilde{X}^d \right\} \tag{3.7'}$$

where:

$$\tilde{X}_t^c = X_t^c - \int_0^t \hat{\psi} \, d\langle X^c, X^c \rangle, \qquad \tilde{X}_t^d = X_t^d - \int_0^t \hat{\psi} \, d\langle X^d, X^d \rangle$$

Moreover, from [7], (Proposition 2.1), $\tilde{X}^d$ belongs to $\mathscr{M}_{\text{loc}}^d(Q, \mathscr{F}^X)$. Therefore, $\tilde{X} = \tilde{X}^c + \tilde{X}^d$ is the decomposition of $\tilde{X}$ in $\mathscr{M}_{\text{loc}}^c(Q, \mathscr{F}^X) + \mathscr{M}_{\text{loc}}^d(Q, \mathscr{F}^X)$.

2) We refer the reader to [1], when the observation consists of a marked point process and a Wiener process plus a drift.

3) Theorem 5 is an extension of results in [5] and [17].

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