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Optimal Stopping and Almost Sure Convergence of Random Sequences

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0. Introduction

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space, $(\mathscr{F}_n)$ an increasing sequence of σ-algebras contained in $\mathscr{F}$. Suppose that all events of probability zero belong to $\mathscr{F}_0$. A nonnegative integer-valued (possibly, $+\infty$) random variable (r.v.) T is called a stopping time (s.t.) if for all n the event $\{T=n\}$ belongs to $\mathscr{F}_n$. We denote by $\overline{\mathfrak{M}}$, $\mathfrak{M}$, and $\mathfrak{M}^b$ the sets of all s.t.'s, a.s. finite s.t.'s, and bounded s.t.'s, resepctively. We shall consider a sequence $X=(X_n)$ of (real-valued) r.v.'s adapted to $(\mathscr{F}_n)$. For $T\in\overline{\mathfrak{M}}$ define X_T by

$$X_T(\omega)=\begin{cases} X_n(\omega), & \text{if } T(\omega)=n \\ \limsup\limits_n X_n(\omega), & \text{if } T(\omega)=+\infty. \end{cases}$$

Let us introduce the class $\overline{\mathfrak{M}}(X)$ of all s.t.'s $T\in\overline{\mathfrak{M}}$ satisfying the condition that the integral $\mathbf{E}X_T$ exists, that means[1], $\mathbf{E}X_T^+<+\infty$ or $\mathbf{E}X_T^-<+\infty$. Set $\mathfrak{M}(X)=\overline{\mathfrak{M}}(X)\cap\mathfrak{M}$ and $\mathfrak{M}^b(X)=\overline{\mathfrak{M}}(X)\cap\mathfrak{M}^b$.

For a class $\mathfrak{N}\subseteqq\overline{\mathfrak{M}}$ a random sequence (X_n) is called a generalized $\mathfrak{N}$-regular supermartingale with respect to $(\mathscr{F}_n)$ if (X_n) is adapted to $(\mathscr{F}_n)$, the integrals $\mathbf{E}X_T$ exist for $T\in\mathfrak{N}$ and the inequality[2]

$$\mathbf{E}(X_T|\mathscr{F}_S)\leqq X_S \quad \text{a.s.}$$

holds for every pair $S, T\in\mathfrak{N}$ such $T\geqq S$. If $\mathfrak{N}$ is the set of nonnegative integers, then we simply speak of generalized supermartingales.

In the problem of optimal stopping one considers the value[3]

$$V=\sup_{T\in\mathfrak{M}(X)}\mathbf{E}X_T$$

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[1] As usual, $X^+=\max(0,X)$ and $X^-=\max(0,-X)$

[2] For any $T\in\overline{\mathfrak{M}}$ define the σ-algebra $\mathscr{F}_T$ as the collection of all $A\in\mathscr{F}_\infty=\sigma(\bigcup_{n\geqq 0}\mathscr{F}_n)$ such that $A\cap\{T=n\}$ belongs to $\mathscr{F}_n$ for every $n\geqq 0$

[3] Of course, $\sup\emptyset=-\infty$ and $\inf\emptyset=+\infty$

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which is interpreted as the maximal gain that can be obtained by stopping the reward sequence (X_n) in an optimal way. Analogously, for any s.t. $S \in \mathfrak{M}(X)$ the value

$$V_S = \sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E} X_T$$

is interpreted as the maximal gain by stopping after S. We now introduce the "value at time infinity"

$$V_\infty = \inf_{S \in \mathfrak{M}(X)} V_S.$$

This value can be regarded as the maximal gain which can still be obtained by stopping (X_n) after an arbitrary long period of time.

In the present note we investigate the "value at time infinity" V_∞. We give an explicite expression for evaluating it. Using this characterisation of V_∞ we come to necessary and sufficient conditions for the a.s. convergence of the sequence (X_n). Connections with other works are pointed out. Our results are stated under most general assumptions. The main results of the paper are Theorem 2.1, Theorem 3.4, and Theorem 4.4.

1. Preliminaries

Obviously the set $\overline{\mathfrak{M}}$ is partially ordered and directed since for arbitrary $S, T \in \overline{\mathfrak{M}}$ the r.v. $S \vee T$[4] also belongs to $\overline{\mathfrak{M}}$. But in general the ordered set $\mathfrak{M}(X)$ does not have this property. We need $\mathfrak{M}(X)$ to be directed to reflect on nets $(EX_T)_{T \in \mathfrak{M}(X)}$.

Lemma 1.1. *Suppose that* $\mathbf{E} \limsup_n X_n$ *exists. Then for all* $S \in \mathfrak{M}$ *there exists an s.t.* T *such that* $T \geqq S$ *and* $T \in \mathfrak{M}(X)$. *In particular, the set* $\mathfrak{M}(X)$ *is directed.*

Proof. First suppose that $\mathbf{E}(\limsup_n X_n)^+ < +\infty$. Let $\varepsilon > 0$, $S \in \mathfrak{M}$ and define

$$T = \min\{n \geqq S \colon X_n \leqq \mathbf{E}((\limsup_n X_n)^+ | \mathscr{F}_n) + \varepsilon\}.$$

Obviously, $T \geqq S$ and T is an s.t. Because of the martingale convergence theorem of P. Levy T is a.s. finite. Hence

$$X_T \leqq \mathbf{E}((\limsup_n X_n)^+ | \mathscr{F}_T) + \varepsilon$$

that means $T \in \mathfrak{M}(X)$ and $\mathbf{E} X_T < +\infty$. For $\mathbf{E}(\limsup_n X_n)^- < +\infty$ the analogous result follows.

Naturally, Lemma 1.1. also holds if $\mathbf{E} \liminf_n X_n$ exists.

[4] Write $a \vee b$ for $\max(a, b)$ and $a \wedge b$ for $\min(a, b)$

Definition 1.1. We introduce the notations

(1) $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \inf_{S\in\mathfrak{M}(X)} \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T = V_\infty$,

(2) $\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \sup_{S\in\mathfrak{M}(X)} \inf_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T$.

(3) We say that the limit of the net $(\mathbf{E}X_T)_{T\in\mathfrak{M}(X)}$ exists if

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$$

and write $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ for it.

Remark 1.1. Note that in view of Lemma 1.1

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \inf_{S\in\mathfrak{M}} \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T \qquad \text{if } \mathbf{E}\limsup_n X_n \text{ (or } \mathbf{E} \liminf_n X_n)$$

exists.

We want to obtain a "Fatou-equation"

$$\begin{gathered}\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\limsup_n X_n \\ \left(\text{resp., } \liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\liminf_n X_n\right).\end{gathered} \tag{1.1}$$

This equation was proved by W. Sudderth [10] under the assumption that the random sequence (X_n) is bounded above (resp., below) by an integrable r.v. R. Chen [3] generalized this result assuming only that the family $(X_T^+)_{T\in\mathfrak{M}}$ (resp., $(X_T^-)_{T\in\mathfrak{M}}$) is uniformly integrable. His proof is only a slight modification of that of W. Sudderth. We will get Eq. (1.1) under most general assumptions. For (1.1) to make sense the integral $\mathbf{E}\limsup_n X_n$ must exist. We obtain a second condition by considering the example in [10]: There (X_n) is a nonnegative uniformly integrable sequence converging to zero. Equation (1.1) does not hold since $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \geqq 1$. One can easily verify that, indeed, $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = +\infty$.

Definition 1.2. (1) $X=(X_n)$ belongs to the class $\mathscr{L}^*$ if $\mathbf{E}\limsup_n X_n$ exists and $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T < +\infty$.

(2) X belongs to the class $\mathscr{L}_*$ if $-X$ belongs to $\mathscr{L}^*$.

(3) We set $\mathscr{L} = \mathscr{L}^* \cap \mathscr{L}_*$.

Definition 1.3. We say that X has the property $(*)$ if for all $S\in\mathfrak{M}$ there exists an s.t. $T\in\mathfrak{M}(X)$ such that $T\geqq S$ and $\mathbf{E}X_T > -\infty$.

Now we give two lemmas which we need for the proof of the announced theorem. Before let us define

$$Y_n = \operatorname*{ess\,sup}_{\substack{T\in\mathfrak{M}(X)\\ T\geqq n}} \mathbf{E}(X_T|\mathscr{F}_n) \tag{1.2}$$

Lemma 1.2. *Assume that X has the property* $(*)$. *Then*

(i) (Y_n) *is a generalized* $\mathfrak{M}$*-regular supermartingale;*

(ii) $Y_S \geqq X_S$ *a.s. for* $S \in \mathfrak{M}(X)$;

(iii) (Y_n) *is the least sequence satisfying statements* (i) *and* (ii);

(iv) $\mathbf{E} Y_S = \sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E} X_T$ *for* $S \in \mathfrak{M}$;

(v) *if, in addition,* $\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T < +\infty$ *then*

$$\liminf_n Y_n \leqq \limsup_n X_n \quad \text{a.s.}$$

Proof. These results are well-known in the theory of optimal stopping if (X_n) is bounded below by an integrable r.v. (cf. J.L. Snell [9], G.W. Haggstrom [6]). Because of the importance for Theorem 2.1 we give the proof in a similar way.

First note that for $S \in \mathfrak{M}$

$$Y_S = \operatorname*{ess\,sup}_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E}(X_T | \mathscr{F}_S).$$

From this and the properties of ess sup it follows that for all $S \in \mathfrak{M}$ there exists a sequence $(T_k) \subseteqq \mathfrak{M}(X)$ such that $T_k \geqq S$ and

$$Y_S = \sup_k \mathbf{E}(X_{T_k} | \mathscr{F}_S) \quad \text{a.s.} \tag{1.3}$$

Hence Y_n is $\mathscr{F}_n$-measurable and because of property $(*)$ the integrals $\mathbf{E} Y_S$ exist and are not equal to $-\infty$ for all $S \in \mathfrak{M}$. Now let us prove the supermartingale inequality. Fix $S \in \mathfrak{M}$. Use (1.3) and choose T_1 such that $T_1 \geqq S$, $T_1 \in \mathfrak{M}(X)$, and $\mathbf{E} X_{T_1} > -\infty$ (property $(*)$). Define a new sequence $(S_k) \subseteqq \mathfrak{M}(X)$ in the following way:

$$S_1 = T_1$$

$$S_{k+1} = \begin{cases} S_k, & \text{if } \mathbf{E}(X_{S_k} | \mathscr{F}_S) \geqq \mathbf{E}(X_{T_{k+1}} | \mathscr{F}_S) \\ T_{k+1} & \text{otherwise.} \end{cases}$$

The sequence $(\mathbf{E}(X_{S_k} | \mathscr{F}_S))$ is increasing now and therefore

$$Y_S = \sup_k \mathbf{E}(X_{S_k} | \mathscr{F}_S) = \lim_k \mathbf{E}(X_{S_k} | \mathscr{F}_S) \quad \text{a.s.}$$

Because of $\mathbf{E} X_{S_1} > -\infty$, from B. Levi's theorem on monotone convergence for $R \in \mathfrak{M}$ with $R \leqq S$

$$\begin{aligned} \mathbf{E}(Y_S | \mathscr{F}_R) &= \mathbf{E}(\lim_k \mathbf{E}(X_{S_k} | \mathscr{F}_S) | \mathscr{F}_R) = \lim_k \mathbf{E}(X_{S_k} | \mathscr{F}_R) \\ &\leqq \sup_k \mathbf{E}(X_{S_k} | \mathscr{F}_R) \leqq Y_R \quad \text{a.s.,} \end{aligned}$$

i.e. the supermartingale inequality holds and (i) is proved. Statement (ii) is obvious. Let (W_n) satisfy (i) and (ii). Then for all $T \in \mathfrak{M}(X)$, $T \geqq n$

$$W_n \geqq \mathbf{E}(W_T | \mathscr{F}_n) \geqq \mathbf{E}(X_T | \mathscr{F}_n) \quad \text{a.s.}$$

and thus

$$W_n \geqq Y_n \quad \text{a.s.}$$

which proves (iii).

For all $S \in \mathfrak{M}$ and $T \in \mathfrak{M}(X)$ such that $T \geqq S$ one has

$$\mathbf{E}\, Y_S \geqq \mathbf{E}\, Y_T \geqq \mathbf{E}\, X_T.$$

Consequently,

$$\mathbf{E}\, Y_S \geqq \sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E}\, X_T.$$

On the other hand, consider the above defined sequence $(S_k) \subseteqq \mathfrak{M}(X)$ such that $S_k \geqq S$, $\mathbf{E}\, X_{S_1} > -\infty$, and $\mathbf{E}(X_{S_k} | \mathscr{F}_S))$ is increasingly converging to Y_S. This gives

$$\mathbf{E}\, Y_S = \mathbf{E}(\lim_k \mathbf{E}(X_{S_k} | \mathscr{F}_S)) = \lim_k \mathbf{E}\, X_{S_k} \leqq \sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E}\, X_T$$

for all $S \in \mathfrak{M}$.

Finally, let us prove (v). Because of $\limsup_{T \in \mathfrak{M}(X)} \mathbf{E}\, X_T < +\infty$ there exists an s.t. $S \in \mathfrak{M}(X)$ such that $\sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E}\, X_T < +\infty$. According to statement (iv) $\mathbf{E}\, Y_S < +\infty$. For this s.t. $S \in \mathfrak{M}(X)$ one can find a sequence of s.t.'s (S_m) such that $S_m \in \mathfrak{M}(X)$ and $S_m \geqq S \vee m$ (cf. Lemma 1.1). Let $\varepsilon > 0$ and define

$$B = B(m, \varepsilon) = \{X_n \leqq Y_n - \varepsilon \text{ for all } n \geqq S_m\}.$$

By way of contradiction suppose that for some $\varepsilon > 0$ and $m \geqq 0$ it holds $\mathbf{P}(B) > 0$. Then for all $T \in \mathfrak{M}(X)$ with $T \geqq S_m$[5]

$$\mathbf{E}\, \chi_B X_T \leqq \mathbf{E}\, \chi_B Y_T - \varepsilon \mathbf{P}(B).$$

Statement (ii) gives

$$\mathbf{E}\, X_T \leqq \mathbf{E}\, Y_T - \varepsilon \mathbf{P}(B)$$

and by statement (i)

$$\mathbf{E}\, X_T \leqq \mathbf{E}\, Y_{S_m} - \varepsilon \mathbf{P}(B).$$

Because of $\mathbf{E}\, Y_{S_m} \leqq \mathbf{E}\, Y_S < +\infty$ and $\varepsilon \mathbf{P}(B) > 0$ this inequality contradicts statement (iv). Therefore

$$\mathbf{P}(B) = \mathbf{P}(\{X_n \leqq Y_n - \varepsilon \text{ for all } n \geqq S_m\}) = 0$$

for all $\varepsilon > 0$ and all $m \geqq 0$. Consequently, for all $\varepsilon > 0$

$$\mathbf{P}(\{X_n \geqq Y_n - \varepsilon \text{ infinitely often}\}) = 1$$

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5 Let χ_A be the indicator function of the set A

which obviously implies

$$\mathbf{P}(\liminf_n Y_n \leqq \limsup_n X_n) = 1$$

Remark 1.2. It is worth while to notice that $\mathfrak{M}(Y) = \mathfrak{M}$ if X has the property (*).

Lemma 1.3. *For all* $S \in \mathfrak{M}$

$$\sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E} X_T = \sup_{\substack{T \in \overline{\mathfrak{M}}(X) \\ T \geqq S}} \mathbf{E} X_T.$$

Proof. This equality, which means that the value V_S does not change using arbitrary s.t.'s, is well-known under conditions of integrability on the sequence (X_n) (cf. [4], or for Markov processes, [8]). Here we give a short proof based on P. Levi's martingale convergence theorem.

It suffices to show that

$$\mathbf{E} X_T \leqq \sup_{\substack{R \in \mathfrak{M}(X) \\ R \geqq S}} \mathbf{E} X_R \tag{1.4}$$

for all $S \in \mathfrak{M}$ and $T \in \overline{\mathfrak{M}}(X)$ with $T \geqq S$.

Let $S \in \mathfrak{M}$, $T \in \overline{\mathfrak{M}}(X)$ with $T \geqq S$, and $\varepsilon > 0$ be fixed. Define

$$R = \min\{n \geqq S : \mathbf{E}(X_T | \mathscr{F}_n) \leqq X_n + \varepsilon\}$$

which is obviously an s.t. Inequality (1.4) is trivial if $\mathbf{E}X_T = -\infty$. Let us assume now that X_T is integrable and return later to the case $\mathbf{E} X_T = +\infty$. On the set $\{T = n\}$ one has

$$R \leqq T = n \quad \text{a.s.}$$

and therefore on $\{T < +\infty\}$

$$R \leqq T < +\infty \quad \text{a.s.} \tag{1.5}$$

It remains to verify that $R < +\infty$ on $\{T = +\infty\}$. Clearly

$$\mathbf{E}(X_T | \mathscr{F}_n) \geqq X_n + \varepsilon \quad \text{a.s.} \quad \text{on } \{R = +\infty\}$$

for all $n \geqq 0$, the martingale convergence theorem yields

$$X_T = \lim_n \mathbf{E}(X_T | \mathscr{F}_n) \geqq \limsup_n X_n + \varepsilon \quad \text{a.s.} \quad \text{on } \{R = +\infty\}.$$

From (1.5) the inclusion $\{R = +\infty\} \subseteqq \{T = +\infty\}$ a.s. holds and, consequently,

$$\limsup_n X_n \geqq \limsup_n X_n + \varepsilon \quad \text{a.s.} \quad \text{on } \{R = +\infty\}.$$

Hence the set $\{R = +\infty\}$ has probability zero, i.e. $\{R < +\infty\}$ a.s. Thus the s.t. R belongs to $\mathfrak{M}$. From the definition of R one has $R \in \mathfrak{M}(X)$ and $\mathbf{E} X_T \leqq \mathbf{E} X_R + \varepsilon$. This gives

$$\mathbf{E} X_T \leqq \sup_{\substack{R \in \mathfrak{M}(X) \\ R \geqq S}} \mathbf{E} X_R + \varepsilon$$

and, after passage to the limit for $\varepsilon \downarrow 0$, inequality (1.4). Finally, suppose $\mathbf{E}X_T = +\infty$. Let a be a positive real number and apply the result just obtained to the integrable r.v. $X_T^a = X_T \wedge a$. Therefore

$$\mathbf{E}X_T^a \leqq \sup_{\substack{R\in\mathfrak{M}\\ R\geqq S}} \mathbf{E}X_R^a = \sup_{\substack{R\in\mathfrak{M}\\ EX_R > -\infty\\ R\geqq S}} \mathbf{E}X_R^a = \sup_{\substack{R\in\mathfrak{M}(X)\\ R\geqq S}} \mathbf{E}X_R^a \leqq \sup_{\substack{R\in\mathfrak{M}(X)\\ R\geqq S}} \mathbf{E}X_R.$$

Since X_T^a is increasing, (1.4) follows from B. Levi's monotone convergence theorem.

2. Inequalities and Equalities for the Value at Infinity

As a direct consequence of Lemma 1.3 we give

Proposition 2.1 (cf. W. Sudderth [10]).

(i) *If* $\mathbf{E}\limsup_n X_n$ *exists, then*

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \geqq \mathbf{E}\limsup_n X_n$$

(ii) *If* $\mathbf{E}\liminf_n X_n$ *exists, then*

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \mathbf{E}\liminf_n X_n.$$

Proof. Of course, (ii) follows applying (i) to $(-X_n)$. From Lemma 1.3

$$\sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T \geqq \mathbf{E}\limsup_n X_n$$

for all $S\in\mathfrak{M}(X)$ since by assumption $+\infty$ belongs to $\overline{\mathfrak{M}}(X)$ and the proof is finished.

Now we formulate the basic result of the paper which is the already mentioned "Fatou-equation".

Theorem 2.1. (i) *If X belongs to $\mathscr{L}^*$, then*

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\limsup_n X_n.$$

(ii) *If X belongs to $\mathscr{L}_*$, then*

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\liminf_n X_n.$$

Proof. It is sufficient to prove (i). Because of Proposition 2.1 it remains to verify the inequality

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \mathbf{E}\limsup_n X_n.$$

Nothing has to be shown if $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = -\infty$ or $\mathbf{E}\limsup_n X_n = +\infty$. Therefore one restricts oneself to the case $\mathbf{E}\limsup_n X_n < +\infty$ and $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T > -\infty$. But, if

$\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T > -\infty$ it is easy to see that in view of Remark 1.1 X is then satisfying property (∗). Consequently, Lemma 1.2 is valid for the sequence (Y_n) defined in (1.2). Thus from the Remarks 1.1 and 1.2 and Lemma 1.2(iv)

$$\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T = \inf_{S \in \mathfrak{M}(Y)} \sup_{\substack{T \in \mathfrak{M}(X) \\ T \geqq S}} \mathbf{E} X_T = \inf_{S \in \mathfrak{M}(Y)} \mathbf{E} Y_S.$$

Lemma 1.2(v) implies the existence of $\mathbf{E} \liminf_n Y_n$. Hence from Proposition 2.1(ii) and Lemma 1.2(v)

$$\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T \leqq \mathbf{E} \liminf_n Y_n \leqq \mathbf{E} \limsup_n X_n,$$

which completes the proof of the theorem.

Now we consider one special case in which the conditions of Theorem 2.1 can be weakened.

Proposition 2.2. *Suppose that for some* $T_0 \in \mathfrak{M}$ *we have* $\mathscr{F}_{T_0} = \mathscr{F}_\infty$.

(i) *If* $\mathbf{E} \limsup_n X_n$ *exists then*

$$\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T = \mathbf{E} \limsup_n X_n.$$

(ii) *If* $\mathbf{E} \liminf_n X_n$ *exists then*

$$\liminf_{T \in \mathfrak{M}(X)} \mathbf{E} X_T = \mathbf{E} \liminf_n X_n.$$

Proof. Assertion (ii) is an immediate consequence of (i). In view of Proposition 2.1(i) it suffices to verify

$$\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T \leqq \mathbf{E} \limsup_n X_n.$$

For $\mathbf{E} \limsup_n X_n = +\infty$ the inequality is clear. Therefore suppose that $\mathbf{E} \limsup_n X_n < +\infty$. Let $\varepsilon > 0$ and define

$$S = \max \{ n \geqq T_0 : X_{n-1} \geqq \limsup_n X_n + \varepsilon \}$$

which is an s.t. because of the assumption. Obviously, $S \in \mathfrak{M}(X)$ and for all $T \in \mathfrak{M}(X)$ such that $T \geqq S$ one has

$$\mathbf{E} X_T \leqq \mathbf{E} \limsup_n X_n + \varepsilon.$$

Hence

$$\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T \leqq \mathbf{E} \limsup_n X_n + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ the assertion follows.

Remarks 2.1. (a) In particular, the condition of Proposition 2.2 is fulfilled in the totally previsible case: $\mathscr{F}_n = \mathscr{F}_\infty$ for all $n \geqq 0$. In this case $\overline{\mathfrak{M}}$ consists of all nonnegativer r.v.'s.

(b) It is interesting to notice that the value $\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ does not change when $(\mathscr{F}_n)$ is replaced by a new family $(\mathscr{G}_n)$ of σ-algebras contained in $\mathscr{F}$. The only conditions are that X belongs to $\mathscr{L}^*$ with respect to both $(\mathscr{G}_n)$ and $(\mathscr{F}_n)$ and, of course, $\mathscr{F}_n^X \subseteqq \mathscr{G}_n$ for all $n \geqq 0$ where $\mathscr{F}_n^X$ is the smallest σ-algebra relative to which X_m is measurable for every $m \leqq n$.

(c) Now we consider the case where $\mathscr{G}_n = \mathscr{F}$ for all $n \geqq 0$. Using Proposition 2.2 we observe that $\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ with respect to $(\mathscr{G}_n)$ cannot be equal to $+\infty$ if X belongs to $\mathscr{L}^*$ (with respect to $(\mathscr{F}_n)$). Thus, if X belongs to $\mathscr{L}^*$ (with respect to $(\mathscr{F}_n)$) the value $\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ does not change by passing over to $(\mathscr{G}_n)$.

(d) Suppose now that X is a Markov sequence and $(\mathscr{G}_n)$ such that $\mathscr{F}_n^X \subseteqq \mathscr{G}_n \subseteqq \mathscr{F}_n$ for all $n \geqq 0$. Then it can be proved that X belongs to $\mathscr{L}^*$ with respect to $(\mathscr{F}_n)$ if X belongs to $\mathscr{L}^*$ with respect to $(\mathscr{G}_n)$. Therefore, if X belongs to $\mathscr{L}^*$ with respect to $(\mathscr{G}_n)$ the extension of $(\mathscr{G}_n)$ to $(\mathscr{F}_n)$ does not change the value $\limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$.

3. Necessary and Sufficient Conditions for the Almost Sure Convergence

Now we investigate the connections between equalities for the value at infinity and a.s. convergence of random sequences. First we formulate sufficient conditions.

Theorem 3.1. *Suppose that* $\mathbf{E} \liminf_n X_n$, $\mathbf{E} \limsup_n X_n$, *and* $\lim_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ *exist. Then*

$$\lim_{T \in \mathfrak{M}(X)} \mathbf{E} X_T = \mathbf{E} \liminf_n X_n = \mathbf{E} \limsup_n X_n. \tag{3.1}$$

If one (and therefore all) of the values is finite, then $\lim_n X_n$ *exists a.s. and is integrable.*

Proof. From Proposition 2.1

$$\liminf_{T \in \mathfrak{M}(X)} \mathbf{E} X_T \leqq \mathbf{E} \liminf_n X_n \leqq \mathbf{E} \limsup_n X_n \leqq \limsup_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$$

and since $\lim_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ exists equality (3.1) holds. The second assertion is obvious.

Now we apply Theorem 3.1 to supermartingales.

Theorem 3.2. *Let* (X_n) *be a generalized* $\mathfrak{M}(X)$*-regular supermartingale and suppose that* $\mathbf{E} \liminf_n X_n$ *and* $\mathbf{E} \limsup_n X_n$ *exist. Then*

$$\mathbf{E} \liminf_n X_n = \mathbf{E} \limsup_n X_n = \inf_{T \in \mathfrak{M}(X)} \mathbf{E} X_T.$$

If one (and therefore all) of the values is finite, then $\lim_n X_n$ *exists a.s. and is integrable.*

Proof. This theorem is a direct consequence of Theorem 3.1: If (X_n) is a generalized $\mathfrak{M}(X)$-regular supermartingale, the net $(\mathbf{E}X_T)_{T\in\mathfrak{M}(X)}$ is decreasing and therefore $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is equal to $\inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$.

We next state connections between generalized $\mathfrak{M}(X)$-regular supermartingales and properties of uniform integrability.

Theorem 3.3. *Let (X_n) be a generalized $\mathfrak{M}(X)$-regular supermartingale. Suppose that one of the following conditions is satisfied:*

(a) $\mathbf{E}\liminf_n X_n$ *exists and is not equal to* $+\infty$.

(b) $\mathbf{E}\liminf_n X_n$ *exists and there is an* s.t. $T\in\mathfrak{M}(X)$ *such that* $\mathbf{E}X_T<+\infty$

(c) $\mathfrak{M}(X)=\mathfrak{M}$.

Then the following two conditions are equivalent:

(1) $\mathbf{E}\liminf_n X_n$ *and* $\mathbf{E}\limsup_n X_n$ *exist and one (and therefore all) of the values* $\mathbf{E}\liminf_n X_n$, $\mathbf{E}\limsup_n X_n$, *and* $\inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *is not equal to* $-\infty$.

(2) *The family* $(X_T^-)_{T\in\mathfrak{M}(X)}$ *is uniformly integrable.*

Proof. First assume condition (a) or (b).

Let (2) be satisfied. By Proposition 2.1

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \mathbf{E}\liminf_n X_n$$

and, since (X_n) is a generalized $\mathfrak{M}(X)$-regular supermartingale,

$$\inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \mathbf{E}\liminf_n X_n.$$

Because $(X_T^-)_{T\in\mathfrak{M}(X)}$ is uniformly integrable one has $\sup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T^- <+\infty$. Consequently, the integral $\mathbf{E}\limsup_n X_n$ exists and

$$-\infty < \inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \mathbf{E}\liminf_n X_n \leqq \mathbf{E}\limsup_n X_n,$$

proving (1).

Conversely, let (1) be satisfied. By Theorem 3.2 and Lemma 1.3

$$\mathbf{E}\liminf_n X_n = \mathbf{E}\limsup_n X_n = \inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \tag{3.2}$$

$$= \inf_{T\in\overline{\mathfrak{M}}(X)} \mathbf{E}X_T$$

$$\leqq \mathbf{E}X_T \tag{3.3}$$

for all $T\in\overline{\mathfrak{M}}(X)$. Let $S\in\mathfrak{M}(X)$ and $A\in\mathscr{F}_S$. Define

$$S_A(\omega)=\begin{cases} S & \text{if } \omega\in A \\ +\infty & \text{otherwise.}\end{cases}$$

Inequality (3.3) is then true for S_A and thus

$$\mathbf{E}\limsup_n X_n \leqq \mathbf{E}\chi_A X_S + \mathbf{E}\chi_{\bar{A}} \limsup_n X_n.$$

By assumption, condition (a) or (b), and equality (3.2) now $\limsup_n X_n$ is integrable. This gives for every $A \in \mathscr{F}_S$

$$\mathbf{E}\chi_A \limsup_n X_n \leqq \mathbf{E}\chi_A X_S$$

which means

$$\mathbf{E}(\limsup_n X_n | \mathscr{F}_S) \leqq X_S \quad \text{a.s.}$$

Hence

$$X_S^- \leqq \mathbf{E}((\limsup_n X_n)^- | \mathscr{F}_S) \quad \text{a.s.}$$

for all $S \in \mathfrak{M}(X)$ proving the uniform integrability of $(X_S^-)_{S \in \mathfrak{M}(X)}$.

Finally, suppose (c). Then $(-X_n^-)$ is a $\mathfrak{M}$-regular supermartingale which, obviously, is satisfying (a). Therefore the theorem is true for $(-X_n^-)$. From this follows immediately that the theorem holds for (X_n), too.

It seems that the implication (1)→(2) was not known previously. Now we give several equivalent conditions to characterize generalized supermartingales.

Proposition 3.1. *Let (X_n) be a generalized supermartingale. The following conditions are equivalent.*

(1) *(X_n^-) is uniformly integrable.*

(2) *$(X_T^-)_{T \in \mathfrak{M}}$ is uniformly integrable.*

(3) *(X_n) is $\mathfrak{M}$-regular, $\inf_{T \in \mathfrak{M}} \mathbf{E} X_T > -\infty$, and $\mathbf{E} \liminf_n X_n$ exists.*

(4) *(X_n) is $\mathfrak{M}$-regular and $\mathbf{E} \liminf_n X_n$ exists and is not equal to $-\infty$.*

If one (and therefore all) of the conditions is satisfied then $\lim_n X_n$ *exists* a.s. *and* $\mathbf{E} \lim_n X_n > -\infty$. *If, moreover, there is an* s.t. $T \in \mathfrak{M}$ *such that* $\mathbf{E} X_T^+ < +\infty$ *then* $\lim_n X_n$ *is integrable.*

Proof. The implications (1)→(2)→(3)→(4)→(1) will be proved. Since (X_n^-) is a submartingale, (1)→(2) is well-known (cf. [7], TV 30).

Let (2) be satisfied. Then (X_n) is $\mathfrak{M}$-regular since, in particular, (X_n^-) is uniformly integrable (cf. [7], TV 17, TV 28). Thus (X_n) is a generalized $\mathfrak{M}$-regular supermartingale such that $(X_T^-)_{T \in \mathfrak{M}}$ is uniformly integrable. According to Theorem 3.3 the condition (3) holds.

Suppose now (3). In view of Proposition 2.1

$$\mathbf{E} \liminf_n X_n \geqq \inf_{T \in \mathfrak{M}} \mathbf{E} X_T > -\infty$$

which proves (3)→(4).

If (4) is satisfied then also $\mathbf{E} \limsup_n X_n$ exists and thus, from Theorem 3.3 condition (1) follows.

Finally, if one of the conditions (1), (2), (3) or (4) holds then (X_n^a) with $X_n^a = X_n \wedge a$ for $a \geqq 0$ is satisfying the conditions of Theorem 3.2. Consequently, $\lim_n X_n^a$ exists a.s.

and is integrable. From this can easily be derived that $\lim_n X_n$ exists a.s. By condition (4), $\mathbf{E}\lim_n X_n$ exists and is not equal to $-\infty$. The additional condition of the existence of an s.t. $T\in\mathfrak{M}$ such that $\mathbf{E}X_T^+ < +\infty$ implies

$$\mathbf{E}\lim_n X_n \leqq \limsup_{T\in\mathfrak{M}} \mathbf{E}X_T = \inf_{T\in\mathfrak{M}} \mathbf{E}X_T < +\infty,$$

i.e. $\lim_n X_n$ is integrable.

We return to the general situation and give necessary and sufficient conditions for the a.s. convergence. First we give a theorem that looks like Lebesgue's theorem on changing the order of limit and integral.

Theorem 3.4. *Let X belong to $\mathscr{L}$. The following conditions are equivalent:*

(1) $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists.*

(2) $\lim_n X_n$ *exists a.s.*

If one of these conditions is satisfied then $\lim_n X_n$ is integrable and

$$\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\lim_n X_n. \tag{3.4}$$

Proof. Theorem 3.1 gives (1)→(2). For the implication (2)→(1) and the equality (3.4) use Theorem 2.1.

We come to the special case considered in Proposition 2.2.

Proposition 3.2. *Suppose that for some $T_0\in\mathfrak{M}$ we have $\mathscr{F}_{T_0}=\mathscr{F}_\infty$ and let $\mathbf{E}\liminf_n X_n$ and $\mathbf{E}\limsup_n X_n$ exist. The following conditions are equivalent.*

(1) $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists and is finite.*

(2) $\lim_n X_n$ *exists* a.s. *and is integrable.*

If one of these conditions is satisfied then

$$\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\lim_n X_n.$$

The proof is a direct consequence of Proposition 2.2. Proposition 3.2 can be applied to the case where $\mathscr{F}_n=\mathscr{F}$ for all $n\geqq 0$, i.e. if the parameter T for the net $(\mathbf{E}X_T)_{T\in\mathfrak{M}(X)}$ is ranging over all nonnegative finite r.v.'s such that the integral $\mathbf{E}X_T$ makes sence.

4. Connections to Amarts

The notion of an amart was introduced by G.A. Edgar and L. Sucheston [5].

Definition 4.1. We set

(1) $\limsup_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T = \inf_{S\in\mathfrak{M}^b(X)} \sup_{\substack{T\in\mathfrak{M}^b(X)\\ T\geqq S}} \mathbf{E}X_T$

(2) $\liminf_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T = \sup_{S\in\mathfrak{M}^b(X)} \inf_{\substack{T\in\mathfrak{M}^b(X)\\ T\geqq S}} \mathbf{E}X_T$

(3) We say that the limit of $(\mathbf{E}X_T)_{T\in\mathfrak{M}^b(X)}$ exists if

$$\liminf_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T = \limsup_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T$$

and write $\lim_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T$ for it.

(4) A random sequence $X=(X_n)$ is called a (generalized) amart if $\lim_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T$ exists and is finite.

We need a theorem which is taken from R. Chen [3]. Since his proof is not the simplest we give another one.

Proposition 4.1. (i) *If* (X_n^-) *is uniformly integrable then*

$$\mathbf{E}\limsup_n X_n \leqq \limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \limsup_{T\in\mathfrak{M}^b} \mathbf{E}X_T.$$

(ii) *If* (X_n^+) *is uniformly integrable then*

$$\liminf_{T\in\mathfrak{M}^b} \mathbf{E}X_T \leqq \liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqq \mathbf{E}\liminf_n X_n$$

Proof. It suffices to prove (i). The uniform integrability of (X_n^-) implies $\liminf_n \mathbf{E}X_n^- < +\infty$ and by Fatou's lemma $\mathbf{E}\liminf_n X_n^- < +\infty$. But

$$(\limsup_n X_n)^- = \liminf_n X_n^-$$

and hence $\mathbf{E}\limsup_n X_n$ exists. Thus by Proposition 2.1 the first inequality holds. Note that by Lemma 1.1 the set $\mathfrak{M}(X)$ is non-void. For proving the second inequality let $S\in\mathfrak{M}^b$ and $T\in\mathfrak{M}$ such that $T\geqq S$ and $\mathbf{E}X_T > -\infty$. The family $(X_{T\wedge n}^-)$ is for $n\geqq 0$ uniformly integrable. By Fatou's lemma

$$\mathbf{E}X_T \leqq \liminf_n \mathbf{E}X_{T\wedge n} \leqq \sup_{n\geqq m} \mathbf{E}X_{T\wedge n}$$

where m is an arbitrary nonnegative integer. Because of $S\in\mathfrak{M}^b$ there can be found $m\geqq 0$ such that $S\leqq m$. Therefore

$$\mathbf{E}X_T \leqq \sup_{\substack{R\in\mathfrak{M}^b\\ R\geqq S}} \mathbf{E}X_R$$

and, consequently,

$$\sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T \leqq \sup_{\substack{T\in\mathfrak{M}^b\\ T\geqq S}} \mathbf{E}X_T.$$

This yields

$$\inf_{S\in\mathfrak{M}(X)} \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T \leqq \inf_{S\in\mathfrak{M}^b} \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T \leqq \inf_{S\in\mathfrak{M}^b} \sup_{\substack{T\in\mathfrak{M}^b\\ T\geqq S}} \mathbf{E}X_T,$$

proving the assertion.

Directly from Proposition 4.1 we now obtain the following

Theorem 4.1. *Let* (X_n) *be a uniformly integrable amart. Then*

(i) $\lim\limits_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ *exists and is finite;*

(ii) $\lim\limits_n X_n$ *exists a.s. and is integrable;*

(iii) $\lim\limits_{T \in \mathfrak{M}(X)} \mathbf{E} X_T = \lim\limits_{T \in \mathfrak{M}^b} \mathbf{E} X_T = \mathbf{E} \lim\limits_n X_n$.

Statement (iii) does not remain valid without the assumption of the uniform integrability as the following example shows.

Example 4.1. Let (Y_n) be a sequence of independent r.v.'s such that $\mathbf{P}(Y_n=1) = \mathbf{P}(Y_n=0)=\frac{1}{2}$. Define $X_n = 2^n \cdot Y_1 \cdot \ldots \cdot Y_n$. Then (X_n) is a nonnegative martingale which is not uniformly integrable. Obviously, $\lim\limits_n X_n = 0$ a.s. and $\lim\limits_{T \in \mathfrak{M}^b} \mathbf{E} X_T = 1$. From Fatou's lemma follows $\mathbf{E} X_T \leqq 1$ for all $T \in \mathfrak{M}$. By Theorem 2.1 $\lim\limits_{T \in \mathfrak{M}(X)} \mathbf{E} X_T$ exists and is equal to zero.

We proceed with a proposition of R. Chen [3].

Proposition 4.2. (i) *Let* $(X_T^+)_{T \in \mathfrak{M}^b}$ *be uniformly integrable. Then*

$$\limsup_{T \in \mathfrak{M}^b} \mathbf{E} X_T \leqq \mathbf{E} \limsup_n X_n.$$

(ii) *Let* $(X_T^-)_{T \in \mathfrak{M}^b}$ *be uniformly integrable. Then*

$$\mathbf{E} \liminf_n X_n \leqq \liminf_{T \in \mathfrak{M}^b} \mathbf{E} X_T.$$

Proof. It is enough to prove (i). First notice that in view of Proposition 4.1 and the uniform integrability of $(X_T^+)_{T \in \mathfrak{M}^b}$

$$\mathbf{E}(\limsup_n X_n)^+ \leqq \mathbf{E} \limsup_n X_n^+ \leqq \limsup_{T \in \mathfrak{M}^b} \mathbf{E} X_T^+ \leqq \sup_{T \in \mathfrak{M}^b} \mathbf{E} X_T^+ < +\infty.$$

Consequently, $\mathbf{E} \limsup\limits_n X_n$ exists and is not equal to $+\infty$. Let c be a positive real number such that $\mathbf{P}(\limsup\limits_n X_n = c) = 0$. Then

$$\begin{aligned}
\limsup_{T \in \mathfrak{M}^b} \mathbf{E} X_T &\leqq \inf_n \sup_{\substack{T \in \mathfrak{M}^b \\ T \geqq n}} \mathbf{E} \chi_{\{X_T^+ \leqq c\}} X_T + \sup_{T \in \mathfrak{M}^b} \mathbf{E} \chi_{\{X_T^+ > c\}} X_T^+ \\
&\leqq \lim_n \mathbf{E} \sup_{m \geqq n} \chi_{\{X_m^+ \leqq c\}} X_m + \sup_{T \in \mathfrak{M}^b} \mathbf{E} \chi_{\{X_T^+ > c\}} X_T^+ \\
&= \mathbf{E} \limsup_n \chi_{\{\limsup_n X_n \leqq c\}} X_n + \sup_{T \in \mathfrak{M}^b} \mathbf{E} \chi_{\{X_T^+ > c\}} X_T^+.
\end{aligned}$$

Now one passes over to the limit as $c \uparrow \infty$. In view of the uniform integrability of $(X_T^+)_{T \in \mathfrak{M}^b}$ the second member on the right hand side tends to zero. Finally, by Fatou's lemma

$$\limsup_{c \uparrow \infty} \mathbf{E} \chi_{\{\limsup_n X_n \leqq c\}} \limsup_n X_n \leqq \mathbf{E} \limsup_n X_n$$

since $\mathbf{E}\limsup_n X_n < +\infty$ and the assertion follows. This proposition yields the following theorem which is also due to R. Chen [3].

Theorem 4.2. *Let $(X_T)_{T\in\mathfrak{M}^b}$ be uniformly integrable. The following conditions are equivalent:*

(1) *(X_n) is an amart.*

(2) *$\lim_{T\in\mathfrak{M}} \mathbf{E}X_T$ exists and is finite.*

(3) *$\lim_n X_n$ exists a.s. and is integrable.*

Proof. Note that $\mathfrak{M}(X)=\mathfrak{M}$ from Fatou's lemma. The implication (1)→(2) is given by Theorem 4.1 and (2)→(3) is known from Theorem 3.4. Finally, Proposition 4.2 yields (3)→(1).

In the following we need the *amart convergence theorem* which is due to D.G. Austin, G.A. Edgar, and A. Ionescu Tulcea [1] and R.V. Chacon [2].

The amart convergence theorem states that $\lim_n X_n$ *exists a.s. and is integrable if (X_n) is an amart satisfying the condition* $\sup_{n\geq 0}\mathbf{E}|X_n| < +\infty$. It is clear that the theorem also is true if we only assume $\limsup_n \mathbf{E}|X_n| < +\infty$. The converse statement of the amart convergence theorem is not true as the following example shows.

Example 4.2. Let (X_n) be as in Example 4.1 and define $Z_{2n+1}=X_n$ and $Z_{2n}=0$ for all $n\geqq 0$. Then $\lim_n Z_n=0$ a.s. but

$$\liminf_{T\in\mathfrak{M}^b} \mathbf{E}Z_T=0 \quad \text{and} \quad \limsup_{T\in\mathfrak{M}^b} \mathbf{E}Z_T=1.$$

Before we present an interesting connection of L_1-bounded amarts to convergent nets $(\mathbf{E}X_T)_{T\in\mathfrak{M}(X)}$ we give a proposition which itself is interesting, too.

Proposition 4.3. (i) *Suppose* $\limsup_n \mathbf{E}X_n^- < +\infty$ *and* $\limsup_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T < +\infty$. *Then X belongs to $\mathscr{L}^*$.*

(ii) *Suppose* $\limsup_n \mathbf{E}X_n^+ < +\infty$ *and* $\liminf_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T > -\infty$. *Then X belongs to $\mathscr{L}_*$.*

Proof. It is sufficient to verify (i). By Fatou's lemma $\mathbf{E}\limsup_n X_n$ exists (cf. proof of Proposition 4.1). It remains to show $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T^+ < +\infty$ which implies $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T < +\infty$. Remark 1.1 gives

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T^+ = \inf_{S\in\mathfrak{M}} \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqq S}} \mathbf{E}X_T^+ \leqq \inf_{S\in\mathfrak{M}} \sup_{\substack{T\in\mathfrak{M}\\ T\geqq S}} \mathbf{E}X_T^+ = \limsup_{T\in\mathfrak{M}} \mathbf{E}X_T^+.$$

But in view of Proposition 4.1

$$\limsup_{T\in\mathfrak{M}} \mathbf{E}X_T^+ \leqq \limsup_{T\in\mathfrak{M}^b} \mathbf{E}X_T^+.$$

Hence the assertion is proven if $\limsup_{T\in\mathfrak{M}^b} \mathbf{E}X_T^+ < +\infty$ can be shown. Choose an integer $k \geqq 0$ with $\sup_{n\geqq k} \mathbf{E}X_n < +\infty$. Notice that then every $T\in\mathfrak{M}^b$ with $T\geqq k$ belongs to $\mathfrak{M}^b(X)$. Let now $S\in\mathfrak{M}^b$ such that $S\geqq k$ and $\sup_{\substack{T\in\mathfrak{M}^b\\ T\geqq S}} \mathbf{E}X_T < +\infty$. Let $T\in\mathfrak{M}^b$ with $T\geqq S$ and choose n such that $n\geqq T$. Define

$$R(\omega)=\begin{cases} T(\omega) & \text{if } \omega\in\{X_T\geqq 0\}\\ n & \text{otherwise.}\end{cases}$$

Thus $R\in\mathfrak{M}^b(X)$ with $R\geqq S$ and one has

$$X_T^+ \leqq X_R + X_n^- .$$

Consequently,

$$\sup_{\substack{T\in\mathfrak{M}^b\\ T\geqq S}} \mathbf{E}X_T^+ \leqq \sup_{\substack{R\in\mathfrak{M}^b(X)\\ R\geqq S}} \mathbf{E}X_R + \sup_{n\geqq k} \mathbf{E}X_n^- .$$

The assumptions now yield the proposition.

Theorem 4.4. *Let* (X_n) *be an amart such that* $\limsup_n \mathbf{E}|X_n| < +\infty$. *Then* $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists and is finite.*

The amart convergence theorem implies the a.s. existence and integrability of $\lim_n X_n$. Now from Proposition 4.3 and Theorem 3.4 the assertion follows.

Unfortunately, the proof of this theorem is based on the amart convergence theorem. A direct proof is not known to us. However, a direct proof of Theorem 4.4 would be of interest because Theorem 4.4 implies the amart convergence theorem in view of Theorem 2.1.

It should be noticed that under the assumptions of Theorem 4.4 $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ and $\lim_{T\in\mathfrak{M}^b(X)} \mathbf{E}X_T$ are not equal in general (cf. Example 4.1). Moreover, the converse statement to Theorem 4.4 does not hold, i.e. if $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is finite then (X_n) need not be an amart (cf. Example 4.2).

Summarizing the results of the paper we conclude that the class of random sequences (X_n) having the property that $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is finite is in several situations more interesting then the class of amarts. In particular, under the assumption $\limsup_n \mathbf{E}|X_n| < +\infty$ the class of amarts is smaller.

Finally, an interesting consequence of Theorem 4.4 should be mentioned: For every supermartingale (X_n) satisfying the condition $\sup_{n\geqq 0} \mathbf{E}X_n^- < +\infty$ we have that $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is finite.

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