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Generalized Exponential Bounds, Iterated Logarithm and Strong Laws

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Summary. New upper and lower exponential bounds are obtained under a more general condition than that of Kolmogorov and these, in turn, elicit iterated logarithm laws

$$\overline{\lim_{n\to\infty}}\,(s_n^2 \log_2 s_n)^{-\frac{1}{2}} \sum_{j=1}^{n} X_j \overset{\text{a.s.}}{=} C\,\varepsilon(0,\infty)$$

for a wider class of bounded, mean zero, independent random variables $\{X_n, n \geqq 1\}$. The constant C need not be the universal number $2^{\frac{1}{2}}$ and may depend upon the underlying distributions. Upper and lower bounds are provided for C in terms of a parameter. The upper bound is also exploited in proving a new strong law of large numbers for independent random variables.

1. Introduction

One of the most beautiful discoveries of probability theory is the celebrated Iterated Logarithm Law. Due in the case of suitably bounded independent random variables, to Kolmogorov [9] this was the culmination of a series of strides by Hausdorff, Hardy, Littlewood and Hinčine. It asserts for independent, zero mean random variables $\{X_n, n \geqq 1\}$ satisfying $|X_n| \leqq c_n s_n$, a.s., $n \geqq 1$ where $c_n^2 = o\left(\frac{1}{\log_2 s_n}\right)$, $s_n^2 = \sum_{i=1}^{n} \sigma_i^2 \to \infty$, $\sigma_n^2 = EX_n^2$ that $\overline{\lim_{n\to\infty}}\,(s_n^2 \log_2 s_n^2)^{-\frac{1}{2}} \sum_{j=1}^{n} X_j \overset{\text{a.s.}}{=} 2^{\frac{1}{2}}$. The crucial instruments in the proof are so-called exponential bounds for the probability that $\sum_{j=1}^{n} X_j$ exceeds $\lambda s_n (\log_2 s_n)^{\frac{1}{2}}$. Kolmogorov's derivation of the lower exponential bound involves partitioning an integral into five parts and although ingenious is rather involved and ad hoc. Here, an improved upper bound (Lemma 1) and simplified lower bounds (Lemmas 2 and 3) are given under the more general condition

$$\lim_{n\to\infty} c_n (\log_2 s_n)^{\frac{1}{2}} = a \geqq 0.$$

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The derivation of the upper bound is especially simple and may be considered an extension and refinement of Lemma 1 of [8] which, in turn, is related to [10, 7]. The development of the lower bound employs the Esscher transformation [5] which has been exploited by Cramer [2]. Feller [6] and more recently Bahadur [1]. The initial argument for the lower bound is very close to that of Feller [6] who, at a crucial point, introduced assumptions whose nature seems difficult to assess. A valiant attempt to concretize these was made in [11] but unfortunately the applications there [p. 2107–8] are invalidated by an error.

Such upper and lower bounds elicit iterated logarithm laws under the prior condition $\lim_{n\to\infty} c_n(\log_2 s_n)^{\frac{1}{2}}=a\geqq 0$. The constant need no longer be the universal number $2^{\frac{1}{2}}$ and when $a>0$ will depend on the underlying distributions most likely via the value of a. It would be of interest to establish this extended universality, if indeed that is the case, or otherwise to negate it by counterexample.

Under the comparable condition $c_n^2=O\left(\dfrac{1}{\log_2 s_n}\right)$, Egorov [3] showed that

$$\overline{\lim_{n\to\infty}} \frac{\left|\sum_1^n X_j\right|}{s_n(\log_2 s_n)^{\frac{1}{2}}} \overset{\text{a.s.}}{=} C<\infty$$

but did not rule out the possibility $C=0$ (when $c_n^2 \sim a^2 \log_2 s_n$) or furnish information about C.[1]

2. Upper Exponential Bound

The inequalities (1), (2), (4) below bear a natural comparison with their Kolmogorov analogues. For example, Kolmogorov's coefficient of t^2 in the right side of (1) is $\dfrac{1}{2}\left(1+\dfrac{c_n t}{2}\right)$ provided $0<t\,c_n\leqq 1$ and this exceeds $g(c_n t)$ under his proviso. Likewise, the coefficient of t^2 in the right side of (4) exceeds $\frac{1}{2}(1-c_n t)$ for $0<c_n t\leqq 1$. The probablistic inequality (2) is of especial interest in the cases(i) $\lambda_n\equiv\lambda$, $x_n=(\log_2 s_n)^{\frac{1}{2}}$ and (ii) $x_n\equiv\varepsilon$, $\lambda_n=\log_2 s_n$, $b=2/\varepsilon^2$ although intermediate cases are not devoid of interest.

Lemma 1. *Let* $S_n=\sum_{j=1}^n X_j$ *where* $\{X_j, 1\leqq j\leqq n\}$ *are independent r.v.'s with* $EX_j=0$, $EX_j^2=\sigma_j^2$, $s_n^2=\sum_1^n \sigma_j^2$; *set* $g(x)=x^{-2}(e^x-1-x)$ *and suppose that* $0<c_j s_j\uparrow$, $1\leqq j\leqq n$.

(i) If $P\{X_j\leqq c_j s_j\}=1$, $1\leqq j\leqq n$ then

[1] As this article was submitted, "On the Law of the Iterated Logarithm" by R. J. Tomkins appeared in the February 1978 Ann. of Prob. showing under more restrictive conditions than those of Corollary 1 that $\limsup S_n/s_n(\log_2 s_n)^{\frac{1}{2}}=\Gamma\leqq 2^{\frac{1}{2}}[\frac{1}{2}+g(v)]$, a.s. where $\Gamma>0$, g is as in Lemma 1 and, in the notation of Corollary 1, $v=a\,2^{\frac{1}{2}}$ (that v is an upper limit whereas a is either an upper bound or limit is irrelevant). The bound for Γ in (10) of Corollary 1 is at most $1+g(a)$ which is less than $2^{\frac{1}{2}}[\frac{1}{2}+g(a\,2^{\frac{1}{2}})]$ for all $a>1-\varepsilon$ where ε is a very small positive number

$$E\, e^{t S_n/s_n} \leqq e^{t^2 g(c_n t)}, \qquad t>0 \tag{1}$$

and if, in addition, $0<c_n x_n \leqq a$, then for all positive λ_n and b,

$$P\{\max_{1\leqq j\leqq n} S_j > \lambda_n x_n s_n\} \leqq e^{-x_n^2[\lambda_n b - b^2 g(ab)]} \tag{2}$$

(ii) If $P\{X_j \geqq -c_j s_j\}=1$, $1\leqq j\leqq n$ then

$$E\, e^{t S_n/s_n} \geqq \exp\left\{t^2 g(-c_n t)\left[1-\frac{t^2 g(-t c_n)}{s_n^4}\sum_{j=1}^n \sigma_j^4\right]\right\}, \qquad t>0 \tag{3}$$

and if, in addition, $\sigma_j \leqq c_n s_n$, $1\leqq j\leqq n$

$$E\, e^{t S_n/s_n} \geqq e^{t^2 g(-t c_n)[1-t^2 c_n^2 g(-t c_n)]} \geqq e^{t^2 g(-t c_n)\left[1-\frac{t^2 c_n^2}{2}\right]}, \qquad t>0. \tag{4}$$

Proof. The function $g(x)$ is non-negative, increasing and convex on $(-\infty, \infty)$. This is obvious for $x>0$ but is not apparent when $x<0$. The identities

$$x\, g'(x)=(x-2)\, g(x)+1, \qquad x\, g''(x)=(x-3)\, g'(x)+g(x) \tag{5}$$

ensure that $x^2 g''(x)=(x^2-4x+6)\, g(x)+x-3$. Thus to verify convexity and monotonicity for $x<0$ it suffices to establish that for $y>0$

$$g(-y)>\frac{y+3}{y^2+4y+6} \quad \text{and} \quad g(-y)>\frac{1}{y+2}. \tag{6}$$

In fact, it suffices to prove the former since this clearly implies the latter. To this end, note that for $0<y<3$

$$e^y-1-y-\frac{y^2}{2}=\frac{y^3}{6}\left[1+\frac{y}{4}+\frac{y^2}{4.5}+\dots\right]<\frac{y^3}{6\left(1-\frac{y}{4}\right)}<\frac{y^3}{6-2y}$$

and so $e^y<(y^2+4y+6)/(6-2y)$ implying

$$g(-y)=\frac{e^{-y}-1+y}{y^2}>\frac{1}{y^2}\left[\frac{6-2y}{y^2+4y+6}+y-1\right]=\frac{y+3}{y^2+4y+6}.$$

On the other hand, for $y\geqq 3$,

$$g(-y)=\frac{e^{-y}-1+y}{y^2}>\frac{y-1}{y^2}\geqq\frac{y+3}{y^2+4y+6}$$

completing the proof of convexity and monotonicity.

The point of departure for (1), (2), (3) is the simple observation

$$E\, e^{t X_j/s_n}=1+E\left[e^{t X_j/s_n}-1-\frac{t X_j}{s_n}\right]=1+t^2 E\frac{X_j^2}{s_n^2} g\left(\frac{t X_j}{s_n}\right). \tag{7}$$

Hence, monotonicity ensures under (i) that if $t>0$

$$E\, e^{t X_j/s_n} \leqq 1+\frac{t^2\sigma_j^2}{s_n^2} g(t c_n) \leqq e^{t^2 g(t c_n)\sigma_j^2/s_n^2} \tag{7'}$$

and (1) follows via independence.

If rather (ii) obtains, then (7) in conjunction with the elementary inequality $(1+u)\,e^{u^2}>e^u$, $u>0$ yields for $t>0$

$$E\,e^{tX_j/s_n}\geqq 1+\frac{t^2\sigma_j^2}{s_n^2}g(-t\,c_n)\geqq \exp\left\{\frac{t^2\sigma_j^2}{s_n^2}g(-t\,c_n)-\frac{t^4\sigma_j^4}{s_n^4}g^2(-t\,c_n)\right\}$$

whence (3) is immediate. Under the additional hypothesis $\sigma_j\leqq c_n s_n$, $1\leqq j\leqq n$ necessarily $s_n^{-4}\sum_{j=1}^{n}\sigma_j^4\leqq c_n^2$ and (4) follows from (3), in view of $g(0)=\frac{1}{2}$.

To establish (2), note that via the submartingale inequality and (1), for $t>0$

$$P\{\max_{1\leqq j\leqq n} S_j>\lambda x_n s_n\}=P\{\max_{1\leqq j\leqq n} e^{tS_j}\geqq e^{\lambda t x_n s_n}\}\leqq e^{-\lambda t x_n s_n}E\,e^{tS_n}$$
$$\leqq e^{-\lambda t x_n s_n+t^2 s_n^2 g(c_n s_n t)} \tag{8}$$

and so setting $t=b\,x_n/s_n$, $b>0$, $x_n>0$

$$P\{\max_{1\leqq j\leqq n} S_j>\lambda x_n s_n\}\leqq e^{-x_n^2[\lambda b-b^2 g(c_n x_n b)]}\leqq e^{-x_n^2[\lambda b-b^2 g(ab)]}.$$

Clearly, nothing precludes λ being a function of n. △

The simple inequality of (2) yields a generalization of the easier half of the Law of the Iterated Logarithm.

Corollary 1. Let $\{X_n, n\geqq 1\}$ be independent r.v.'s with $EX_n=0$, $EX_n^2=\sigma_n^2$, $s_n^2=\sum_1^n\sigma_i^2\to\infty$. (i) If $\sigma_n^2=o(s_n^2)$ and $X_n\leqq c_n s_n\uparrow$, a.s., $n\geqq 1$ where $c_n>0$ then

$$c_n\sqrt{\log_2 s_n}\to 0\Rightarrow \overline{\lim_{n\to\infty}}\frac{\sum_{j=1}^n X_j}{s_n\sqrt{\log_2 s_n}}\leqq\sqrt{2},\quad \text{a.s.} \tag{9}$$

$$c_n\sqrt{\log_2 s_n}\to a\in(0,\infty)\Rightarrow \overline{\lim_{n\to\infty}}\frac{\sum_{j=1}^n X_j}{s_n\sqrt{\log_2 s_n}}\leqq\min_{b>0}\left[\frac{1}{b}+b\,g(ab)\right],\quad \text{a.s.}$$

or

$$0<c_n\sqrt{\log_2 s_n}\leqq a \tag{10}$$

In particular when $a=1$ the minimum in (10) is $e-1$. (ii) If $|X_n|\leqq c_n s_n\uparrow$, a.s., $n\geqq 1$ where $c_n\sqrt{\log_2 s_n}\to a\in[0,\infty)$ then $\overline{\lim}_{n\to\infty}\left|\sum_1^n X_j\right|/s_n\sqrt{\log_2 s_n}\leqq K$, a.s. where $K=\sqrt{2}$ if $a=0$ and $K=\min_{b>0}\left[\frac{1}{b}+b\,g(ab)\right]\leqq\min\left[a+\frac{e-2}{a},\ 1+g(a)\right]$ otherwise.

Proof. If $\lambda>b^{-1}+b\,g(ab)$ where $g(x)=x^{-2}(e^x-1-x)$ then for α larger than but sufficiently close to one, $\lambda/\alpha^2>b^{-1}+b\,g(ab)$. Define an increasing sequence of integers $\{n_k, k\geqq 1\}$ by $s_{n_k}\leqq\alpha^k<s_{n_k+1}$. Since $\sigma_n^2=o(s_n^2)$ entails $s_{n+1}^2\sim s_n^2$, necessarily $s_{n_k+1}^2=s_{n_k}^2+\sigma_{n_k+1}^2=s_{n_k}^2+o(s_{n_k}^2)$ and so $s_{n_k}\sim\alpha^k$. Suppose now that either

$c_n\sqrt{\log_2 s_n}\to a_0\geqq 0$ or $0<c_n\sqrt{\log_2 s_n}\leqq a$. Then for any $a>a_0$ and all large n, necessarily $c_n x_n\leqq a$ with $x_n=\sqrt{\log_2 s_n}$ and so (2) obtains for this choice of x_n, a and all large n. Thus, replacing λ_n by λ/α^2 in (2), there exists an $\varepsilon>0$ such that for all large k

$$P\left\{\max_{1\leqq j\leqq n_k} S_j>\frac{\lambda}{\alpha^2}s_{n_k}(\log_2 s_{n_k})^{\frac{1}{2}}\right\}\leqq\exp\left\{-\left[\frac{\lambda b}{\alpha^2}-b^2 g(ab)\right]\log_2 s_{n_k}\right\}$$
$$\leqq\exp\{-(1+\varepsilon)\log_2 s_{n_k}\}\leqq\frac{1}{(k\log\alpha)^{1+(\varepsilon/2)}}$$

whence by the Borel-Cantelli lemma

$$P\{S_n>\lambda s_n(\log_2 s_n)^{\frac{1}{2}},\text{i.o.}\}\leqq P\{\max_{n_{k-1}<n\leqq n_k} S_n>\lambda s_{n_{k-1}}(\log_2 s_{n_{k-1}})^{\frac{1}{2}},\text{i.o.}\}$$
$$\leqq P\left\{\max_{1\leqq n\leqq n_k} S_n>\frac{\lambda}{\alpha^2}s_{n_k}(\log_2 s_{n_k})^{\frac{1}{2}},\ \text{i.o.}\right\}=0.$$

Consequently, for all $b>0$, with probability one

$$\overline{\lim_{n\to\infty}}\frac{\sum_{j=1}^{n} X_j}{s_n\sqrt{\log_2 s_n}}\leqq\frac{1}{b}+b\,g(ab).$$

This proves (10) when $0<c_n\sqrt{\log_2 s_n}\leqq a$. Under the alternative hypothesis, letting $a\downarrow a_0$, the preceding holds with a replaced by a_0. If $a_0=0$ then $g(0)=\frac{1}{2}$ and the right side has a minimum of $\sqrt{2}$. When $a_0=1$ (or $a=1$), (5) reveals that $g'(1)=1-g(1)$ whence the minimum value $e-1$ occurs at $b=1$. In case (ii) the hypothesis ensures $\sigma_n^2/s_n^2\leqq c_n^2\to 0$ and the conclusion follows by applying (i) to both $\{-X_n, n\geqq 1\}$ and $\{X_n, n\geqq 1\}$.

Finally, to verify that the right side of (10) is bounded by $\min\left[a+\frac{e-2}{a},\ 1+g(a)\right]$, note that the minimizing value b_0 of $r(b)\equiv b^{-1}+b\,g(ab)$ satisfies $b^{-2}=abg'(ab)+g(ab)$ and hence also (recall (5)) $b^2(ab-1)g(ab)=1-b^2$. Consequently, either $1>b_0>\frac{1}{a}$ (if $a>1$) or $1<b_0<\frac{1}{a}$ (if $a<1$) and so the right side of (10) is bounded by $\min\left[r(1),\ r\left(\frac{1}{a}\right)\right]$. △

3. Lower Exponential Bound

The form of the lower bound is simplest in the classical case $\lim_{n\to\infty} c_n(\log_2 s_n)^{\frac{1}{2}}=a=0$ and hence the alternatives $a=0$ and $a>0$ will be dealt with in separate lemmas.

The crucial tool is the Esscher transformation which leads to the basic formula (11). At this juncture Lemma 1 provides indispensable bounds for the logarithm of the moment generating function (m.g.f.) $\psi_n(t)$ and its derivatives and the combination yields the desired lower bounds.

For any positive integer n, let $\{X_{n,j},\ 1 \leqq j \leqq n\}$ constitute independent r.v.'s with d.f.'s $F_{n,j}$ and finite moment generating functions $\varphi_{n,j}(t) = \exp\{\psi_{n,j}(t)\}$ for $0 \leqq t < t_0$. Suppose that $S_{n,n} = \sum_{j=1}^{n} X_{n,j}$ has mean 0, variance one and d.f. F_n. For any t in $[0, t_0)$, define *associated d.f.'s* $F_{n,j}^{(t)}$ by

$$F_{n,j}^{(t)}(x) = \frac{1}{\varphi_{n,j}(t)} \int_{-\infty}^{x} e^{ty}\, dF_{n,j}(y)$$

and let $\{X_{n,j}(t),\ 1 \leqq j \leqq n\}$ be (fictitious) independent r.v.'s with d.f.'s $\{F_{n,j}^{(t)},\ 1 \leqq j \leqq n\}$. Since the characteristic function (c.f.) of $X_{n,j}(t)$ is $\varphi_{n,j}(t+iu)/\varphi_{n,j}(t)$, setting $\psi_n(t) = \sum_{j=1}^{n} \psi_{n,j}(t)$, the c.f. of $S_n(t) = \sum_{j=1}^{n} X_{n,j}(t)$ is given by

$$E\, e^{iuS_n(t)} = \sum_{j=1}^{n} \frac{\varphi_{n,j}(t+iu)}{\varphi_{n,j}(t)} = e^{\psi_n(t+iu) - \psi_n(t)}.$$

Thus, the mean and variance of $S_n(t)$ are $\psi_n'(t)$ and $\psi_n''(t)$ respectively and moreover the d.f. of $S_n(t)$ is

$$F_n^{(t)}(x) = e^{-\psi_n(t)} \int_{-\infty}^{x} e^{ty}\, dF_n(y)$$

whence for any t in $[0, t_0)$ and real w,

$$P\{S_{n,n} > w\} = e^{\psi_n(t) - t\psi_n'(t)} \int_{\frac{w - \psi_n'(t)}{\sqrt{\psi_n''(t)}}}^{\infty} e^{-ty\sqrt{\psi_n''(t)}}\, dF_n^{(t)}\left(y\sqrt{\psi_n''(t)} + \psi_n'(t)\right). \tag{11}$$

If ψ_n and its derivatives can be approximated with sufficient accuracy, (11) holds forth the possibility of obtaining a lower bound for the probability that a sum of independent r.v.'s with zero means exceeds a multiple of its standard deviation.

Lemma 2. *Let $\{X_j, 1 \leqq j \leqq n\}$ be independent r.v.'s with $EX_j = 0$, $EX_j^2 = \sigma_j^2$, $s_n^2 = \sum_1^n \sigma_j^2$ and $P\{|X_j| \leqq d_j\} = 1$ where $0 < d_j \uparrow$. If $S_n = \sum_{i=1}^{n} X_i$ and $\lim_{n\to\infty} \frac{d_n x_n}{s_n} = 0$ where $x_n > x_0 > 0$ then for every γ in $(0,1)$, some C_γ in $(0, \frac{1}{2})$ and all large n*

$$P\{S_n > (1-\gamma)^{\frac{3}{2}}\, s_n x_n\} \geqq C_\gamma\, e^{-x_n^2(1-\gamma^2)/2}. \tag{12}$$

Proof. Let $\varphi_j(t)$ denote the m.g.f. of X_j and set $S_{n,n} = S_n/s_n = \sum_{j=1}^{n} X_j/s_n$ and $c_n = d_n/s_n$. Since, in the notation leading to (11), $\varphi_{n,j}(t) = \varphi_j(t/s_n)$, $1 \leqq j \leqq n$ and $g_1(x) = x^{-1}(e^x - 1)\uparrow$, it follows for $t > 0$ and $1 \leqq j \leqq n$ that

$$\varphi'_{n,j}(t) = \frac{d}{dt} E\, e^{tX_j/s_n} = E \frac{X_j}{s_n}(e^{tX_j/s_n} - 1) \underset{\geqq}{\leqq} t\, g_1(\pm t\, c_n)\, \sigma_j^2/s_n^2,$$

$$\varphi''_{n,j}(t) = E \frac{X_j^2}{s_n^2} e^{tX_j/s_n} \underset{\geqq}{\leqq} e^{\pm t c_n} \sigma_j^2/s_n^2$$

and so noting $\varphi_{n,j}(t) \geqq 1$, (7′) and $\sigma_j^2 \leqq c_n^2 s_n^2$, $1 \leqq j \leqq n$

$$\psi'_{n,j}(t) = \frac{\varphi'_{n,j}(t)}{\varphi_{n,j}(t)} \leqq t\, g_1(t\, c_n)\, \sigma_j^2/s_n^2$$

$$\geqq \frac{t\, g_1(-t\, c_n)\, \sigma_j^2/s_n^2}{1 + t^2 c_n^2 g(t\, c_n)}, \tag{13}$$

$$\psi''_{n,j}(t) = \frac{\varphi''_{n,j}(t)}{\varphi_{n,j}(t)} - [\psi'_{n,j}(t)]^2 \leqq e^{t c_n} \sigma_j^2/s_n^2$$

$$\geqq \frac{\sigma_j^2}{s_n^2} [\exp\{-t\, c_n - t^2 c_n^2 g(t\, c_n)\} - t^2 c_n^2 g_1^2(t\, c_n)].$$

where g is as in Lemma 1. Hence if $\psi_n(t) = \sum_{j=1}^{n} \psi_{n,j}(t)$,

$$\psi'_n(t) = \sum_{j=1}^{n} \psi'_{n,j}(t) \begin{array}{l} \leqq t\, g_1(t\, c_n) \\ \geqq t\, g_1(-t\, c_n)/[1 + t^2 c_n^2 g(t c_n)] \end{array} \tag{14}$$

Moreover, since $|g_1(x) - 1| < \frac{|x|}{2}\left[1 - \frac{|x|}{3}\right]^{-1}$ for $0 < |x| < 3$ and

$$g(x) < \frac{1}{2} + \frac{x}{6}\left(1 - \frac{x}{4}\right)^{-1} \quad \text{for } 0 < x < 4, \text{ if } \lim_{n \to \infty} t_n c_n = 0$$

$$\psi''_n(t_n) = \sum_{j=1}^{n} \psi''_{n,j}(t_n) = 1 + O(t_n c_n). \tag{15}$$

Thus, via (4), (14), (15) and $g(0) = \frac{1}{2}$, $g_1(0) = 1$, for any γ in $(0, 1)$ and all large n

$$\psi_n(t_n) - t_n \psi'_n(t_n) \geqq t_n^2 [g(-t_n c_n) - t_n^2 c_n^2 g^2(-t_n c_n) - g_1(t_n c_n)] \geqq \frac{-t_n^2}{2}(1 + \gamma),$$

$$v_n \equiv \frac{(1-\gamma)\, t_n - \psi'_n(t_n)}{\sqrt{\psi''_n(t_n)}} = -\gamma\, t_n(1 + o(1)) \leqq \frac{-\gamma\, t_n}{2}.$$

Consequently, taking $w = (1-\gamma)\, t_n$ in (11),

$$P\{S_n > (1-\gamma)\, s_n t_n\} \geqq e^{\psi_n(t_n) - t_n \psi'_n(t_n)} \int_{v_n}^{0} e^{-t y \sqrt{\psi''_n(t_n)}}\, dF_n^{(t_n)}(y \sqrt{\psi''_n(t_n)} + \psi'_n(t_n))$$

$$\geqq e^{\frac{-t_n^2}{2}(1+\gamma)} \int_{\frac{-\gamma t_n}{2}}^{0} dF_n^{(t_n)}(y \sqrt{\psi''_n(t_n)} + \psi'_n(t_n)) \geqq C_\gamma\, e^{\frac{-t_n^2}{2}(1+\gamma)} \tag{16}$$

since $\frac{S_n(t_n)-\psi_n'(t_n)}{\sqrt{\psi_n''(t_n)}} \equiv \sum_{j=1}^{n} Z_{n,j} \xrightarrow{d} N_{0,1}$ in view of

$$EZ_{n,j}=0, \quad \sum_{j=1}^{n} EZ_{n,j}^2=1 \text{ and } |Z_{nj}|=\left|\frac{X_{n,j}(t_n)-EX_{n,j}(t_n)}{\sqrt{\psi_n''(t_n)}}\right|$$

$$\leqq \frac{2c_n}{\sqrt{\psi_n''(t_n)}}=o(1).$$

Finally, set $t_n=(1-\gamma)^{\frac{1}{2}} x_n$ in (16) to obtain (12). △

Remark. If $x_n \to \infty$, then for every γ in $(0,1)$, the constant $C_\gamma > \frac{1}{2}-\varepsilon$, all $\varepsilon>0$ provided $n \geqq$ some integer N_ε.

Clearly, Lemma 2 may be substituted for Kolmogorov's lower bound in proving the other half of the Law of the Iterated Logarithm. It suffices to follow the main lines of the proof of Theorem 1.

Lemma 3. *Let $S_n=\sum_{j=1}^{n} X_j$ where $\{X_j, 1\leqq j \leqq n\}$ are independent random variables with $EX_j=0$, $EX_j^2=\sigma_j^2$, $s_n^2=\sum_1^n \sigma_j^2$ and $P\{|X_j|\leqq d_j\}=1$ where $0<d_j\uparrow$. Set $h(u)=u^2[g_1(u)-g(-u)+u^2g^2(-u)]$ where $g_1(u)=u^{-1}(e^u-1)$ and $g(u)=u^{-2}(e^u-1-u)$. If $x_n\to\infty$ and $d_nx_n/s_n \to a>0$, then for all γ in $(0,1)$ and all u in $(0,u_0)$*

$$P\left\{S_n > \frac{1-\gamma}{a}\left(\frac{1-e^{-u}}{e^u-u}\right) s_n x_n\right\} \geqq \left(\tfrac{1}{2}+o(1)\right) e^{\frac{-x_n^2}{a^2}[h(u)+o(1)]}. \tag{17}$$

Moreover, $u_0=0.5347\ldots$ is the root of the equation $e^{-u}=(e^u-u)(e^u-1)^2$.

Proof. Setting $c_n=d_n/s_n$, the lower bound for $\psi_{n,j}''$ can be improved via (13), (7′) to

$$\frac{\sigma_j^2}{s_n^2} e^{tc_n} \geqq \psi_{n,j}''(t) \geqq \frac{\sigma_j^2}{s_n^2}\left[\frac{e^{-tc_n}}{1+t^2c_n^2 g(tc_n)} - t^2 c_n^2 g_1^2(tc_n)\right].$$

Thus, if $t=ux_n/a$, $u>0$, it follows since $c_n x_n \to a$ that

$$e^u+o(1) \geqq \psi_n''\left(\frac{ux_n}{a}\right) \geqq \frac{e^{-u}}{1+u^2 g(u)} - u^2 g_1^2(u)+o(1) \tag{18}$$

and the lower bound is positive for $0<u<u_0$ and n sufficiently large. Moreover, recalling (4) and (14)

$$\psi_n\left(\frac{ux_n}{a}\right) - \frac{ux_n}{a}\psi_n'\left(\frac{ux_n}{a}\right)$$

$$\geqq \frac{u^2 x_n^2}{a^2}\left[g\left(\frac{-ux_nc_n}{a}\right) - \frac{u^2}{a^2} c_n^2 x_n^2 g\left(\frac{-ux_nc_n}{a}\right) - g_1\left(\frac{u}{a}x_nc_n\right)\right]$$

$$= -x_n^2 \frac{u^2}{a^2}[g_1(u)-g(-u)+u^2g^2(-u)+o(1)]$$

$$= -\frac{x_n^2}{a^2}[h(u)+o(1)] \tag{19}$$

Furthermore, if $w=\frac{(1-\gamma)\,a^{-1}\,u\,g_1(-u)}{1+u^2\,g(u)}\,x_n$, then via (14)

$$w-\psi_n'\left(\frac{u}{a}x_n\right)\leqq w-\frac{u\,a^{-1}\,x_n\,g_1(-x_n\,c_n\,u/a)}{1+u^2\,a^{-2}\,x_n^2\,c_n^2\,g(x_n\,c_n\,u/a)}=\frac{x_n}{a}\,\frac{u\,g_1(-u)}{1+u^2\,g(u)}[-\gamma+o(1)].$$

Thus, taking $t=t_n=u\,x_n/a$ in (11)

$$P\{S_n>w s_n\}\geqq\exp\left\{\psi_n\left(\frac{u\,x_n}{a}\right)-\frac{u\,x_n}{a}\,\psi_n'\left(\frac{u\,x_n}{a}\right)\right\}$$

$$\cdot\int\limits_{\frac{w-\psi_n'(u x_n/a)}{\sqrt{\psi_n''(u x_n/a)}}}^{0} dF_n^{(u x_n/a)}\left(y\sqrt{\psi_n''\left(\frac{u\,x_n}{a}\right)}+\psi_n'\left(\frac{u\,x_n}{a}\right)\right)$$

whence (17) follows via (19), the Central Limit Theorem and the fact that the lower limit of integration $\to-\infty$. The CLT obtains for $0<u<u_0$ since

$$\left|\frac{X_{nj}(t_n)-EX_{nj}(t_n)}{\sqrt{\psi_n''(t_n)}}\right|\leqq\frac{2c_n}{\sqrt{\psi_n''(t_n)}}=o(1)\ \text{recalling (18).}\quad\triangle$$

It may be noted that $h(u)$ is increasing for $u>0$. In fact via the identities $g_1'(u)=g_1(u)-g(u)$ and $u\,g'(-u)=(u+2)\,g(-u)-1$, $u>0$ it follows readily that for $u>0$

$$\begin{aligned}h'(u)&=2u[g_1(u)-g(-u)+u^2\,g^2(-u)]\\&\quad+u^2[g_1'(u)+g'(-u)+2u\,g^2(-u)-2u^2\,g(-u)\,g'(-u)]\\&=2u[g_1(u)-g(-u)+u^2\,g^2(-u)]\\&\quad+u^2[g_1(u)-g(u)+2u\,g^2(-u)-2u^2\,g(-u)\,g'(-u)]\\&\quad+(u^2+2u)\,g(-u)-u\\&=u^2[g_1(u)-g(-u)+u^2\,g^2(-u)]\\&\quad+u[2g_1(u)-1]+2u^3\,g(-u)\,[1-u\,g(-u)]>0\end{aligned}$$

since all three bracketed expressions are non-negative for $u>0$.

Moreover, $\frac{1-e^{-u}}{e^u-u}$ is increasing for $0<u<0.75$ and a fortiori for u in $(0, u_0)$ where $u_0=0.5347\ldots$ is as in Lemma 3.

Theorem 1. *Let* $S_n=\sum_{i=1}^n X_i$ *where* $\{X_n, n\geqq 1\}$ *are independent random variables satisfying* $EX_n=0$, $EX_n^2=\sigma_n^2$, $s_n^2=\sum_{i=1}^n\sigma_i^2\to\infty$ *and* $P\{|X_n|\leqq d_n\}=1$, $n\geqq 1$ *with* $d_n\uparrow$ *and*

$$\lim_{n\to\infty}\frac{d_n(\log_2 s_n)^{\frac{1}{2}}}{s_n}=a. \tag{20}$$

(i) If $a \geqq \sqrt{h(u_0)} = 0.5215\ldots$ where u_0 is the root of $e^{-u} = (e^u - u)(e^u - 1)^2$, then

$$\overline{\lim_{n\to\infty}} \frac{S_n}{s_n(\log_2 s_n)^{\frac{1}{2}}} \geqq \frac{1-e^{-u_0}}{a(e^{u_0}-u_0)} = \frac{0.3533\ldots}{a}, \quad \text{a.s.} \tag{21}$$

(ii) If rather, $a = \sqrt{h}(u)$ for some u in $(0, u_0)$, then

$$\overline{\lim_{n\to\infty}} \frac{S_n}{s_n(\log_2 s_n)^{\frac{1}{2}}} \geqq \frac{1}{a}\left(\frac{1-e^{-u}}{e^u - u}\right), \quad \text{a.s.} \tag{22}$$

Proof. The hypothesis ensures that $\sigma_n^2/s_n^2 \leqq d_n^2/s_n^2 = o(1)$ whence as in the proof of Corollary 1 there exists a sequence of integers $\{n_k,\ k \geqq 1\}$ with $s_{n_k} \sim \alpha^k$, $\alpha > 1$. Choose positive quantities c, γ, u such that

$$\begin{aligned} &c = a\left(1 - \frac{1}{\alpha^2}\right)^{-\frac{1}{2}}, \quad h(u) < a^2 \\ &\gamma < \min\left[1, \frac{c^2}{h(u)} - 1\right] \end{aligned} \tag{23}$$

and then define independent events

$$A_k = \left\{S_{n_k} - S_{n_{k-1}} > \frac{1-\gamma}{c}\left(\frac{1-e^{-u}}{e^u - u}\right) g_k h_k\right\}, \quad k \geqq 1$$

where

$$g_k^2 \equiv s_{n_k}^2 - s_{n_{k-1}}^2 \sim s_{n_k}^2\left(1 - \frac{1}{\alpha^2}\right),$$

$$h_k^2 \equiv \log_2 g_k \sim \log_2 s_{n_k} < (1+\gamma)\log k$$

for all large k. Via (20)

$$\frac{d_{n_k} h_k}{g_k} \sim \frac{d_{n_k}(\log_2 s_{n_k})^{\frac{1}{2}}}{s_{n_k}\left(1 - \frac{1}{\alpha^2}\right)^{\frac{1}{2}}} \sim \frac{a}{\left(1 - \frac{1}{\alpha^2}\right)^{\frac{1}{2}}} = c$$

and so taking $x_{n_k} = h_k$ in Lemma 3 and recalling (23)

$$\begin{aligned} P\{A_k\} &\geqq \tfrac{1}{3}\exp\left\{\frac{-h_k^2}{c^2}[h(u) + o(1)]\right\} \\ &\geqq \tfrac{1}{3}\exp\left\{-\left[\frac{(1+\gamma)\,h(u)}{c^2} + o(1)\right]\log k\right\} \geqq \frac{1}{3k} \end{aligned}$$

for all large k. Thus, by the Borel-Cantelli lemma

$$P\{A_k, \text{i.o.}\} = 1. \tag{24}$$

Next, choose α so large that

$$\frac{1-\gamma}{a}\left(1-\frac{1}{\alpha^2}\right)\frac{(1-e^{-u})}{e^u-u}-\frac{2K}{\alpha}>\frac{(1-\gamma)^2}{a}\,\frac{(1-e^{-u})}{e^u-u}$$

and set $t_n=(\log_2 s_n)^{\frac{1}{2}}$, whence for all large k

$$\frac{1-\gamma}{c}\left(\frac{1-e^{-u}}{e^u-u}\right)g_k h_k-2K\,s_{n_{k-1}}\,t_{n_{k-1}}\,t_{n_{k-1}}$$

$$\sim\left[\frac{(1-\gamma)}{a}\left(1-\frac{1}{\alpha^2}\right)\frac{(1-e^{-u})}{e^u-u}-\frac{2K}{\alpha}\right]s_{n_k}\,t_{n_k}>\frac{(1-\gamma)^2}{a}\,\frac{(1-e^{-u})}{e^u-u}\,s_{n_k}\,t_{n_k}.$$

Hence, if $B_k=\{|S_{n_{k-1}}|\leqq 2K\,s_{n_{k-1}}\,t_{n_{k-1}}\}$ where $K=\min_{b>0}[b^{-1}+b\,g(ab)]$, then

$$A_k B_k\subset\left\{S_{n_k}>\frac{(1-\gamma)^2}{a}\,\frac{(1-e^{-u})}{e^u-u}\,s_{n_k}\,t_{n_k}\right\}$$

again for all large k. However, (ii) of Corollary 1 guarantees $P\{B_k^c,\text{ i.o.}\}=0$ which in conjunction with (24) entails

$$P\left\{S_{n_k}>\frac{(1-\gamma)^2}{a}\,\frac{(1-e^{-u})}{e^u-u}\,s_{n_k}\,t_{n_k},\text{ i.o.}\right\}\geqq P\{A_k B_k,\text{ i.o.}\}=1.$$

Thus, with probability one

$$\overline{\lim_{n\to\infty}}\,\frac{S_n}{s_n t_n}\geqq\overline{\lim_{k\to\infty}}\,\frac{S_{n_k}}{s_{n_k}t_{n_k}}\geqq\frac{(1-\gamma)^2}{a}\,\frac{(1-e^{-u})}{e^u-u}. \tag{25}$$

If, as in case (i), $a^2\geqq h(u_0)$, then (25) holds for all $u<u_0$ and all small, positive γ whence (21) obtains as $u\uparrow u_0$ and $\gamma\downarrow 0$. If rather $0<a^2=h(u)<h(u_0)$, then in analogous fashion (22) follows from (25). △

A combination of Theorem 1, Corollary 1 and the Kolmogorov zero-one law yields.

Theorem 2. *Let* $S_n=\sum_{j=1}^n X_j$ *where* $\{X_n,\ n\geqq 1\}$ *are independent random variables satisfying* $EX_n=0$, $EX_n^2=\sigma_n^2$, $s_n^2=\sum_{j=1}^n\sigma_j^2\to\infty$ *and* $P\{|X_n|\leqq d_n\}=1$, $n\geqq 1$ *with* $d_n\uparrow$ *and* $\lim_{n\to\infty}\frac{d_n(\log_2 s_n)^{\frac{1}{2}}}{s_n}=a\geqq 0$. *Then*

$$P\left\{\overline{\lim_{n\to\infty}}\,\frac{S_n}{s_n(\log_2 s_n)^{\frac{1}{2}}}=C\right\}=1$$

where

$$\frac{0.3533}{a}\leqq C\leqq\min_{b>0}[b^{-1}+b\,g(ab)]$$

$$\leqq\min\left[a+\frac{e-2}{a},\,1+g(a)\right]\quad\textit{if }a\geqq\sqrt{h(u_0)}=0.5215.$$

$$\frac{1}{a}\left(\frac{1-e^{-u}}{e^u-u}\right) \leqq C \leqq \min_{b>0}\,[b^{-1}+b g(ab)]$$

$$\leqq \min\left[a+\frac{e-2}{a},\,1+g(a)\right] \quad \textit{if } 0<a=\sqrt{h(u)}<\sqrt{h(u_0)}.$$

As $a=\sqrt{h(u)}\to 0$ (that is, $u\to 0$) it follows easily from $h(u)\sim\frac{u^2}{2}$ that the lower bound for C approaches $\sqrt{2}$ which coincides with the upper bound when $a=0$.

Remark. The requirement that $d_n\uparrow$ is inessential in Theorems 1 and 2. Since $b_n = s_n/(\log_2 s_n)^{\frac{1}{2}}\uparrow\infty$ (for all large n), the hypothesis $d_n/b_n\to a$ ensures that $d'_n\equiv \max_{1\leqq i\leqq n} d_i$ likewise satisfies $d'_n/b_n\to a$ in view of

$$a\leftarrow\frac{d_n}{b_n}\leqq\frac{d'_n}{b_n}\leqq\frac{1}{b_n}\max_{1\leqq i\leqq m} d_i+\max_{m\leqq i\leqq n}\frac{d_i}{b_i}\xrightarrow{n\to\infty}\sup_{i>m}\frac{d_i}{b_i}\xrightarrow{m\to\infty} a$$

Thus, even if d_n is not monotone, Lemmas 1, 2 and 3 apply for all large n since $|X_j|\leqq d'_j\uparrow$, $1\leqq j\leqq n$, a.c.

4. A Strong Law of Large Numbers

It is well known that independent, random variables $\{X_n,\ n\geqq 1\}$ with $EX_n=0$, $EX_n^2=\sigma_n^2$, $s_n^2=\sum_{i=1}^n \sigma_i^2\to\infty$ satisfy the strong law

$$\frac{S_n}{s_n(\log s_n)^\delta}\xrightarrow{\text{a.s.}} 0 \tag{26}$$

for $\delta>\frac{1}{2}$. It is less well known but follows from results of [4, 12] that (26) obtains even when $\delta>0$ provided (as is necessary) $\sum_{n=1}^\infty P\{|X_n|>\varepsilon s_n(\log s_n)^\delta\}<\infty$, $\varepsilon>0$.

Conditions under which (26) can be strengthened to

$$\frac{S_n}{s_n(\log_2 s_n)^\delta}\xrightarrow{\text{a.s.}} 0 \tag{27}$$

appear in

Theorem 3. *Let $\{S_n,\ n\geqq 1\}$ be the partial sums of independent random variables $\{X_n,\ n\geqq 1\}$ with $EX_n=o, E|X_n|^\alpha\leqq a_{n,\alpha}$, $A_n\equiv A_{n,\alpha}=\left(\sum_{i=1}^n a_{i,\alpha}\right)^{\frac{1}{\alpha}}\to\infty$ where $1<\alpha\leqq 2$ and $A_{n+1,\alpha}/A_{n,\alpha}\leqq\gamma<\infty$, all $n\geqq 1$. If for some β in $\left[0,\frac{1}{\alpha}\right)$ and positive quantities δ, c*

$$\sum_{n=1}^\infty P\{|X_n|>\delta A_n(\log_2 A_n)^{1-\beta}\}<\infty, \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{1}{A_n^2(\log_2 A_n)^{2(1-\beta)}} EX_n^2 I_{[cA_n(\log_2 A_n)^{-\beta} < |X_n| \leqq \delta A_n(\log_2 A_n)^{1-\beta}]} < \infty \tag{29}$$

then

$$\frac{S_n}{A_n(\log_2 A_n)^{1-\beta}} \xrightarrow{\text{a.s.}} 0. \tag{30}$$

Proof. Set $Y_n = X_n I_{[|X_n| \leqq cA_n(\log_2 A_n)^{-\beta}]}$, $W_n = X_n I_{[|X_n| > \delta A_n(\log_2 A_n)^{1-\beta}]}$, $V_n = X_n - Y_n - W_n$. Now since $\beta < 1/\alpha$,

$$\begin{aligned}\left|\sum_{i=1}^{n} EW_i\right| &\leqq \sum_{i=1}^{n} (E|X_i| I_{[\delta A_i(\log_2 A_i)^{1-\beta} < |X_i| \leqq A_n(\log_2 A_n)^{-\beta}]} + E|X_i| I_{[|X_i| > A_n(\log_2 A_n)^{-\beta}]}) \\ &\leqq A_n(\log_2 A_n)^{-\beta} \sum_{i=1}^{n} P\{|X_i| > \delta A_i(\log_2 A_i)^{1-\beta}\} \\ &\quad + \frac{(\log_2 A_n)^{\beta(\alpha-1)}}{A_n^{\alpha-1}} \sum_{i=1}^{n} E|X_i|^\alpha I_{[|X_i| > A_n(\log_2 A_n)^{-\beta}]} \\ &= o(A_n(\log_2 A_n)^{1-\beta})\end{aligned}$$

and so, in view of (28)

$$\frac{1}{A_n(\log_2 A_n)^{1-\beta}} \sum_{i=1}^{n} (W_i - EW_i) \xrightarrow{\text{a.s.}} 0.$$

Secondly, (29) and Kronecker's lemma guarantee

$$\frac{1}{A_n(\log_2 A_n)^{1-\beta}} \sum_{i=1}^{n} (V_i - EV_i) \xrightarrow{\text{a.s.}} 0.$$

Thus, since $EX_n = 0$, it suffices to verify that

$$\frac{1}{A_n(\log_2 A_n)^{1-\beta}} \sum_{i=1}^{n} (Y_i - EY_i) \xrightarrow{\text{a.s.}} 0. \tag{31}$$

To this end, note that if $n_k = \inf\{n \geqq 1 : A_n \geqq \gamma^k\}$

$$A_{n_k} \leqq \gamma A_{n_k - 1} < \gamma^{k+1} \leqq A_{n_{k+1}}$$

and so $\{n_k, k \geqq 1\}$ is strictly increasing. Moreover, for all $k \geqq 1$

$$\frac{A_{n_k}}{A_{n_{k-1}}} < \frac{\gamma^{k+1}}{\gamma^{k-1}} = \gamma^2 > 1.$$

Therefore, setting $U_n = \sum_{i=1}^{n} (Y_i - EY_i)$, for all $\varepsilon > 0$

$$\begin{aligned}&P\{U_n > 2\gamma^2 \varepsilon A_n(\log_2 A_n)^{1-\beta}, \text{i.o.}\} \\ &\quad \leqq P\{\max_{n_{k-1} < n \leqq n_k} U_n > 2\gamma^2 \varepsilon A_{n_{k-1}}(\log_2 A_{n_{k-1}})^{1-\beta}, \text{i.o.}\} \\ &\quad \leqq P\{\max_{1 \leqq n \leqq n_k} U_n > \varepsilon A_{n_k}(\log_2 A_{n_k})^{1-\beta}, \text{i.o.}\}.\end{aligned} \tag{32}$$

Now, $|Y_n - EY_n| \leqq c_n t_n$ where $c_n = \dfrac{2c A_n (\log_2 A_n)^{-\beta}}{t_n}$ and

$$\begin{aligned} t_n^2 = EU_n^2 &\leqq \sum_{i=1}^{n} EX_i^2 I_{[|X_i| \leqq cA_i(\log_2 A_i)^{-\beta}]} \\ &\leqq \frac{c^{2-\alpha} A_n^{2-\alpha}}{(\log_2 A_n)^{\beta(2-\alpha)}} \sum_{i=1}^{n} E|X_i|^\alpha = \frac{c^{2-\alpha} A_n^2}{(\log_2 A_n)^{\beta(2-\alpha)}}. \end{aligned} \tag{33}$$

Consequently, setting

$$\lambda_n = \frac{\varepsilon A_n^2 (\log_2 A_n)^{1-2\beta}}{t_n^2}, \qquad x_n = \frac{t_n (\log_2 A_n)^\beta}{A_n}$$

it follows that

$$\begin{aligned} &\lambda_{n_k} x_{n_k}^2 = \varepsilon(\log_2 A_{n_k}) \to \infty, \\ &c_{n_k} x_{n_k} = 2c, \qquad \lambda_{n_k} x_{n_k} t_{n_k} = \varepsilon A_{n_k} (\log_2 A_{n_k})^{1-\beta}, \\ &x_{n_k}^2 = \frac{t_{n_k}^2 (\log_2 A_{n_k})^{2\beta}}{A_{n_k}^2} \leqq c^{2-\alpha} (\log_2 A_{n_k})^{\beta\alpha} = o(\log_2 A_{n_k}) = o(\lambda_{n_k} x_{n_k}^2) \end{aligned}$$

recalling (33) and $\beta < 1/\alpha$. Thus, taking $b = 3/\varepsilon$ in (2) of Lemma 1

$$P\{\max_{1 \leqq n \leqq n_k} U_n > \varepsilon A_{n_k} (\log_2 A_{n_k})^{1-\beta}\} \leqq e^{-2\log_2 A_{n_k}} \leqq \frac{1}{(k \log \gamma)^2}$$

and so (32) and the Borel-Cantelli lemma ensure

$$\overline{\lim_{n \to \infty}} \frac{U_n}{A_n (\log_2 A_n)^{1-\beta}} \leqq 0, \qquad \text{a.s.} \tag{34}$$

Since $\{-(Y_n - EY_n), n \geqq 1\}$ have the same bounds and variances as $\{Y_n - EY_n, n \geqq 1\}$, (34) likewise obtains with $-U_n$ replacing U_n thereby proving (31) and the theorem. $\triangle$

Corollary 2. Let $\{X_n, n \geqq 1\}$ be independent random variables with $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $s_n^2 = \sum_i^n \sigma_i^2 \to \infty$ and $s_{n+1}/s_n \leqq \gamma < \infty$, $n \geqq 1$. If for some β in $[0, \frac{1}{2})$ and some positive c, δ

$$\sum_{n=1}^{\infty} P\{|X_n| > \delta s_n (\log_2 s_n)^{1-\beta}\} < \infty, \tag{35}$$

$$\sum_{n=1}^{\infty} \frac{1}{s_n^2 (\log_2 s_n)^{2(1-\beta)}} EX_n^2 I_{[cs_n(\log_2 s_n)^{-\beta} < |X_n| \leqq \delta s_n (\log_2 s_n)^{1-\beta}]} < \infty \tag{36}$$

then

$$\frac{S_n}{s_n (\log_2 s_n)^{1-\beta}} \xrightarrow{\text{a.s.}} 0. \tag{37}$$

Note that Corollary 2 precludes $\beta=\frac{1}{2}$. In fact, (35) and (36) when $\beta=\frac{1}{2}$ comprise two of the three conditions for the Law of the Iterated Logarithm in Theorem 1 of [13].

Corollary 3. *If* $\{X_n,\ n\geqq 1\}$ *are independent random variables with* $EX_n=0$, $EX_n^2=\sigma_n^2$, $s_n^2=\sum_1^n \sigma_i^2\to\infty$ *and* $|X_n|\leqq c_\beta s_n(\log_2 s_n)^{-\beta}$, *a.s.,* $n\geqq 1$ *where* $0\leqq\beta<\frac{1}{2}$, $c_\beta>0$ *then* (37) *obtains provided in the case* $\beta=0$ *that* $s_{n+1}=O(s_n)$.

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