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The Variation of a Stable Path is Stable

PRISCILLA E. GREENWOOD

Summary. Let $X(t)$ be a separable symmetric stable process of index α. Let P be a finite partition of $[0, 1]$, and $\mathscr{P}$ a collection of partitions. The variation of a path $X(t)$ is defined in three ways in terms of the sum $\sum_{t_i \in P} |X(t_i) - X(t_{i-1})|^\beta$ and the collection $\mathscr{P}$. Under certain conditions on $\mathscr{P}$ and on the parameters α and β, the distribution of the variation is shown to be a stable law. Under other conditions the distribution of the variational sum converges to a stable distribution.

1. Introduction

Let $X(t)$ be a separable symmetric stable process on R^1 of index α, $0<\alpha\leqq 2$. $X(t)$ has characteristic function $e^{-t|\lambda|^\alpha}$. Let P be a finite partition, $\{t_1<\cdots<t_k\}$ of $[0,1]$, and $\mathscr{P}$ a collection of such partitions. Let $V(P,\beta)$ denote the variational sum $\sum_{t_i\in P} |X(t_i)-X(t_{i-1})|^\beta$, where $\beta>0$. The variation of the stable path $X(t)$ can be calculated in at least three ways: as the supremum over $\mathscr{P}$ of $V(P,\beta)$; as the limit as $n\to\infty$ of the supremum over $\mathscr{P}_n$ of $V(P,\beta)$ where $\mathscr{P}_n=\{P\in\mathscr{P}$: mesh of $P\leqq 1/n\}$; if $\mathscr{P}$ is a sequence $\{P_n\}$, as the lim sup as $n\to\infty$ of $V(P,\beta)$. We give sufficient conditions on $\mathscr{P}$ that the variation, calculated in each of these ways, be a stable random variable of index α/β if $\beta\leqq 1$. The case $\beta>1$ presents special problems, some of which are resolved in Section 5. We use and refine results of Blumenthal and Getoor, and of Bochner [1, 2, 3].

2. Definitions

We impose closure conditions on $\mathscr{P}$ which are satisfied, for example, by the collection of all finite partitions and by the collection of all finite rational partitions.

If $P=\{t_1,\ldots,t_k\}\subset[0,1]$, let $rP+s$ denote the partition

$$\{r t_1+s,\ldots,r t_k+s\}\cap[0,1].$$

We say that $\mathscr{P}$ is closed under translation and multiplication by $r=\frac{1}{3},\frac{1}{2},\frac{2}{3}$ if for each $P\in\mathscr{P}$ some refinement of each of rP, $(1/r)P$, $P+r$, $P-r$ is $\in\mathscr{P}$.

We define $\mathscr{P}$ to be closed under piecing at $r=\frac{1}{2},\frac{1}{3}$ if whenever P_1 and P_2 are in $\mathscr{P}$, some refinement of the partition $(P_1\cap[0,r])\cup(P_2\cap[r,1])\cup r$ is also in $\mathscr{P}$.

When we use these conditions with $\beta>1$, we intend that the words "some refinement of" should be omitted.

If $\mathscr{P}$ is a sequence of partitions we require that the closure conditions hold in the forward direction of the sequence, and define $\mathscr{P}$ to be closed with respect to translation, multiplication, and piecing for $r=\frac{1}{3},\frac{1}{2},\frac{2}{3}$ if whenever P_n and P_m are in $\mathscr{P}$ there is a refinement further along in the sequence for each of rP_n, $1/r\,P_n$, P_n+r, P_n-r, and $(P_n\cap[0,r])\cup(P_m\cap[r,1])\cup r$. An example of such a sequence is $P_n=\{i/n,\ i=0,\ldots,n\}$.

Let $X \stackrel{d}{=} Y$ mean that the random variables X and Y have the same distribution. It follows from the form of the characteristic function that for $a>0$, $X(t) \stackrel{d}{=} a^{1/\alpha} X(t/a)$.

The mesh μ of a partition P will be the maximum length of the interval between successive points of P.

Finally, we introduce distinguishing notation for the three types of variation which we consider. The supremum over P in $\mathscr{P}$ of the variational sum $V(P,\beta)=\sum_{t_i \in P} |X(t_i)-X(t_{i-1})|^\beta$ is denoted by β-var X.

The limit as $n \to \infty$ of the supremum over $\mathscr{P}_n$ of $V(P,\beta)$ where $\mathscr{P}_n=\{P \in \mathscr{P}$: mesh of $P \leqq 1/n\}$ is denoted by lim β-var $X(\mathscr{P}_n)$.

In case $\mathscr{P}$ is a sequence of partitions P_n, the lim sup as $n \to \infty$ of $V(P_n, \beta)$ is denoted by lim sup β-var $X(P_n)$.

β-var $X[a,b]$ denotes the supremum over P in $\mathscr{P}$ of $V(P \cap [a,b], \beta)$.

3. A Method Using the Definition of Stable Random Variable

Theorem 1. *Let $X(t)$ be the symmetric stable process on R^1 of index α, $0<\alpha \leqq 2$, let $0<\beta \leqq 1$, and let $\mathscr{P}$ be a collection of finite partitions of $[0,1]$, closed under translation and multiplication by $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and under piecing at $\frac{1}{3}, \frac{1}{2}$. Then the distribution of β-var X is a stable law of index α/β and*

i) *if $0<\alpha/\beta<1$ then the distribution is a proper stable law on $[0,\infty)$;*

ii) *if $\alpha/\beta=1$ then β-var X is either a constant with probability one, or is ∞ with probability one;*

iii) *if $\alpha/\beta>1$, $P\{\beta$-var $X=\infty\}=1$.*

Proof. That β-var X is a random variable is established by Blumenthal and Getoor [1]. We will show that $Y=\beta$-var X satisfies the defining relation for a stable random variable,

$$S_n \stackrel{d}{=} c_n Y, \quad \text{where } S_n=\sum_{i=1}^{n} Y_i,$$

the Y_i are independent and distributed like Y, and $c_n=n^{\beta/\alpha}$. It suffices that the relation hold for $n=2, 3$ [4, p. 215]. An equivalent relation is $r^{\beta/\alpha} Y_1+(1-r)^{\beta/\alpha} Y_2 \stackrel{d}{=} Y$, for $r=\frac{1}{2}, \frac{1}{3}$, where $Y_1 \stackrel{d}{=} Y_2 \stackrel{d}{=} Y$ and Y_1, Y_2 are independent. The proof of this relation is given in three parts: For $r=\frac{1}{2}, \frac{1}{3}$,

$$\beta\text{-var } X=\beta \text{ var } X[0,r]+\beta\text{-var } X[r,1], \tag{1}$$

where the summands are independent;

$$\beta\text{-var } [r,1] \stackrel{d}{=} \beta\text{-var } [0,1-r]; \tag{2}$$

$$\beta\text{-var } [0,r] \stackrel{d}{=} r^{\beta/\alpha}\, \beta\text{-var } [0,1], \tag{3}$$

for $r=\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$.

For fixed P and X, since $\beta \leqq 1$, the variational sum is only increased if P is replaced by a refinement of $P \cup \{\frac{1}{3}, \frac{1}{2}\}$. Therefore we assume $r \in P$ and write

$$V(P,\beta)=V(P \cap [0,r], \beta)+V(P \cap [r,1], \beta).$$

The condition that $\mathscr{P}$ is closed under piecing at $r=\frac{1}{3}, \frac{1}{2}$ enables us to calculate β-var X, for fixed X, by taking the supremum of these two terms separately. Since the random variables β-var $X[0,r]$ and β-var $X[r,1]$ depend on values of $X(t)$ in non-overlapping time intervals, they are independent, and (1) is proved.

Because the process is stationary,

$$V(P\cap[r,1],\beta)\stackrel{d}{=}\sum_{t_{i-1},t_i\in P\cap[r,1]}|X(t_i-r)-X(t_{i-1}-r)|^\beta.$$

For each path the supremum over $P\in\mathscr{P}$ of the right side is the same as β-var $X[0,1-r]$, since $\mathscr{P}$ is closed under translation by r. The supremum of the left side is β-var $X[r,1]$, so we have (2).

For fixed P, $V(P\cap[0,r],\beta)\stackrel{d}{=}$

$$r^{\beta/\alpha}\sum_{t_{i-1},t_i\in P\cap[0,r]}|X(t_i/r)-X(t_{i-1}/r)|^\beta. \tag{4}$$

The closure of $\mathscr{P}$ under multiplication by r enables us to calculate the supremum of (4) over $P\in\mathscr{P}$ for fixed X by taking the supremum of $r^{\beta/\alpha}V(P,\beta)$. Statement (3) follows.

Now combining (1) and (2) and applying (3) to each term we have

$$\begin{aligned}\beta\text{-var } X &\stackrel{d}{=}\beta\text{-var } X[0,r]+\beta\text{-var } X[0,1-r]\\ &\stackrel{d}{=}r^{\beta/\alpha}\,\beta\text{-var } X+(1-r)^{\beta/\alpha}\,\beta\text{-var } X,\end{aligned}$$

where the terms on the right are independent random variables. This is the relation we wished to prove. Note that only the proof of (1) requires that $\beta\leqq 1$.

Let F be the distribution of $Y=\beta$-var X. Since Y is nonnegative, either F is concentrated at a single point, possibly ∞, or the Laplace transform of F is $e^{-c\lambda^\gamma}$ where $0<\gamma<1$, [cf. 4, p. 424]. In the latter case $\gamma=\alpha/\beta$.

Suppose F is concentrated at a. The supremum is 0 only with probability 0, so $a>0$, and $F*F$ is concentrated at $2a$. But $Y_1+Y_2\stackrel{d}{=}2^{\beta/\alpha}Y$, so a is ∞ except possibly when $\beta/\alpha=1$. We conclude that when $\beta/\alpha<1$, F is concentrated at ∞, whereas if $\beta/\alpha=1$, F may be concentrated at any point of $[0,\infty]$. This concludes the proof of (ii) and (iii).

In case $0<\alpha/\beta<1$ the above argument does not tell us whether F is the proper stable law of index α/β on $[0,\infty)$ or whether F is concentrated at ∞. The following theorem of Blumenthal and Getoor [1] gives the answer.

Theorem A. Let $X(t)$ be the symmetric stable process in R^n of index α, $0<\alpha\leqq 2$, and let $\mathscr{P}'=\{\text{all finite partitions of }[0,1]\}$. Then $P\{\beta\text{-var } X=\infty\}=1$ or 0 according as $\beta\leqq\alpha$ or $\beta>\alpha$.

We use only the second half of this theorem. (The above includes an independent proof of the result for $\beta<\alpha$.) If $\mathscr{P}\subset\mathscr{P}'$ then β-var X over $\mathscr{P}\leqq\beta$-var X over $\mathscr{P}'$. Therefore if $\beta>\alpha$, $P\{\beta\text{-var } X=\infty\}=0$ for any collection of finite partitions and F is not concentrated at ∞. We conclude that F is the proper stable law on $[0,\infty)$.

If $\mathscr{P}$ is a sequence of partitions, closed under translation and multiplication by $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and piecing at $\frac{1}{3}, \frac{1}{2}$, which proceeds by refinements, and $\beta\leqq 1$, then lim sup

β-var $X(P_n)$ = limit β-var $X = \beta$-var X, the supremum over the sequence $\mathscr{P}$ regarded as a set, so that Theorem 1 applies. If $\mathscr{P}$ does not proceed by refinements, the identity $\limsup \beta\text{-var } X(P_n) \stackrel{d}{=} \beta\text{-var } X$, for $\beta \leqq 1$, $\beta \neq \alpha$ follows from Theorem 2, which can also be proved as an easy corollary to Theorem 1.

Theorem 2. *Let $\mathscr{P}$ be a sequence of finite partitions of* $[0, 1]$, *closed with respect to translation and multiplication by $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and piecing at $\frac{1}{3}, \frac{1}{2}$. Let $\beta \leqq 1$. Then Theorem* 1 *holds with β-var X replaced by* $\limsup \beta\text{-var } X(P_n)$.

Proof. The proof parallels the proof of Theorem 1 with the obvious modifications. In place of Theorem A we need to know that $P\{\limsup \beta\text{-var } X = \infty\} = 0$ if $\beta > \alpha$. This follows easily from Theorem A. As a set, $\mathscr{P} \subset \mathscr{P}'$ so that

$$\limsup \beta\text{-var } X(P_n) \leqq \beta\text{-var } X \text{ over } \mathscr{P} \leqq \beta\text{-var } X \text{ over } \mathscr{P}'.$$

4. A Method Using Laplace Transforms

When $\mathscr{P}$ is a sequence we expect that the distribution of the variation can sometimes be established by showing that the distributions of $Y_n = \sum_{t_i \in P_n} |X(t_i) - X(t_{i-1})|^\beta$ converge to a stable law, while the random variables Y_n converge almost surely. An example is given by Theorem 3, which differs from Theorem 2 in allowing $\beta > 1$ and in restricting the mesh of the partitions. We require the following lemma:

Lemma. *Let X_n be a sequence of random variables and let* $\limsup_{n \to \infty} X_n = X$. *Let F_n be the distribution of X_n, and suppose the F_n converge to a limit distribution F. Let G be the distribution of X. Then $G(y) \leqq F(y)$ for every y. If X_n converges to X almost surely, then $G = F$.*

Theorem 3. *Let $X(t)$ be a symmetric stable process on R^1 with index α, $0 < \alpha < 2$. Let $\mathscr{P}$ be a sequence of partitions P_n such that P_n contains n points and the mesh μ_n of P_n goes to* 0 *as n goes to ∞. Let $Y_n = \sum_{t_i \in P_n} |X(t_i) - X(t_{i-1})|^\beta$. Then*

i) *if $\beta > \alpha$, the distribution of Y_n converges to the one-sided stable law of index α/β. If in addition $\beta \leqq 1$ and P_n proceeds by refinements then $Y_n \to Y$ almost surely, where Y is distributed by the limiting distribution;*

ii) *if $\alpha/2 < \beta < \alpha$, the distribution of Y_n converges to* 0 *on* $[0, \infty)$ *and*

$$P\{\limsup_{n \to \infty} Y_n = \infty\} = 1;$$

iii) *if $\alpha = \beta$ then* (ii) *holds.*

Proof. Let H be the distribution of $X(1)$. That $x^\alpha(1 - H(x)) \to b$, a constant > 0 as $x \to \infty$ is a well-known property of stable laws [c.f. 4, p. 547]. Since H is symmetric, the distribution of $|X(1)|^\beta$ is $G(x) = 2H(x^{1/\beta}) - 1$ on $[0, \infty)$, and

$$x^{\alpha/\beta}(1 - G(x)) \to 2b \qquad \text{as} \quad x \to \infty. \tag{5}$$

The distribution of $|X(t_i) - X(t_{i-1})|^\beta$ is the same as that of $|X(1)|^\beta (t_i - t_{i-1})^{\beta/\alpha}$ and the terms of the sum Y_n are independent. Denoting $(t_i - t_{i-1})$ by Δt_i, the distribution of Y_n is

$$G(\Delta t_1^{-\beta/\alpha} x) * G(\Delta t_2^{-\beta/\alpha} x) * \cdots * G(\Delta t_n^{-\beta/\alpha} x).$$

Let $\varphi = \mathfrak{L} G$. Then the Laplace transform of the convolution is $\prod_{i=1}^{n} \varphi(\lambda\, \Delta t_i^{\beta/\alpha})$, and that of the distribution with density $1-G$ is $(1-\varphi(\lambda))/\lambda$. Therefore (5) implies

$$(1-\varphi(\lambda))/\lambda \sim c\,\lambda^{\alpha/\beta-1} \quad \text{as} \quad \lambda \to 0 \text{ [4, p. 424]}. \tag{6}$$

For fixed λ consider

$$-\log \prod_{i=1}^{n} \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) = -\sum_{i=1}^{n} \log[\varphi(\lambda\, \Delta t_i^{\beta/\alpha}) - 1 + 1]. \tag{7}$$

We use the inequality $|\log(1+x) - x| \leqq \frac{1}{2} x^2$, for small $|x|$, with $x = \varphi - 1$. Given $\varepsilon > 0$, let n be large enough so that $1 - \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \leqq (c+\varepsilon)(\lambda\, \Delta t_i^{\beta/\alpha})^{\alpha/\beta}$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}
&\sum_{i=1}^{n} \left|\log(\varphi(\lambda\, \Delta t_i^{\beta/\alpha}) - 1 + 1) - (\varphi(\lambda\, \Delta t_i^{\beta/\alpha}) - 1)\right| \\
&\quad \leqq \tfrac{1}{2} \sum_{i=1}^{n} |1 - \varphi(\lambda\, \Delta t_i^{\beta/\alpha})|^2 \leqq \tfrac{1}{2}\, n\, \lambda^{2\alpha/\beta} \sum \Delta t_i^2 (c'+\varepsilon)^2 \\
&\quad \to 0 \quad \text{as} \quad n \to 0 \quad \text{since} \quad \max_{i=1,\ldots,n} \Delta t_i \to 0.
\end{aligned}$$

It follows that

$$-\sum_{i=1}^{n} \log \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \sim \sum_{i=1}^{n} 1 - \varphi(\lambda\, \Delta t_i^{\beta/\alpha}).$$

Now choose n so large that

$$(c-\varepsilon)\, \lambda^{\alpha/\beta}\, \Delta t_i \leqq 1 - \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \leqq (c+\varepsilon)\, \lambda^{\alpha/\beta}\, \Delta t_i, \qquad i = 1, \ldots, n.$$

Summation over i yields

$$(c-\varepsilon)\, \lambda^{\alpha/\beta} \leqq \sum_{i=1}^{n} 1 - \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \leqq (c+\varepsilon)\, \lambda^{\alpha/\beta},$$

i.e.

$$\sum_{i=1}^{n} 1 - \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \to c\, \lambda^{\alpha/\beta} \quad \text{as} \quad n \to \infty.$$

Combining this with (7) we have

$$-\log \prod_{i=1}^{n} \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \to c\, \lambda^{\alpha/\beta},$$

or

$$\prod_{i=1}^{n} \varphi(\lambda\, \Delta t_i^{\beta/\alpha}) \to \exp(-c\, \lambda^{\alpha/\beta}).$$

It follows from the continuity theorem for Laplace transforms that the distribution of Y_n converges to F where $\mathfrak{L} F = \exp(-c\, \lambda^{\alpha/\beta})$.

If $\beta \leqq 1$ and $\mathscr{P}$ proceeds by refinements then for each sample path the sequence Y_n is nondecreasing and therefore converges. The last statement of (i) now follows from the Lemma.

If $\beta<\alpha$ we use a similar argument. Since $1-G(x)\sim 2b\,x^{-\alpha/\beta}$ and $\alpha/\beta>1$, $F(x)=\int_0^x 1-G(y)\,dy\to c$ as $x\to\infty$, where $0<c<\infty$. It follows that $\mathfrak{L}F(\lambda)=[1-\varphi(\lambda)]/\lambda\to c$ as $\lambda\to 0$, i.e. $1-\varphi(\lambda)\sim c\,\lambda$.

Proceeding as in case (i), given $\varepsilon>0$ we have as $n\to\infty$,

$$\sum_{i=1}^n |1-\varphi(\lambda\,\Delta t_i^{\beta/\alpha})|^2 \leqq (c+\varepsilon)^2\,\lambda^2 \sum \Delta t_i^{2\beta/\alpha}\to 0$$

if $2\beta/\alpha>1$, so that again

$$-\sum_{i=1}^n \log\varphi(\lambda\,\Delta t_i^{\beta/\alpha})\sim \sum_{i=1}^n 1-\varphi(\lambda\,\Delta t_i^{\beta/\alpha}).$$

For large enough n,

$$(c-\varepsilon)\,\lambda\sum_{i=1}^n \Delta t_i^{\beta/\alpha}\leqq \sum_{i=1}^n 1-\varphi(\lambda\,\Delta t_i^{\beta/\alpha}).$$

Since $\beta/\alpha<1$ and $\max\limits_{i=1,\dots,n} \Delta t_i\to 0$ the left side $\to\infty$, and so does $-\log\prod\limits_{i=1}^n \varphi(\lambda\,\Delta t_i^{\beta/\alpha})$. Therefore $\prod\limits_{i=1}^n \varphi(\lambda\,\Delta t_i^{\beta/\alpha})\to 0$ as $n\to\infty$ for each λ, and the continuity theorem implies that the distribution of Y_n converges to 0 for each x in $[0,\infty)$. It follows from the Lemma that $P\{\limsup\limits_{n\to\infty} Y_n=\infty\}=1$.

If $\beta=\alpha$, then $y(1-G(y))\to b$ as $y\to\infty$, and $F(x)=\int_0^x 1-G(y)\,dy\sim b\log x$. A Tauberian theorem [4, p. 421] implies that $[1-\varphi(\lambda)]/\lambda\sim c\log(1/\lambda)$ as $\lambda\to 0$.

Again proceeding as in case (i), given $\varepsilon>0$, as $n\to\infty$,

$$\sum_{i=1}^n |1-\varphi(\lambda\,\Delta t_i)|^2\leqq (c+\varepsilon)^2\,\lambda^2\sum_{i=1}^n\left(\Delta t_i\log\frac{1}{\lambda\,\Delta t_i}\right)^2$$
$$\leqq \sum_{i=1}^n \Delta t_i^2(\lambda\,\Delta t_i)^{-2\varepsilon}\to 0.$$

For large enough n,

$$(c-\varepsilon)\,\lambda\log\left(\frac{1}{\lambda\max\limits_{i=1,\dots,n}\Delta t_i}\right)\sum\Delta t_i\leqq (c-\varepsilon)\,\lambda\sum\Delta t_i\log\left(\frac{1}{\lambda\,\Delta t_i}\right)$$
$$\leqq \sum_{i=1}^n 1-\varphi(\lambda\,\Delta t_i).$$

As $n\to\infty$, the left side and consequently also the right side $\to\infty$. As in case (ii) we conclude that the distribution of Y_n converges to 0, and $P\{\limsup\limits_{n\to\infty} Y_n=\infty\}=1$.

Brownian motion, the stable process with index $\alpha=2$ can also be investigated by the method of Theorem 3. In the notation used there, $F(x)\to c>0$ as $x\to\infty$ for every $\beta>0$. This leads to the result that if $\mu_n\to 0$ as $n\to\infty$, the distribution of Y_n converges to a distribution concentrated at ∞ if $\beta<2$, at 0 if $\beta>2$, and at a number $d>0$ if $\beta=2$. We conclude that Y_n does not converge to a stable random variable in the case $\beta>\alpha=2$.

Some information about the rate of growth of the variational sum if $\alpha=\beta<2$ and $P_n=\{i/n, i=0, \ldots, n\}$ is given by Theorem 4.

Theorem 4. *Let $X(t)$ be a symmetric stable process of index α, $0<\alpha<2$. The distribution of $Z_n=\sum_{i=1}^{n}|X(i/n)-X((i-1)/n)|^\alpha/\log n$ converges to a distribution concentrated at c, $0<c<\infty$.*

Proof. Let G be the distribution of $|X(1)|^\alpha$ and $F(x)=\int_0^x 1-G(y)\,dy$. The distribution of Z_n is $G^{n^*}(x(n\log n))$. As in part (iii) of Theorem 3, $y(1-G(y))\to b$ as $y\to\infty$, $F(x)\sim b\log x$, and $[1-\varphi(\lambda)]/\lambda\sim c\log 1/\lambda$, where $\mathfrak{L}G=\varphi$. As $n\to\infty$,

$$-\log\varphi^n(\lambda/(n\log n))\sim n[1-\varphi(\lambda/(n\log n))]\sim n\,c(\lambda/(n\log n))$$
$$\cdot\log((n\log n)/\lambda)=c\,\lambda[\log n+\log\log n+\log 1/\lambda]/\log n\to c\,\lambda.$$

Therefore $\varphi^n(\lambda/(n\log n))\to e^{-k\lambda}$. It follows from the continuity theorem for Laplace transforms that $G^{n^*}((n\log n)\,x)$ converges to the distribution concentrated at c.

5. Corollaries Using the Method of Section 3 and Results of Section 4

Corollary 1. *Let $X(t)$ be a symmetric stable process of index α, $0<\alpha<2$, and let $\mathscr{P}_n=\{$all finite partitions of $[0,1]$ of* mesh *$\mu\leqq 1/n\}$. Let Y_n be the supremum over $P\in\mathscr{P}_n$ of $V(P,\beta)$, $\beta>\alpha$. Then the variation* lim β-var *$X(\mathscr{P}_n)$ defined by $Y=\operatorname*{limit}_{n\to\infty} Y_n$ is a proper stable random variable of index α/β.*

Proof. The proof follows that of Theorem 1. Statements (2) and (3) hold just as in Theorem 1. We must show that (1) holds.

Let A be the set of paths which are continuous at r. Since the stable process has no fixed discontinuities, $P\{A\}=1$. For each path in A, $Y_n\leqq Y_n[0,r]+Y_n[r,1]+\sup_{P\in\mathscr{P}_n}|X(t_j)-X(t_{j-1})|^\beta$, where t_{j-1} and t_j are the points of P neighboring r, or $t_{j-1}=t_j=r$ if $r\in P$. As $n\to\infty$, $t_j-t_{j-1}\to 0$, and so does $\sup_{P\in\mathscr{P}_n}|X(t_j)-X(t_{j-1})|^\beta$.

Almost surely, then,

$$\text{limit } Y_n\leqq\text{limit } Y_n[0,r]+\text{limit } Y_n[r,1].$$

To obtain the reverse inequality we write for each point in A,

$$Y_n+\sup_{P\in\mathscr{P}_n}(|X(t_j)-X(r)|^\beta+|X(r)-X(t_{j-1})|^\beta)\geqq Y_n[0,r]+Y_n[r,1],$$

and apply a similar argument. We conclude that (1) holds.

As in Theorem 1, Y satisfies the defining relation for a stable random variable. Let F be the distribution of Y. F is either concentrated at a point, possibly ∞, or is the proper one-sided stable law of index α/β. Since $\alpha<\beta$, as in Theorem 1 F cannot be concentrated at a point of $(0,\infty)$. Theorem A implies, again, that F is not concentrated at ∞. That F is not concentrated at 0 follows from Theorem 3. Let $\{P_n\}$ be a sequence of partitions such that P_n has n points and the mesh of P_n decreases to 0 as $n\to\infty$. Then Theorem 3 part (i), together with the Lemma, implies that the distribution G of lim sup β-var $X(P_n)$ is bounded by a proper one-sided stable law; in particular $G(0)=0$. For each sample path, lim β-var $X(\mathscr{P}_n) \geqq$ lim sup β-var $X(P_n)$. We conclude that $F(0)=0$.

We note that this proof does not extend to Brownian motion since for $\alpha=2$, $\beta>\alpha$, and P_n becoming dense in $[0,1]$, lim sup β-var $X(P_n)=0$ almost surely.

Corollary 2. *Let* $\mathscr{P}=\{$*all finite partitions of* $[0,1]\}$, $0<\alpha<2$, $\beta>\alpha$, *and* β-var $X=\sup_{P\in\mathscr{P}} V(P,\beta)$. *Let G be the distribution of* β-var X. *Then G is in the domain of normal attraction of the stable law F of Corollary* 1, *and* $1-G(x) \sim c\,x^{-\alpha/\beta}$ *as* $x\to\infty$.

Proof. First we introduce some notation in addition to that of Corollary 1. If P is a finite partition, let $\mathscr{P}_n \cup P = \{$partitions in $\mathscr{P}_n$ which include the points of $P\}$. Let F_n denote the distribution of Y_n, $F_n(P)$ the distribution of the supremum of $V(Q,\beta)$ over Q in $\mathscr{P}_n \cup P$, and $F(P)$ the limit as $n\to\infty$ of $F_n(P)$. Since the collection of partitions $\mathscr{P}_n \cup P$ decreases with increasing n, the supremum of $V(Q,\beta)$ is nonincreasing, $F_n(P)$ is nondecreasing, and $F(P)$ is the distribution of the limiting random variable.

Let $r\in[0,1]$. From Corollary 1 we have limit $Y_n=$ limit $Y_n[0,r]+$ limit $Y_n[r,1]$ almost surely. In terms of the new notation, if we regard $\{r\}$ as a partition, $F=F(r)$. More generally, if P is any finite partition $F=F(P)=\lim F_n(P)$. Since $\mathscr{P}_n \supset \mathscr{P}_n \cup P$ and $F_n(P)$ is nondecreasing as n increases, we have $F_n \leqq F_n(P) \leqq F$. This inequality holds, in particular, when P is P_n, the regular partition of mesh $1/n$. We conclude that $F_n(P_n)$ converges to F. But $\mathscr{P}_n \cup P_n = \mathscr{P} \cup P_n$, so $F(P_n)$ converges to F.

Now $F(P_n)$ is the n-fold convolution of the distribution of β-var $X[0,1/n]$ which, as we saw in Section 1, is distributed like $n^{-\beta/\alpha}$ β-var X; i.e. $F(P_n)=[G(n^{\beta/\alpha}x)]^{n*}$ converges to the stable law F. By definition, then, G is in the domain of normal attraction of F. The accompanying asymptotic relation is well known [4].

The next corollary is an extension of Theorem 3 part (i). Its proof uses Corollary 1 as well as previous results.

Corollary 3. *Let* $X(t)$ *be a symmetric stable process of index* α, $0<\alpha<2$, *and let* $\mathscr{P}$ *be a sequence of partitions* P_n *of* $[0,1]$ *such that* P_n *contains n points and* $\mu_n=$ mesh *of* P_n *decreases to* 0 *as* $n\to\infty$. *Then the distribution of* lim sup β-var $X(P_n)$ *is bounded between two proper stable laws of index* α/β, *for every* $\beta>\alpha$.

Proof. With $\mathscr{P}_n$ as in Corollary 1, for each path

$$\lim \beta\text{-var } X(\mathscr{P}_n) \geqq \limsup \beta\text{-var } X(P_n).$$

If F and G are the distributions of these two random variables in the order written, then $F\leqq G$. Corollary 1 says F is a proper stable law of index α/β. On the other hand Theorem 3 part (i), together with the Lemma, says G is bounded above by such a stable law.

6. Discussion

The restriction $\beta\leqq 1$ in Theorem 1 was imposed in order to allow $\frac{1}{2}$ and $\frac{1}{3}$ to be included in all partitions. If $\mathscr{P}$ is all finite partitions of $[0,1]$, it is possible that the distribution of β-var X remains unchanged if $\frac{1}{2}$ or $\frac{1}{3}$ is added to every partition. If this is the case then the method of Theorem 1 will give a new proof of Theorem A whenever $\beta\leqq\alpha$ and will specify that β-var X is distributed by a stable law when $\beta>\alpha$.

The scaling relation (3) can be shown to hold for each t in $[0,1]$. When β-var X is distributed by a proper stable law with Laplace transform $\exp(-c\lambda^{\alpha/\beta})$ it follows from (3) that the transform of the law of β-var $X[0,t]$ is $\exp(-tc\lambda^{\alpha/\beta})$. It now follows from the stationary and Markov properties of the stable process $X(t)$ that β-var $X[0,t]$ can be viewed as a one-sided stable process or stable subordinator of index α/β. In fact an alternative approach to Theorem 1 is to show that β-var $X[0,t]$ is a differential process which satisfies (3) for each t. Blumenthal and Getoor have observed this in a slightly different context [2, p. 509].

It is not clear what happens in Theorem 3 part (ii) if $\beta \leqq \alpha/2$. If P_n is the sequence $P_n=\{i/n,\ i=0,\ldots,n\}$, however, the method of Theorem 4 leads to the result $P\{\limsup \beta\text{-var } X=\infty\}=1$ for every $\beta<\alpha<2$.

The earliest results of this type appear to be those of P. Lévy [5] on the 2-variation of the Brownian path. Lévy showed that if $\mathscr{P}$ is all finite partitions then 2-var $X=\infty$ almost surely, whereas if $\mathscr{P}$ is an increasing sequence of partitions which become dense in $[0,1]$, i.e. the mesh goes to 0, then lim-2-var $X=c>0$ with probability 1. The method of Theorem 1 indicates that two such possibilities exist when $\alpha=\beta\leqq 1$. However we can conclude from part (iii) of Theorem 3, together with Theorem A, that Brownian motion is the *only* stable process of the family for which the two types of α-variation have different distributions. If $0<\alpha<2$, then both the supremum of $\sum_{t_i\in P} |X(t_i)-X(t_{i-1})|^\alpha$ over all finite partitions and the lim sup through a sequence of partitions which become dense, are almost surely infinite.

There remains the possibility that for α in $(0,2]$, $\beta>1$ and $\beta>\alpha$, the distribution of β-var X may differ from that of lim sup β-var $X(P_n)$ when the limit is taken through a sequence of partitions which become dense. However Corollaries 2 and 3 indicate that for $0<\alpha<2$, the two distributions have similar asymptotic behavior.

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P. E. Greenwood
Dept. of Mathematics
The University of British Columbia
Vancouver 8, Canada

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