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On the Central Limit Theorem in R^k

The Remainder Term for Special Borelsets*

HARALD BERGSTRÖM

Summary. Let F^{*n} denote the nth convolution of a distribution function F on R^k and suppose that F has zero moments of the first order and finite second order moment matrix. It is well-known that $F^{*n}(\sqrt{n}\cdot)$ converges to a Gaussian d.f. Φ as $n\to+\infty$. These d.f.s determine measures $F^{*n}(\sqrt{n}A)$ and $\Phi(A)$ for Borelsets A. We present a method that admits the estimation of the remainder-term $F^{*n}(\sqrt{n}A)-\Phi(A)$ when A belongs to a certain class of Borelsets. This class contains all convex sets. If F has finite absolute third order moments then the remainder-term is of the order $n^{-\frac{1}{2}}$. Also the remainder term's dependence on the dimension k is given. These results strengthen and generalize earlier results in the same direction.

1. Introduction

Consider a distribution function F on R^k where F has zero moments of the first order and second order moment matrix M. Let F^{*n} denote the nth convolution of F with itself. It is well-known that $F^{*n}(\sqrt{n}\cdot)$ converges to the Gaussian distribution function Φ with the same moments of the first and second orders as F, provided that M is non-singular. In [4] I proved that

$$\delta_n=\sup_x |F^{*n}(\sqrt{n}\,x)-\Phi(x)|$$

is of the order $\left(\frac{1}{\sqrt{n}}\right)$ if F has finite absolute moments of the third order. Also I gave a more precise estimation of δ_n. Now $F^{*n}(\sqrt{n}\cdot)$ induces a measure μ_n defined for any Borelset A on R^k. We shall also use the notation $F^{*n}(\sqrt{n}\cdot)$ for this measure and thus put

$$\mu_n(A)=F^*(\sqrt{n}\,A).$$

In the same way $\Phi(A)$ denotes the measure of A induced by Φ.

We may now ask for estimations of the difference

$$\delta_n(A)=|F^{*n}(\sqrt{n}\,A)-\Phi(A)|$$

for Borelsets A. Already in [6] Esseen proved that

$$\delta_n(A)\leqq C n^{-\frac{k}{k+1}}$$

with a constant C for a closed sphere A if M is the unit matrix and F has finite moments of the fourth order. In [7] Rao stated (without proof) under the same conditions that

$$\delta_n(A)=[(\log n)^\alpha n^{-\frac{1}{2}}]$$

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with $\alpha=(k-1)/2(k-1)$ for any convex set. B. von Bahr [1] gave estimations of δ_n for general Borelsets where, however, the dependence of n is rather complicated and δ_n also depends heavily on the set A. For convex sets he gave the uniform estimation

$$\delta_n \leqq C n^{-\frac{1}{2}}$$

if F has bounded absolute moments of the order $s>k>1$.

In [2] Bhattacharya, assuming the existence of higher moments than the third moment gave estimations of δ_n for certain Borelsets.

Let B be a Borel set on the unit sphere $|x|=1$. We say that the Borelset A belongs to the class $\Gamma(B)$ if

$$A \subset A+\alpha z$$

for any $z \in B$ and any non-negative number α. Clearly the class $\Gamma(B)$ has the following properties

$$A \in \Gamma(B) \rightarrow A+x \in \Gamma(B) \tag{1.1}$$

for any $x \in R^k$,

$$A \in \Gamma(B) \rightarrow \beta A \in \Gamma(B) \tag{1.2}$$

for any positive number β (βA denotes the set $\{x\colon x=\beta y,\, y \in A\}$). In Theorem 3.1 I give an estimation of

$$\sup_{A \in \Gamma(B)} \left|F^{*n}(\sqrt{n}\, A)-\Phi(A)\right| \tag{1.3}$$

under the condition that F has finite moments of the third order and show that this supremum is of the order $n^{-\frac{1}{2}}$. Considering differences and sums of sets belonging to $\Gamma(B)$ we can get estimations of the remainder term for fairly general sets A for instance for all convex Borel sets. The quantity (13) depends on B and k in a rather simple way which is shown in Theorem 3.2. In Section 4 we estimate the remainder-term for sums of dependent random variables.

The method that I use here can be applied also for convolutions of unequal components and also in cases when the limit distribution is not the Gaussian distribution but another infinitely divisible distribution, for instance a stable distribution.

2. A Fundamental Lemma

Let Φ be the Gaussian distribution function with the non-singular second order moment-matrix M and zero moments of the first order and denote the density function of Φ by D. Hence

$$D(x)=(2\pi)^{-\frac{k}{2}} \Delta^{-\frac{1}{2}} \exp -\tfrac{1}{2} x' M^{-1} x, \tag{2.1}$$

$M=(\mu_{ij})$, μ_{ij} second order moment, $\Delta=\det M$.

There is an orthogonal matrix L which transforms M^{-1} into the diagonal matrix where the elements λ_j of the main diagonal of the matrix are the eigen-values of M^{-1}. By this transformation $D(x)$ is transformed into the density

$$D_0(x)=(2\pi)^{-\frac{k}{2}} \Delta^{-\frac{1}{2}} \exp -\tfrac{1}{2} \sum_{j=1}^{k} \lambda_j x_j^2. \tag{2.2}$$

We observe that $M^{-1}=\left(\frac{\Delta_{ij}}{\Delta}\right)$ where Δ_{ij}, $i,j=1,2,\dots,k$ are the cofactors of Δ. We also introduce the trace S of M^{-1}

$$S=\sum_{j=1}^{k}\frac{\Delta_{jj}}{\Delta} \tag{2.3}$$

and observe that also

$$S=\sum_{j=1}^{k}\lambda_j. \tag{2.4}$$

For a set B on the unit sphere $|x|=1$ on R^k we introduce the cone

$$C(x,B)=\{t\colon t=x+\alpha z,\ z\in B,\ \alpha\in[0,+\infty)\}. \tag{2.5}$$

Now we state our fundamental lemma which is a generalization of Lemma 2 in [4].

Lemma 2.1. *Let B, $\Gamma(B)$ and $C(x,B)$ be defined as above and put*

(i) $g(\rho)=\sup_{|x|=\rho S^{-\frac{1}{2}}}\Phi[C(x,B)]$

(ii) $M(\varepsilon)=\sup_{x\in R^k}\left|[F-\Phi]*\Phi\left(\frac{\cdot}{\varepsilon}\right)(A+x)\right|,\ \varepsilon>0,$

(iii) $(2-\beta)\,g(\rho_0)=1+\frac{1}{\gamma}$

where β and ρ_0 are positive numbers and $\beta<1$. Then

$$\sup_{x\in R^k}|F(A+x)-\Phi(A+x)|\leqq \operatorname{Max}\left[\frac{\rho_0\,\varepsilon}{\beta\sqrt{2\pi}},\ \gamma\left(M(\varepsilon)+\frac{\varepsilon k}{\pi}\right)\right]$$

for $A\in\Gamma(B)$, provided that $\gamma>0$.

Remark. If B contains an open set then $\sup_{\rho} g(\rho)>\frac{1}{2}$ so that β and ρ can be chosen as positive numbers so that $\gamma>0$.

In order to prove Lemma 2.1 we shall need two lemmas which we now give without proofs.

Lemma 2.2. *For any Borelset A on R^k we have*

$$|\Phi(A+y)-\Phi(A)|\leqq(2\pi)^{-\frac{1}{2}}|y|\,S^{\frac{1}{2}}.$$

Lemma 2.3. *For any Borelsets A and C on R^k and any $\varepsilon>0$ we have*

$$\left|\int_C[\Phi(A+\varepsilon v)-\Phi(A)]\,D(v)\,dv\right|\leqq\frac{k\varepsilon}{\pi}.$$

We are now going to prove Lemma 2.1. Put

$$\delta=\sup_{x\in R^k}|F(A+x)-\Phi(A+x)|$$

and first consider the case when $F(A+x)$ is larger than $\Phi(A+x)$ at the supremum. Then we choose x_0 such that $\delta_1=F(A+x^0)-\Phi(A+x^0)$ is arbitrarily close to δ.

Observing that $F(A+t) \geqq F(A+x^0)$ for $t \in C(x^0, B)$ since $A \in \Gamma(B)$ and that $|F(A+t)-\Phi(A+t)| \leqq \delta$ for all t we obtain for any point x

$$\begin{aligned} M(\varepsilon) &\geqq \frac{1}{\varepsilon^k} \int_{R^k} [(F(A+t)-\Phi(A+t)] D\left(\frac{x-t}{\varepsilon}\right) dt \\ &\geqq [F(A+x^0)-\Phi(A+x^0)] \cdot \frac{1}{\varepsilon^k} \int_{C(x^0, B)} D\left(\frac{x-t}{\varepsilon}\right) dt \\ &\quad - \frac{1}{\varepsilon^k} \int_{C(x^0, B)} [\Phi(A+t)-\Phi(A+x^0)] D\left(\frac{x-t}{\varepsilon}\right) dt \\ &\quad - \delta \cdot \frac{1}{\varepsilon^k} \int_{R^k - C(x^0, B)} D\left(\frac{x-t}{\varepsilon}\right) dt. \end{aligned}$$

In this relation we make the transformation $\dfrac{x-t}{\varepsilon} = -v$ and observe that $C(x^0, B)$ then is transformed into $C\left(\dfrac{x^0-x}{\varepsilon}, B\right)$. Then we obtain

$$\begin{aligned} M(\varepsilon) \geqq (\delta+\delta_1) &\int_{C\left(\frac{x^0-x}{\varepsilon}, B\right)} D(v)\, dv - \delta \\ - &\int_{C\left(\frac{x^0-x}{\varepsilon}, B\right)} [\Phi(A+x+\varepsilon v)-\Phi(A+x^0)]\, D(v)\, dv. \end{aligned} \tag{2.6}$$

By Lemma 2.2 and 2.3 we find that the last integral is not larger than

$$(2\pi)^{-\frac{1}{2}} |x-x^0|\, S^{\frac{1}{2}} \int_{C\left(\frac{x^0-x}{\varepsilon}, B\right)} D(v)\, dv + \frac{\varepsilon k}{\pi}.$$

Putting $|x-x^0| = \beta\, \delta S^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}}$ and observing the definition of $g(\rho)$ we get from (2.6)

$$M(\varepsilon) \geqq (\delta_1+\delta-\beta\,\delta)\, g\left[\frac{\beta\,\delta(2\pi)^{\frac{1}{2}}}{\varepsilon}\right] - \delta - \frac{\varepsilon k}{\pi}$$

or since δ_1 is arbitrarily close to δ

$$M(\varepsilon) \geqq \delta(2-\beta)\, g\left[\frac{\beta\,\delta(2\pi)^{\frac{1}{2}}}{\varepsilon}\right] - \delta - \frac{\varepsilon k}{\pi}. \tag{2.7}$$

Either we have

$$\frac{\beta\,\delta(2\pi)^{\frac{1}{2}}}{\varepsilon} \leqq \rho_0$$

and then

$$\delta \leqq \frac{\varepsilon\, \rho_0 (2\pi)^{-\frac{1}{2}}}{\beta}$$

or

$$\frac{\beta\,\delta(2\pi)^{\frac{1}{2}}}{\varepsilon} > \rho_0$$

and then we obtain from (2.7) regarding (iii)

$$\delta \leqq \gamma \left[M(\varepsilon) + \frac{\varepsilon k}{\pi} \right].$$

Hence the lemma is proved in this case. If $\delta = \sup_x \Phi(A+x) - F(A+x)$ we use the fact that $F(A+t) \leqq F(A+x^0)$ on the cone

$$\bar{C}(x^0, B) = \{t: t = x^0 - \alpha z, z \in B, \alpha \in [0, +\infty)\}.$$

Then the proof can be carried through in the same way as in the first case. The lemma is proved.

Remark. It is easily seen that (2.6) remains true if we change $C\left(\frac{x^0 - x}{\varepsilon}, B\right)$ against any sub-Borelset of this cone. This fact can be used in order to get better estimations in some cases (cf. [3]).

3. An Estimation of the Remainder Term

Theorem 3.1. *Let $\Gamma(B)$ and $C(x, B)$ be defined as above and suppose that*

$$\lim_{\rho \to +\infty} \sup_{|x| = \rho S^{-\frac{1}{2}}} \Phi[C(x, B)] > \tfrac{1}{2}.$$

Further let F be a distribution function on R^k with zero moments of the first order and second order moment matrix M which is non-singular. Put

$$\beta_{3j} = \int_{R^k} |x_j|^3 \, dF(x), \qquad \beta_3 = \sum_{j=1}^{k} \beta_{3j}$$

and assume that $\beta_3 < +\infty$. Then

(i) $\sup_{A \in \Gamma(B)} |F^{*n}(\sqrt{n} A) - \Phi(A)| \leqq C \beta_3 S^{\frac{3}{2}} n^{-\frac{1}{2}}$

where Φ is the Gaussian distribution function with zero moments of the first order and second order moment matrix M, S is the trace of M^{-1} and C is a constant only depending on k and B.

Proof. At first we notice that

$$\beta_3 S^{\frac{3}{2}} \geqq k^{\frac{5}{2}}. \tag{3.1}$$

From (3.1) it follows that the inequality of Theorem 3.1 holds for $n \leqq n_0$ if $C \geqq n_0^{\frac{1}{2}} k^{-\frac{5}{2}}$.

Having established the fundamental lemma we can now proceed as in [4] and prove the theorem by the help of induction. Thus putting

$$\varepsilon_\nu^2 = \varepsilon^2 + \frac{\nu}{n}, \qquad (\bar{\varepsilon}_{n-\nu-1})^2 = \frac{n-\nu-1}{n}, \qquad \varepsilon_\nu > 0, \qquad \bar{\varepsilon}_{n-\nu-1} \geqq 0$$

$$F^{*n-\nu-1}(\sqrt{n}\,\cdot) = \Phi\left(\frac{\cdot}{\bar{\varepsilon}_{n-\nu-1}}\right) + r_{n-\nu-1}(\cdot)$$

for $v=0,1,\dots,n-1$, we get

$$[F^{*n}(\sqrt{n}\cdot)-\Phi(\cdot)]*\Phi\left(\frac{\cdot}{\varepsilon}\right)=\sum_{v=0}^{n-1}F^{*n-v-1}(\sqrt{n}\cdot)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)]*\Phi\left(\frac{\cdot}{\varepsilon_v}\right)$$

$$=\sum_{v=0}^{n-1}\Phi\left(\frac{\cdot}{\bar{\varepsilon}_{n-v-1}}\right)*\Phi\left(\frac{\cdot}{\bar{\varepsilon}_v}\right)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)] \quad (3.2)$$

$$+\sum_{v=0}^{n-2}r_{n-v-1}(\cdot)*\Phi\left(\frac{\cdot}{\varepsilon_v}\right)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)].$$

Now

$$\Phi\left(\frac{\cdot}{\bar{\varepsilon}_{n-v-1}}\right)*\Phi\left(\frac{\cdot}{\bar{\varepsilon}_v}\right)=\Phi\left(\frac{\cdot}{\bar{\varepsilon}_{n-1}}\right).$$

Hence putting

$$H_{n-v-1}(\cdot)=r_{n-v-1}(\cdot)*\Phi\left(\frac{\cdot}{\varepsilon_v}\right) \quad (3.3)$$

we obtain from (3.3)

$$F^{*n}(\sqrt{n}\cdot)-\Phi^{*n}(\sqrt{n}\cdot)*\Phi\left(\frac{\cdot}{\varepsilon}\right)=n\,\Phi\left(\frac{\cdot}{\varepsilon_{n-1}}\right)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)]$$
$$+\sum_{v=0}^{n-2}H_{n-v-1}(\cdot)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)]. \quad (3.4)$$

Now assume that the inequality in Theorem 3.1 holds true for $n\leqq n'$. It was shown above that it certainly does for $n'=n_0$ if $C\geqq n_0^{\frac{1}{2}}k^{-\frac{5}{2}}$. Consider then $n=n'+1$. In (3.4) we obtain for any Borel set A

$$[F^{*n}(\sqrt{n}\cdot)-\Phi^{*n}(\sqrt{n}\cdot)]*\Phi\left(\frac{\cdot}{\varepsilon}\right)(A)=n\,\Phi\left(\frac{\cdot}{\varepsilon_{n-1}}\right)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)](A)$$
$$+\sum_{v=0}^{n-2}H_{n-v-1}(\cdot)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)](A). \quad (3.5)$$

Since F and Φ have the same moments of the first and second orders we obtain expanding $H(A-t)$ by Taylor's formula

$$|H(\cdot)*[F(\sqrt{n}\cdot)-\Phi(\sqrt{n}\cdot)](A)|=\left|\int_{R^k}H(A-t)[dF(t\sqrt{n})-\Phi(t\sqrt{n})]\right|$$
$$\leqq\frac{1}{3!}\int_{R^k}\sup_{x\in R^k}\left|\left(\sum_{v=1}^{k}t_v\frac{\partial}{\partial x_v}\right)^3 H(A-x)\right|d[F(t\sqrt{n})+\Phi(t\sqrt{n})]. \quad (3.6)$$

For any non-negative integers $m_1,\dots,m_k$ with $m_1+\dots+m_2=m$ we have

$$\left|\frac{\partial^m H(A-x)}{\partial x_1^{m_1}\dots\partial x_2^{m_k}}\right|=\frac{1}{\varepsilon_v^k}\left|\int_{R^k}r\,(A-v)\frac{\partial^m}{\partial x_1^{m_1}\dots\partial x_k^{m_k}}D\left(\frac{x-v}{\varepsilon_v}\right)dv\right|$$

$$\leqq\varepsilon_v^{-m}\sup_{x\in R^k}|r\,(A-x)|\int_{R^k}\left|\frac{\partial^m}{\partial v_1^{m_1}\dots\partial v_k^{m_k}}D(v)\right|dv.$$

In [4] p. 114 we have given the inequality

$$\left\{\int_{R^k}\left|\frac{\partial^m}{\partial v_1^{m_1}\dots\partial v_k^{m_k}}D(v)\right|dv\right\}^2 \leqq \left|\frac{\partial^m}{\partial x_1^{m_1}\dots\partial x_k^{m_k}}\prod_{i=1}^k\left(\sum_{j=1}^k\frac{\Delta_{ij}}{\Delta}x_j\right)^{m_i}\right|. \tag{3.7}$$

Observing that

$$|\Delta_{ij}|\leqq(\Delta_{ii}\Delta_{jj})^{\frac{1}{2}}$$

since Φ is non singular, we find that the right hand side of (3.7) is not larger than

$$m!\,\Pi\left(\frac{\Delta_{jj}}{\Delta}\right)^{m_j}.$$

Thus we get from (3.7)

$$\left|\frac{\partial^m}{\partial x_1^{m_1}\dots\partial x_k^{m_k}}H(A-x)\right|\leqq(m!)^{\frac{1}{2}}\left(\prod_{j=1}^k\frac{\Delta_{jj}}{\Delta}\right)^{\frac{m_j}{2}}\varepsilon_\nu^{-m}\sup_{x\in R^k}|r(A-x)|.$$

Combining this inequality and (3.6) we obtain

$$\begin{aligned}|H(\cdot)*[F(\sqrt{n})-\Phi(\sqrt{n})](A)|&\leqq 6^{-\frac{1}{2}}\varepsilon_\nu^{-3}\sup_{x\in R^k}|r(A-x)|\\ &\cdot\int_{R^k}\left[\sum_{j=1}^k|t_j|\left(\frac{\Delta_{jj}}{\Delta}\right)^{\frac{1}{2}}\right]^3 d[F(t\sqrt{n})+\Phi(t\sqrt{n})].\end{aligned} \tag{3.8}$$

By well-known inequalities we easily get

$$\left[\sum_{j=1}^k|t_j|\left(\frac{\Delta_{jj}}{\Delta}\right)^{\frac{1}{2}}\right]^3\leqq k^{\frac{1}{2}}\left[\sum_{j=1}^k\left(\frac{\Delta_{jj}}{\Delta}\right)\right]^{\frac{3}{2}}\sum_{j=1}^k|t_j|^3=k^{\frac{1}{2}}S^{\frac{3}{2}}\sum_{j=1}^k|t_j|^3.$$

Since

$$\int_{R^k}|t_j|^3\,dF(t\sqrt{n})=n^{-\frac{3}{2}}\beta_{3j}$$

and by [1] 4

$$\int_{R^k}|t_j|^3\,d\Phi\left(\frac{t}{\sqrt{n}}\right)\leqq 3^{\frac{1}{2}}(\mu_{jj})^{\frac{3}{2}}n^{-\frac{3}{2}}\leqq 3^{\frac{1}{2}}\beta_{3j}n^{-\frac{3}{2}}$$

we thus obtain from (3)

$$\begin{aligned}&|H_{n-\nu-1}(\cdot)*[F(\sqrt{n}\,\cdot)-\Phi(\sqrt{n}\,\cdot)](A)|\\ &\quad\leqq 6^{-\frac{1}{2}}(1+3^{\frac{1}{2}})(n^{\frac{1}{2}}\varepsilon_\nu)^{-3}k^{\frac{1}{2}}S^{\frac{3}{2}}\beta_3\sup|r_{n-\nu-1}(A-x)|\end{aligned} \tag{3.9}$$

for $\nu=0,\dots,n-2$. It is easily seen that this inequality remains true for $\nu=0$ if we define $H_{n-1}(\cdot)=\Phi\left(\frac{\cdot}{\varepsilon_{n-1}}\right)$ and thus put $r_{n-1}(\cdot)$ equal to the unit distribution and then $\sup|r_{n-1}|=1$. Hence we get from (3.5)

$$\begin{aligned}\left|[F^{*n}(\sqrt{n}\,\cdot)-\Phi^{*n}(\sqrt{n}\,\cdot)]*\Phi\left(\frac{\cdot}{\varepsilon}\right)(A)\right|&\leqq 6^{-\frac{1}{2}}(1+3^{\frac{1}{2}})\,n^{-\frac{3}{2}}k^{\frac{1}{2}}S^{\frac{3}{2}}\beta_3\\ &\cdot\left\{n\cdot\varepsilon_{n-1}^{-3}+\sum_{\nu=0}^{n-2}\varepsilon_\nu^{-3}\sup_{x\in R^k}|r_{n-\nu-1}(A-x)|\right\}.\end{aligned} \tag{3.10}$$

Now assuming that the inequality in Theorem 3.1 holds true for $n \leqq n'$ and observing that

$$A \in \Gamma(B) \to \left(\frac{n}{n-\nu-1}\right)^{\frac{1}{2}} (A-x) \in \Gamma(B)$$

we get

$$|r_{n-\nu-1}(A-x)| \leqq C \beta_3 S^{\frac{3}{2}} (n-\nu-1)^{-\frac{1}{2}}.$$

It was shown in [3] p. 150

$$\sum_{\nu=0}^{n-2} \varepsilon_\nu^{-3} (n-\nu-1)^{-\frac{1}{2}} \leqq 2 n^{\frac{1}{2}} \varepsilon^{-1} + n^{-\frac{1}{2}} \varepsilon^{-3} \left(1-\frac{1}{n}\right)^{-\frac{1}{2}}$$

and the right hand side is smaller than $4 n^{\frac{1}{2}} \varepsilon^{-1}$ for $\varepsilon \geqq n^{-\frac{1}{2}}, n \geqq 2$[1].

Regarding these inequalities we obtain from (3.11)

$$\left|[F^{*n}(\sqrt{n}\cdot) - \Phi^{*n}(\sqrt{n}\cdot)] * \Phi\left(\frac{\cdot}{\varepsilon}\right)(A)\right| \leqq M'(\varepsilon) \tag{3.11}$$

with

$$M'(\varepsilon) = 6^{\frac{1}{2}} (1+3^{\frac{1}{2}}) k^{\frac{1}{2}} [\varepsilon_{n-1}^{-3} + 4 C \beta_3 S^{\frac{3}{2}} \varepsilon^{-1} n^{-\frac{1}{2}}] \beta_3 S^{\frac{3}{2}} n^{-\frac{1}{2}}.$$

To this inequality we apply Lemma 2.1 and get

$$|F^{*n}(\sqrt{n} A) - \Phi(A)| \leqq \operatorname{Max}\left\{\frac{\rho_0 \varepsilon}{\beta \sqrt{2\pi}}, \gamma \left[M'(\varepsilon) + \frac{\varepsilon k}{\pi}\right]\right\} \tag{3.12}$$

where ρ_0, β and γ are given in this lemma.

Now choose

$$\varepsilon = a \beta_3 S^{\frac{3}{2}} n^{-\frac{1}{2}}$$

where $a \geqq 1$ and a is independent of n. Then $\varepsilon_{n-1} \geqq 1$ and

$$\gamma M'(\varepsilon) \leqq \gamma 6^{-\frac{1}{2}} (1+3^{\frac{1}{2}}) k^{\frac{1}{2}} \left[1+\frac{4C}{a}\right] \beta_3 S^{\frac{3}{2}} n^{-\frac{1}{2}}$$

$$\gamma \frac{\varepsilon k}{\pi} = \gamma \frac{a k}{\pi} \beta_3 S^{\frac{3}{2}} n^{-\frac{1}{2}}$$

$$\frac{\rho_0 \varepsilon}{\beta \sqrt{2\pi}} = \frac{\rho_0 a}{\beta \sqrt{2\pi}} \beta^3 S^{\frac{3}{2}} n^{-\frac{1}{2}}.$$

If we choose C so large that

$$\gamma \cdot 6^{-\frac{1}{2}} (1+3^{\frac{1}{2}}) k^{\frac{1}{2}} \left(1+\frac{4C}{a}\right) + \frac{\gamma a k}{\pi} \leqq C. \tag{3.13}$$

$$\frac{\rho_0 a}{\beta \sqrt{2\pi}} \leqq C \tag{3.14}$$

we find that the right hand side of (3.12) is not larger than $C \beta_3 S^{\frac{3}{2}} n^{-\frac{1}{2}}$ and so the theorem follows.

1 This estimation may be improved. The second term on the right hand side of the inequality is small compared to the first one for those ε which are considered.

We shall now examine the dependence of C on the dimension k. It follows from (3.13) that we have to choose

$$a = a_0 k^{\frac{1}{2}}$$

with a number $a_0 > 1$ and then C is at least of the order $0(k^{\frac{3}{2}})$. We may then let ρ_0 have the order $0(k)$. However ρ_0 may have a larger order for certain sets B. As a complement to Theorem 3.1 we show

Theorem 3.2. *Let the eigenvalues λ_j, $j=1,2,\ldots,k$ of M^{-1} satisfy the condition*

$$\inf_{i,j} \frac{\lambda_i}{\lambda_j} = d^2$$

and let γ, β and μ be given positive numbers, $\beta < 1$. If

$$\text{(i)}\ (2-\beta) \sup_{|x| = \mu k \delta(k) d^{-1} S^{-\frac{1}{2}}} \Phi[C(x,B)] \geqq 1 + \frac{1}{\gamma}$$

with $\delta(k) \geqq 1$, then the constant C in the inequality (i) *of Theorem* 3.1 *can be chosen as*

$$C = C_0 k^{\frac{3}{2}} \delta(k) d^{-1}$$

with an absolute constant C_0. (Hence C_0 only depends on β, γ and μ. The factor d is introduced for later purpose.)

Proof. We may choose

$$\rho_0 = \mu k \delta(k) d^{-1}.$$

Observing that $d \leqq 1$, $\delta(k) \geqq 1$ we find that (3.13) and (3.14) then are satisfied for

$$a = 8\gamma \cdot 6^{-\frac{1}{2}} (1 + 3^{\frac{1}{2}}) k^{\frac{1}{2}},$$

$$C_0 \geqq 2\gamma \cdot 6^{-\frac{1}{2}} (1 + 3^{\frac{1}{2}}) \left(1 + \frac{8\gamma}{\pi}\right)$$

$$C_0 \geqq 8\gamma\mu \cdot 6^{-\frac{1}{2}} (1 + 3^{\frac{1}{2}}) \cdot \beta^{-1} (2\pi)^{-\frac{1}{2}}.$$

Of course, these estimations are not very good for numerical computations. (If $\delta(k)$ is of order larger than $0(1)$ it is easily seen that the left hand side of (3.14) essentially depends on $k^{\frac{1}{2}}(C/a)$.) We may now ask: Which sets B do satisfy the inequality (i) of Theorem 3.2 for given μ, $\delta(k)$, d, s, β and γ. Clearly B depends on the dimension k. The next theorem gives a sufficient condition for (i) and a description of sets B that holds generally for any dimension.

Theorem 3.3. *Let the cone $C(0,B)$, 0 being the zero-point, contain the "circular" cone with the angle of departure 2φ, $0 \leqq \varphi \leqq \frac{\pi}{2}$,*

$$\{y\colon y = \alpha z,\ \alpha \geqq 0,\ |z| = 1,\ z z' \geqq \cos\varphi\}$$

where z' is some point on $|z|=1$ and

$$\sin\varphi \geqq \frac{1}{\delta(k)}.$$

Further suppose that

$$\inf_{i,j} \frac{\lambda_i}{\lambda_j} = d^2.$$

Then to given positive numbers β and γ, $\beta < 1$ there exists a positive number μ such that (i) *of Theorem* 3.2 *holds*

Remark. We observe that we then have the estimation

$$|F^{*n}(\sqrt{n}\,A) - \Phi(A)| \leqq C_0\, k^{\frac{3}{2}}\, \delta(k)\, d^{-1}\, S^{\frac{3}{2}}\, \beta_3\, n^{-\frac{1}{2}}$$

for $A \in \Gamma(B)$ which is uniform in respect to all classes $\Gamma(B)$ and all F such that a circular one with the angle of departure 2φ, $\sin\varphi \geqq \dfrac{1}{\delta(k)}$ belongs to B and F has first order moments equal to 0 non-singular moment matrix and finite third order absolute moments [2].

Proof. Let the conditions of the theorem be satisfied. The cone $C(x, B)$ contains the cone

$$\{y\colon y = x + \alpha z,\ |z| = 1,\ \alpha \geqq 0,\ z z' \geqq \cos\varphi\}$$

and this cone contains the sphere

$$\{y\colon |y| \leqq |x| \sin\varphi\}.$$

Hence we have

$$\Phi[C(x, B)] \geqq \int_{|v| \leqq |x| \sin\varphi} D(v)\, dv. \tag{3.15}$$

By the orthogonal transformation $v \to Lv$ with $L'L =$ unit matrix we first transform this integral into

$$\int_{|v| \leqq |x| \sin\varphi} D_0(v)\, dv$$

where $D_0(v)$ is defined by (2.2). Then making the transformation

$$P v \to v$$

in the last integral where P has the diagonal elements $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \ldots, \lambda_k^{\frac{1}{2}}$ and all other elements equal to 0, we find that the last integral can be written

$$(2\pi)^{-\frac{k}{2}} \int_{|P^{-1} v| \leqq |x| \sin\varphi} \exp -\tfrac{1}{2}|v|^2\, dv.$$

The ellipsoid

$$|P^{-1} v| \leqq |x| \sin\varphi$$

contains the sphere

$$|v| \leqq \lambda^{\frac{1}{2}}\, |x| \sin\varphi$$

where $\lambda = \min\limits_j \lambda_j$. Hence we obtain from (3.16)

$$\Phi[C(x, B)] \geqq (2\pi)^{-\frac{k}{2}} \int_{|v| \leqq \lambda^{\frac{1}{2}} |x| \sin\varphi} \exp -\tfrac{1}{2}|v|^2\, dv.$$

2 If $\delta(k) = o(1)$ we may choose $C_0 = 4$ for large k.

Consider now the case

$$\rho=\frac{\mu k}{d}\,\delta(k), \qquad \sin\varphi \geqq \frac{1}{\delta(k)}, \qquad \min_{i,j}\frac{\lambda_i}{\lambda_j}=d^2$$

and observe that

$$\lambda^{\frac{1}{2}} S^{-\frac{1}{2}} \geqq \frac{d}{\sqrt{k}}.$$

Hence we get for $|x|=\rho S^{-\frac{1}{2}}$,

$$\lambda^{\frac{1}{2}}|x|\sin\varphi \geqq \mu\sqrt{k},$$

and

$$\sup_{|x|=\rho S^{-\frac{1}{2}}} \Phi[C(x,B)] \geqq (2\pi)^{-\frac{k}{2}} \int_{|v|\leqq\mu\sqrt{k}} \exp -\tfrac{1}{2}|u|^2\,dv.$$

Introducing spherical coordinates in the last integral we then get for $\mu>1$

$$\sup_{x=\rho S^{-\frac{1}{2}}} \Phi[C(x,B)] \geqq \frac{1}{\Gamma\left(\frac{k}{2}\right)} \int_{0\leqq y\leqq\frac{1}{2}(\mu\sqrt{k})^2} y^{\frac{k-2}{2}} \exp(-y)\,dy$$

$$=1-\frac{1}{\Gamma\left(\frac{k}{2}\right)} \int_{y>\frac{1}{2}(\mu\sqrt{k})^2} y^{\frac{k-2}{2}} (\exp -y)\,dy = 1-I_k$$

where

$$I_k=\frac{1}{\Gamma\left(\frac{k}{2}\right)} (\tfrac{1}{2}\mu^2 k)^{\frac{k-2}{2}} (\exp -\tfrac{1}{2}\mu^2 k) \int_0^\infty \left(1+\frac{2y}{\mu^2 k}\right)^{\frac{k-2}{2}} \exp(-y)\,dy$$

$$\leqq \frac{1}{\Gamma\left(\frac{k}{2}\right)} (\tfrac{1}{2}\mu^2 k)^{\frac{k-2}{2}} \exp(-\tfrac{1}{2}\mu^2 k) \int_0^\infty \exp -y\left(1-\frac{1}{\mu^2}\right) dy.$$

By the use of Stirling's formula we get

$$\Gamma\left(\frac{k}{2}\right) > (2\pi)^{\frac{1}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}-\frac{1}{2}} \exp -\frac{k}{2}.$$

Hence the last term in the inequality above is smaller than

$$\frac{1}{\mu^2-1}\cdot\frac{1}{\sqrt{\pi k}} \exp -\tfrac{1}{2}[\mu^2-1-ln\,\mu^2]\,k$$

and this quantity is not larger than

$$\frac{1}{\mu^2-1}\cdot\frac{1}{\sqrt{\pi k}}$$

for $\mu > 1$. Thus the inequality (i) of Theorem 3.2 is satisfied if

$$(2-\beta)\left[1-\frac{1}{(\mu^2-1)\sqrt{\pi k}}\right] \geqq 1+\frac{1}{\gamma} \tag{3.16}$$

and $\mu > 1$. Clearly $\mu > 1$ can be chosen independently of k such that this inequality holds for given β and μ. For large k we may choose $\mu = 1 + 0(k^{-\frac{1}{4}})$.

4. Convex Sets

Theorem 4.1. *Let F be a distribution function on R^k with first order moments equal to* 0, *non-singular second order moment matrix M and finite absolute third order moments β_{3j}. Denote the trace of M^{-1} by S the eigenvalues of M^{-1} by λ_j, $j=1, \ldots, k$ and put*

$$\beta_3 = \sum_{j=1}^{k} \beta_{3j}, \qquad d^2 = \inf_{i,j} \frac{\lambda_i}{\lambda_j}.$$

Let Φ be the Gaussian d.f. which has first order moments 0 *and second order moment matrix M. Then there exists a positive constant C_0 such that*

(i) $$|F^{*n}(\sqrt{n}\, A) - \Phi(A)| \leqq C_0\, k^{\frac{7}{2}}\, S^{\frac{3}{2}}\, \beta_3\, d^{-1}\, n^{-\frac{1}{2}}$$

for any convex set A such that all its translates $A+x$ are F-measurable. The constant C_0 is independent on F, k, S, β_3, d and n.

Remark. A and its translates are certainly F-measurable if A is a Borelset.

Proof. Since the estimation (i) is uniform in respect to the sets A it is sufficient to prove it for finite sets A. Any finite convex set A is Φ-measurable. In fact, we can find a polyeder B that contains A and a polyeder B' that is contained in A such that $\Phi(B)$ and $\Phi(B')$ are arbitrarily close to each other. Hence there exist Borelsets D and D' such that $A \subset D$, $D' \subset A$ and $\Phi(D) = \Phi(D')$. Since $\Phi(B') \leqq \Phi(A) \leqq \Phi(B)$ and the polyeders B' and B can be chosen such that $\Phi(B')$ and $\Phi(B)$ are arbitrarily close it is also sufficient to prove (i) for closed polyeders A of the dimension k and clearly a polyeder is a Borelset. Since the convex polyeder has the dimension k it has an interior that is an open set. We can then enclose a regular simplex in A. Let $p^{(j)}$, $j=1, \ldots, k+1$ be the normal unit vectors to the sides of the simplex, directed from the simplex. Any direction

$$p = \sum_{j=1}^{k} \alpha_j p_j, \qquad \alpha_j \geqq 0$$

(p unit vector) determines a hyperplane $P(p)$ of support of A having p as normal and at least one point say x in common with A and separating A and the set $\{y\colon p\, y > p \cdot x\}$. All these directions determine the set

$$B_{k+1} = \left\{y\colon y = z + \alpha\, p,\ \alpha \geqq 0,\ z \in P(p) \cap A,\ p = \sum_{j=1}^{k} \alpha_j p_j,\ \alpha_j \geqq 0\right\}. \tag{4.1}$$

In the same way we define sets B_j corresponding to the directions p_ν, $\nu = 1, \ldots, k+1$ $\nu \neq j$. We shall now show that B_{k+1} is a set of that type that has been considered in Theorem 3.1. Then we have to determine directions $q(|q|=1)$ such that

$$B_{k+1}+q\beta\subset B_{k+1} \tag{4.2}$$

for any positive number β. Denote the angle between the direction $-p_{k+1}$ and any of the directions $p_j, j\neq k+1$ by v. Then

$$-p_j\cdot p_{k+1}=\cos v.$$

We easily find that $\cos v=1/k$.

If now $x\in P(p^{(j)})\cap A$ for $j\neq k+1$ then the angle between q and $p^{(j)}$ must not be larger than $\pi/2$ in order that (4.2) holds. That means that the angle between $-p^{(k+1)}$ and q must not be larger than $\frac{\pi}{2}-v$. But if this condition is satisfied then $q\cdot p^{(j)}\geqq 0$ for all $j\neq k+1$ and

$$q\cdot p=\sum_{j=1}^{k}\alpha_j p^{(j)}\geqq 0$$

for $\alpha_j\geqq 0$. Hence all directions q which form at most the angle $\frac{\pi}{2}-v$ with $-p^{(k+1)}$ satisfy 4.2. We observe that

$$\sin\left(\frac{\pi}{2}-v\right)=\frac{1}{k}. \tag{4.3}$$

We can now apply Theorem 3.3 with $\delta(k)=k$ to B_j and get

$$|F^{*n}(\sqrt{n}\,B_j)-\Phi(B_j)\leqq C_0'\,k^{\frac{5}{2}}\,d^{-1}\,S^{\frac{3}{2}}\,\beta_3\,n^{-\frac{1}{2}}.$$

The common boundary between any to sets B_j, B_v can be included in one but only one of these sets so that all B_j are disjoint. Observing that then

$$|F^{*n}(\sqrt{n}\,A)-\Phi(A)|\leqq\sum_{j=1}^{k+1}|F^{*n}(\sqrt{n}\,B_j)-\Phi(B_j)|$$

we get the desired inequality with

$$C_0=C_0'\cdot\frac{k+1}{k}\leqq 2\,C_0'.$$

5. Sums of Dependent Random Variables

Consider normed sums

$$\eta_{n,k}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_j^{(i)}$$

of random variables such that the random vectors

$$\xi^{(i)}=(\xi_1^{(i)},\dots,\xi_k^{(i)})$$

are independent for $i=1,2,\dots$ and have the same distribution function F. Further suppose that $\xi^{(i)}$ has zero mean value vector and non-singular covarianz matrix M. Let Φ be the normal distribution function with zero mean value vector and covarianz matrix and let the eigenvalues λ_j satisfy the relation

$$\inf_{i,j} \frac{\lambda_i}{\lambda_j} = d^2 > 0.$$

Putting

$$A = \left\{ v : \sum_{j=1}^{k} v_j \leqq a \right\}$$

we get

$$P[\eta_{n,k} \leqq a] = F^{*n}(\sqrt{n}\, A).$$

Applying Theorem 3.3 and observing that we can put $\varphi = \pi/2$ here we get

$$|P(\eta_{n,k} \leqq a) - \Phi(\eta_{n,k} \leqq a) \leqq C_0\, k^{\frac{3}{2}}\, \beta_3\, S^{\frac{3}{2}}\, d^{-1}\, n^{-\frac{1}{2}}$$

provided that $E[\xi_j^{(i)}] = \beta_{3j} < +\infty$ (β_3 and S are defined as above).

Remark. Just as this paper was to be published I got an offprint of a paper by Sazonov. On the speed of convergence in the multidimensional central limit theorem, Sankhya, Ser. A, Vol. 30, Part 1968, p. 181 – 204. There he studies convex Borelsets and then get essentially the same results as I have given in Section 4 though the remainder term is presented in forms that differ from mine. Sazonov partly uses my method [4] but gives a direct estimation of the Gaussian measure of the homogenous shell of a convex body.

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Professor Dr. H. Bergström
Department of Mathematics
Chalmers Institute of Technology
and the University of Göteborg
Sven Hultins gata
Göteborg, Sweden

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