Z. Wahrscheinlichkeitstheorie verw. Geb. 14, 93–95 (1969)

A Note on σ-Finite Invariant Measures

LESLIE K. ARNOLD

Summary. We consider a one-to-one, bi-measurable, non-singular transformation ϕ of a finite measure space onto itself. We obtain two conditions which are equivalent to the existence of a σ-finite measure μ which is invariant with respect to ϕ and equivalent to the given measure m. The first is a generalization of a condition used by Ornstein in his construction of a transformation for which there does not exist any measure μ as above. The second condition asserts that the entire space is the union of a countable collection $\{F\}$ of subsets, each of which has the following property: if we countably decompose F in such a way that each set in the decomposition of F has an image (under some power of ϕ) which is also a subset of F, then the sum of the m-measures of the images is finite (even though the images need not be disjoint).

In this note we obtain two conditions which are equivalent to the existence of a σ-finite, invariant, measure μ which is equivalent to a given measure m. The first is related to previous results of Ornstein [1] and Kakutani and Hajian [2]. The second is an extension of the author's results in [3].

Throughout the paper we let $(X, \mathscr{B}, m)$ be a finite measure space. Any subset of X which we mention is assumed to be measurable. We let ϕ be a one-to-one, bi-measurable, non-singular transformation of X onto X. Thus, B is measurable implies $\phi(B)$ and $\phi^{-1}(B)$ are measurable, and $m(B)=0$ implies $m(\phi(B))= m(\phi^{-1}(B))=0$. The measures $m\phi^p$ are defined on $\mathscr{B}$ by $m\phi^p(B)=m(\phi^p(B))$. Any other measure μ defined on $\mathscr{B}$ is equivalent to m, written $\mu\equiv m$, if μ and m have precisely the same sets of 0 measure. We are interested in the existence of measures μ defined on $\mathscr{B}$ which are σ-finite, invariant (i.e., $\mu\phi(B)=\mu(B)$ for all B), and equivalent to m.

We shall say a set B is a copy of a set A if A and B are equivalent by countable decomposition, i.e., there are countable decompositions $\{A_i\}$ and $\{B_i\}$ of A and B respectively, and integers $p(i)$ such that $\phi^{p(i)}(A_i)=B_i$ for $i=1,2,\ldots$.

In [1] Ornstein constructed a transformation T on (almost all of) the unit interval which satisfies the conditions we have just assumed for ϕ, but which has no σ-finite invariant measure μ equivalent to Lebesgue measure. He proved that T has this property by showing that T has the following property P: for any integer N and any set S of Lebesgue measure greater than 9/10, there is a set $M\subset S$ of Lebesgue measure 1/8 such that there are N disjoint copies of M in S.

We generalize property P slightly and state it for our transformation ϕ.

Property GP. *There exist $\varepsilon>0$ and $\delta>0$ such that if S is any set with $m(X-S)<\varepsilon$, and N is any positive integer, then there is a set $M\subset S$ with $m(M)>\delta$ such that M has N pairwise disjoint copies contained in S.*

As Ornstein pointed out, it is easy to see that if ϕ has property GP, then it cannot have a σ-finite, invariant measure $\mu\equiv m$. For if μ is such a measure, then there is a set S with $\mu(S)<\infty$ and $m(X-S)<\varepsilon$. Property GP implies S contains sets E_i, $i=1,2,\ldots$, such that $m(E_i)>\delta$, but $\mu(E_i)\to 0$ as $i\to\infty$. This is impossible. However, we also have the following theorem.

Theorem 1. *If ϕ does not have property* GP, *then it does have a σ-finite, invariant measure $\mu\equiv m$.*

Proof. Since ϕ does not have property GP, there is a sequence $\{E_i\}$ of sets whose union is X with the following property: for any $\delta>0$, there are positive integers N_i such that if $B\subset E_i$ and B has more than N_i pairwise disjoint copies in E_i, then $m(B)<\delta$. This implies for each i that no $B\subset E_i$ with $m(B)>0$ can have infinitely many pairwise disjoint copies all contained in E_i.

Let E be any one of the E_i. We assert E is bounded, i.e., there is no $A\subset E$ such that $m(A)<m(E)$ and A is a copy of E. Suppose such an A exists. In the proof of the lemma on p. 87 of [3] it was shown that $B=E-A$ has infinitely many pairwise disjoint copies in E. Briefly, if

$$A=\bigcup_{j=1}^{\infty}\phi^{n(j)}(G_j)$$

where $\{G_j\}$ is a decomposition of E, $\{n(j)\}$ is a sequence of integers, and the $\phi^{n(j)}(G_j)$ are pairwise disjoint, we define $\tau\colon E\to A$ by $\tau(x)=\phi^{n(j)}(x)$ for x in G_j. The copies of B are the sets $\tau^k(B)$ for $k=1,2,\ldots$. But by our choice of the E_i, $m(B)=0$. So $m(A)=m(E)$.

We have shown X is the countable union of bounded sets. By a theorem of Halmos [4], a σ-finite invariant measure $\mu\equiv m$ exists. Q.E.D.

We note that we have also shown that ϕ has a σ-finite, invariant measure $\mu\equiv m$ if and only if X is the union of a countable sequence $\{E_i\}$ of sets such that for each i, if $B\subset E_i$ and B has infinitely many pairwise disjoint copies contained in E_i, then $m(B)=0$. This is, in some sense, an analog of Kakutani and Hajian's theorem on weakly wandering sets [2; Theorem 1].

Our second result is stated in the following theorem. Like conditions (b) and (c) of [3; Theorem 1], the condition of this theorem restricts the ability of ϕ to map sets of small measure into sets of large measure.

Theorem 2. *There exists a σ-finite invariant measure $\mu\equiv m$ if and only if X is the countable union of a sequence of sets $\{F_i\}$ each of which satisfies the following condition: if $\{A_j\}$ is a countable decomposition of F_i and if $\{n(j)\}$ is a sequence of integers such that $\phi^{n(j)}(A_j)\subset F_i$, then*

$$\sum_{j=0}^{\infty} m\,\phi^{n(j)}(A_j)<\infty.$$

Proof. (*Necessity.*) Suppose μ exists. In [3; Theorem 1] we showed this implies the existence of a countable sequence $\{F_i\mid -\infty<i<\infty\}$ of sets such that $A\subset F_i$ and $\phi^p(A)\subset F_i$ (for any integer p) implies $m\,\phi^p(A)<2m(A)$. The sets F_i are given by

$$F_i=\{x\mid 2^i\leqq f(x)<2^{i+1}\}$$

where f is a Radon-Nikodym derivative of μ with respect to m. Clearly these F_i satisfy the condition of the present theorem.

(*Sufficiency*.) Suppose X is the countable union of sets F_i satisfying the condition of the theorem. We assert each F_i is bounded, i.e., there is no $A \subset F_i$ with $m(A) < m(F_i)$ such that A is a copy of F_i. Suppose there is such an A. Then, as in the preceding theorem, the set $B = F_i - A$ has infinitely many pairwise disjoint copies B_j for $j = 1, 2, \ldots$ all contained in F_i. This means that for each j there is a decomposition $\{B_{j,k} | k = 1, 2, \ldots\}$ of B_j and integers $\{p(j,k): k = 1, 2, \ldots\}$ such that

$$B = \bigcup_{k=1}^{\infty} \phi^{p(j,k)}(B_{j,k})$$

and the sets in this union are pairwise disjoint, for $k = 1, 2, \ldots$. Put

$$D = F_i - \bigcup_{j=1}^{\infty} B_j.$$

Then $\{D\} \cup \{B_{j,k} | k = 1, 2, \ldots, j = 1, 2, \ldots\}$ is a decomposition of F_i, and

$$m(D) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m\, \phi^{p(j,k)}(B_{j,k}) = m(D) + \sum_{j=1}^{\infty} m(B) = +\infty.$$

This contradicts the assumption about F_i, hence F_i must be bounded. By the theorem of Halmos [4] mentioned above, there is a σ-finite, invariant measure $\mu = m$. This completes the proof.

References

1. Ornstein, D. S.: On invariant measures. Bull. Amer. Math. Soc. **66**, 297–300 (1960).
2. Kakutani, S., Hajian, A.: Weakly wandering sets and invariant measures. Trans. Amer. Math. Soc. **110**, 136–151 (1964).
3. Arnold, L.K.: On σ-finite invariant measures. Z. Wahrscheinlichkeitstheorie verw. Geb. **9**, 85–97 (1968).
4. Halmos, P.: Invariant measures. Ann. of Math., II. Ser. **48**, 735–754 (1947).

Dr. L.K. Arnold
Daniel H. Wagner, Associates
Station Square One
Paoli, Pennsylvania 19301, USA

(Received October 3, 1968)