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Compositions, Inverses and Thinnings of Random Measures *

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Compositions and inverses of measures on the real line are defined as measures whose cumulative distribution functions (c.d.f.'s) are compositions and inverses, respectively, of the c.d.f.'s of the measures involved. We study the continuity of the composition and inverse operators on measures. We then show how a large class of thinnings of point processes and random measures can be characterized by compositions of random measures. We present several convergence theorems for such compositions. These contain, as special cases, the classical thinning theorem of Renyi and many of its contemporary extensions.

Key Words: Random measures – Point processes – Thinning – Rarefaction – Composition and inverses of measures – Vague convergence of measures.

1. Introduction

The first result on the convergence of thinned point processes appeared in Renyi (1956). It is as follows. Consider a renewal process whose distances between points have a finite expectation α. Independently retain each point of the process with probability p and delete it with probability $1-p$. Change the time scale of the process of retained points so that p^{-1} is the new time unit. Then as $p \to 0$ the resulting thinned process converges in distribution to a Poisson process with intensity α^{-1}.

During the last two decades, a number of such convergence theorems have been proved for the thinning of (i) renewal processes under more general thinnings [2, 7, 14, 17, 18, 20, 21, 25] and [26]; (ii) point processes that obey a law of large numbers [2, 6, 13, 15] and [19]; and (iii) point processes on topological spaces [11, 12] and [16]. These thinning studies have focused on (or are highly dependent

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on) the "locations of points" of the processes. Consequently, the various thinned processes appear to be very different mathematically. However, by describing the thinned process by the "numbers of points in regions" rather than by their locations, one can see (Sect. 4) that all of these thinned processes on the real line are compositions of the form $\xi = \eta \circ \zeta$, where ζ is the "initial process" and η is the "thinning process". The thinned processes in [11, 12] and [16] on topological spaces are equal in distribution to similar compositions (Sect. 5). Furthermore, from these descriptions of thinning it is now clear how such thinnings can be defined for random measures (Sect. 4).

The thinning theorems mentioned above are basically convergence theorems for various compositions $\xi_n = \eta_n \circ \zeta_n$ where η_n and ζ_n are convergent sequences of random measures. Our main result, Theorem 3.2, describes the convergence of such compositions in a general setting. Its proof is based on the continuity of the composition operator on measures (Theorem 2.3), and the continuity of the operator of taking inverses of measures (Theorem 2.1). It yields many results for thinnings (Sect. 4). Among other things our results address the question: if the thinned measure ξ_n is to converge to a desired measure ξ, then how must the thinning measure η_n behave? We discuss thinnings further in the last Section 5.

2. Inverses and Compositions of Measures

In this section we describe the continuity of the inverse and composition operators on measures. We first introduce some notation.

Let $(R, \mathscr{B}(R))$ denote the real numbers and their Borel sets. We denote by $\mathscr{M}$ the set of measures on $\mathscr{B}(R)$ that are finite on compact sets. We endow $\mathscr{M}$ with the vague topology [1] and [8]: the coarsest topology on $\mathscr{M}$ that makes the mappings $\mu \to \int f(x)\, d\mu(x)$, for $f \in \mathscr{C}_c$, continuous. Here $\mathscr{C}_c$ denotes the set of continuous functions on R that have compact support. For μ_n and μ in $\mathscr{M}$ the following statements are equivalent: (i) $\mu_n \to \mu$, (ii) $\int f(x)\, d\mu_n(x) \to \int f(x)\, d\mu(x)$ for each $f \in \mathscr{C}_c$, and (iii) $\mu_n A \to \mu A$ for each bounded A in $\mathscr{B}(R)$ with $\mu \partial A = 0$ (where ∂A denotes the boundary of A). The Borel σ-field $\mathscr{B}(\mathscr{M})$ generated by the vague topology is the same as the smallest σ-field that makes the mappings $\mu \to \mu A$, for $A \in \mathscr{B}(R)$, measurable.

The cumulative distribution function (c.d.f.) centered at zero of a $\mu \in \mathscr{M}$ is defined by

$$\begin{aligned} \mu(x) &= -\mu(x, 0] \quad \text{for } x \leqq 0 \\ &= \mu(0, x] \quad \text{for } x > 0. \end{aligned}$$

The $\mu(x)$ is right-continuous, nondecreasing and $\mu(0) = 0$. There is a one-to-one correspondence between such functions and $\mathscr{M}$ that is based on $\mu(a, b] = \mu(b) - \mu(a)$ for $a < b$ in R. (Replacing 0 by $a \in R$ in the preceding yields the c.d.f. centered at a of μ.) Note that if $\mu_n(x) \to \mu(x)$ for each $x \in C_\mu$, then $\mu_n \to \mu$. The converse holds if $0 \in C_\mu$. Herein $C_\mu = \{x \in R : \mu\{x\} = 0\}$, and $D_\mu = R \setminus C_\mu$ are the continuity and the discontinuity sets, respectively, of μ.

We now discuss inverse measures. Here and throughout this article we let m denote the Lebesgue measure on R. Let μ be in $\mathscr{M}$. The image of m under the mapping $\mu(x)$ is the measure

$$m\mu^{-1}A = m\{x\in R: \mu(x)\in A\} \quad \text{for } A\in\mathscr{B}(R).$$

We shall denote $m\mu^{-1}$ simply by μ^{-1}, and shall call it the inverse of μ associated with its c.d.f. centered at 0. (One can define an analogous inverse of μ associated with a c.d.f. of μ centered at any $a\in R$. This will generally be different from μ^{-1}.)

Note that μ^{-1} may be infinite on some compact sets and hence not in $\mathscr{M}$. In particular, if $\mu(\infty)=\lim_{x\to\infty}\mu(x)<\infty$, then

$$\mu^{-1}(0,\mu(\infty)]=\infty \quad \text{and} \quad \mu^{-1}(\mu(\infty),\infty)=0.$$

A similar comment applies to $\mu(-\infty)$. It is easily seen that $\mu^{-1}\in\mathscr{M}$ if and only if $\mu\in\mathscr{M}_\infty=\{\mu\in\mathscr{M}: \mu(\pm\infty)=\pm\infty\}$. In this case, μ^{-1} is also in $\mathscr{M}_\infty$. Clearly for $\mu\in\mathscr{M}_\infty$, $x\in R$, and $a<b$ in R,

$$\mu^{-1}(x)=\tilde{\mu}(x)-\tilde{\mu}(0) \quad \text{and} \quad \mu^{-1}(a,b]=\tilde{\mu}(b)-\tilde{\mu}(a),$$

where $\tilde{\mu}(x)=\inf\{t\in R: \mu(t)>x\}$ is the right-continuous inverse of $\mu(x)$. For simplicity of exposition, we shall frequently confine our discussion to measures in $\mathscr{M}_\infty$.

Observe that different measures may have the same inverse. Also, using the well-known relation $\mu(x)=\inf\{t\in R: \tilde{\mu}(t)>x\}$, we can write

$$(\mu^{-1})^{-1}(x)=\mu(x+\tilde{\mu}(0))-\mu(\tilde{\mu}(0)) \quad \text{for } x\in R.$$

That is, $(\mu^{-1})^{-1}A=\mu\{A+\tilde{\mu}(0)\}$. Consequently, $(\mu^{-1})^{-1}=\mu$ if and only if μ is invariant under the translation $\tilde{\mu}(0)$. Note that $\tilde{\mu}(0)=0$ if and only if $\mu(x)>0$ for each $x>0$.

We now consider the mapping $\mu\to\mu^{-1}$ from $\mathscr{M}_\infty$ to $\mathscr{M}_\infty$, where $\mathscr{M}_\infty$ is endowed with the relativized vague topology. (Note that

$$\mathscr{M}_\infty=\bigcap_n\{\mu\in\mathscr{M}: \mu(0,\infty)\geqq n, \mu(-\infty,0)\geqq n\}\in\mathscr{B}(\mathscr{M}).)$$

(2.1) **Theorem.** *The inverse mapping $\mu\to\mu^{-1}$ from $\mathscr{M}_\infty$ to $\mathscr{M}_\infty$ is Borel measurable, and it is continuous at those μ for which $\mu\{0\}=0$.*

Proof. For each a and x in R,

$$\{\mu\in\mathscr{M}_\infty: \tilde{\mu}(x)<a\}=\{\mu\in\mathscr{M}_\infty: \mu(a-)>x\}\in\mathscr{M}_\infty\cap\mathscr{B}(\mathscr{M})=\mathscr{B}(\mathscr{M}_\infty).$$

Thus $\mu\to\tilde{\mu}(x)$ from $\mathscr{M}_\infty$ to R is measurable for each x. It follows that $\mu\to\mu^{-1}(x)$ is measurable for each x, and this implies that $\mu\to\mu^{-1}$ is measurable.

Now suppose $\mu_n\to\mu$ in $\mathscr{M}_\infty$ and $\mu\{0\}=0$. Fix $x\in C_{\tilde{\mu}}$ and $\varepsilon>0$. Pick $a,b\in C_\mu$ such that

$$\tilde{\mu}(x)-\varepsilon<a<\tilde{\mu}(x)<b<\tilde{\mu}(x)+\varepsilon.$$

Pick α and β such that

$$\mu(a)<\alpha<x<\beta<\mu(b).$$

This is possible since $\tilde{\mu}(x)$ is a point of increase of $\mu(\cdot)$. Pick N such that for each $n\geqq N$,

$$|\mu_n(a)-\mu(a)|<\alpha-\mu(a) \quad \text{and} \quad |\mu_n(b)-\mu(b)|<\mu(b)-\beta.$$

This is possible since $\mu_n(t) \to \mu(t)$ when 0 and t are in C_μ. The monotonicity of $\mu_n(\cdot)$ and the above inequalities yield

$$\mu_n(\tilde{\mu}(x)-\varepsilon) < \mu_n(a) < \alpha$$
$$< x < \beta < \mu_n(b) < \mu_n(\tilde{\mu}(x)+\varepsilon).$$

Since $\mu_n(t) < x$ implies $t < \tilde{\mu}_n(x)$, and $\mu_n(t) > x$ implies $t \geqq \tilde{\mu}_n(x)$, then from the preceding inequalities,

$$\tilde{\mu}(x) - \varepsilon < \tilde{\mu}_n(x) \leqq \tilde{\mu}(x) + \varepsilon.$$

Thus $\tilde{\mu}_n(x) \to \tilde{\mu}(x)$ for each $x \in C_{\tilde{\mu}}$. From this we have

$$\mu_n^{-1}(a, b] = \tilde{\mu}_n(b) - \tilde{\mu}_n(a) \to \tilde{\mu}(b) - \tilde{\mu}(a) = \mu^{-1}(a, b]$$

for each $a < b$ in $C_{\tilde{\mu}} = C_{\mu^{-1}}$. Thus $\mu_n^{-1} \to \mu^{-1}$ in $\mathcal{M}_\infty$ and this completes the proof.

The map $\mu \to \mu^{-1}$ may not be continuous at μ if $\mu\{0\} > 0$. To see this note that:

$$\mu_n = m + \delta_{a_n} \to \mu = m + \delta_0, \quad \text{as} \quad a_n \downarrow 0,$$

where δ_a denotes the Dirac measure with unit mass at a (recall that m is the Lebesgue measure). However,

$$\mu_n^{-1} A = m\{A \setminus [a_n, 1+a_n]\} \to m\{A \setminus [0, 1]\} \neq \mu^{-1} A = m\{A \setminus [-1, 0]\}.$$

We now discuss compositions of measures. We define the composition of two measures λ and μ in $\mathcal{M}$ to be the measure $\lambda \circ \mu$ whose c.d.f. is

$$\lambda \circ \mu(x) = \lambda(\mu(x)) \quad \text{for } x \in R.$$

Clearly $\lambda \circ \mu \in \mathcal{M}$ and

$$\lambda \circ \mu(a, b] = \lambda(\mu(a), \mu(b)] \quad \text{for } a < b \text{ in } R.$$

As one would anticipate, many of the properties of compositions of functions carry over to compositions of measures. Some of these are as follows:

(i) $m \circ \mu = \mu = \mu \circ m$.
(ii) $(\lambda \circ \mu)^{-1}(a, b] = \mu^{-1}(\tilde{\lambda}(a), \tilde{\lambda}(b)]$ for $a < b$ in R.
(iii) $\mu^{-1} \circ \mu = m$ if and only if $\mu(x)$ is strictly increasing.
(iv) If $\tilde{\mu}(0) = 0$, then $\mu \circ \mu^{-1} = m$ if and only if $\mu(x)$ is continuous.
(v) $(\lambda \circ \mu)^{-1} = \mu^{-1} \circ \lambda^{-1}$ if and only if μ^{-1} is invariant under the translation $\tilde{\lambda}(0)$.
(vi) Let $\nu = \sum_n y_n \delta_{s_n}$ be an atomic measure in $\mathcal{M}_\infty$ where $\ldots, s_{-1} \leqq 0 < s_0 \leqq s_1, \ldots$; and let $\mu = \sum_n \delta_{s_n}$ and $\lambda = \sum_n y_n \delta_{n+1}$. Then $\nu = \lambda \circ \mu$, and $\nu^{-1} = \sum x_{n+1} \delta_{t_n}$, where $x_n = s_n - s_{n-1}$ and $t_n = \lambda(n+1)$.

The continuity of the composition mapping $(\lambda, \mu) \to \lambda \circ \mu$ from $\mathcal{M} \times \mathcal{M}$ (with the product topology) to $\mathcal{M}$ is described in Theorem 2.3 below. For this we need the following result.

(2.2) **Lemma.** *If λ and μ in $\mathcal{M}$ satisfy $\lambda D_{\tilde{\mu}} = 0$, then*

$$\lambda[\mu(x-), \mu(x)] = 0 \qquad \textit{for each } x \in C_{\lambda \circ \mu}.$$

Proof. For each $x \in C_{\lambda \circ \mu}$,

$$\lambda \circ \mu(x) = \lambda \circ \mu(x-) = \lim_{t \uparrow x} \lambda(\mu(t))$$

$$= \begin{cases} \lambda(\mu(x-)-) & \text{if } \mu(t) < \mu(x-) \text{ for each } t < x \\ \lambda(\mu(x-)) & \text{otherwise.} \end{cases}$$

In the latter case $\mu(x-) \in D_{\tilde{\mu}}$, and since $\lambda D_{\tilde{\mu}} = 0$ then $\lambda(\mu(x-)) = \lambda(\mu(x-)-)$. Thus the assertion follows.

(2.3) **Theorem.** *The composition mapping $(\lambda, \mu) \to \lambda \circ \mu$ from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{M}$ is Borel measurable, and it is continuous at those (λ, μ) for which $\lambda D_{\tilde{\mu}} = 0$ and $\mu\{0\} = 0$.*

Proof. The measurability of the composition mapping follows by an argument as in [3, p. 232].

Now suppose $\lambda_n \to \lambda$ and $\mu_n \to \mu$ in $\mathcal{M}$, where $\lambda D_{\tilde{\mu}} = 0$ and $\mu\{0\} = 0$. Fix $\varepsilon > 0$ and $x < y$ in $C_{\lambda \circ \mu}$. Pick, a, b, c, d in $C_{\lambda \circ \mu}$ that satisfy

$$a < \mu(x-) \leqq \mu(x) < b < c < \mu(y-) \leqq \mu(y) < d,$$

and are such that each of the sets

$$(a, \mu(x)], \ (\mu(x), b], \ (c, \mu(y)], \ (\mu(y), d]$$

has λ-measure less than ε. This is possible because of Lemma 2.2. Pick an integer N such that for each $n \geqq N$

$$a < \mu_n(x) < b, \qquad c < \mu_n(y) < d,$$
$$|\lambda_n(a, d] - \lambda(a, d]| < \varepsilon, \quad \text{and} \quad |\lambda_n(b, c] - \lambda(b, c]| < \varepsilon.$$

Then for each $n \geqq N$,

$$\begin{aligned} |\lambda_n \circ \mu_n(x, y] - \lambda \circ \mu(x, y]| &= |\lambda_n(\mu_n(x), \mu_n(y)] - \lambda(\mu(x), \mu(y)]| \\ &\leqq |\lambda_n(a, d] - \lambda(\mu(x), \mu(y)]| + |\lambda_n(b, c] - \lambda(\mu(x), \mu(y)]| \\ &\leqq |\lambda_n(a, d] - \lambda(a, d]| + \lambda(a, \mu(x)] + \lambda(\mu(y), d] \\ &\quad + |\lambda_n(b, c] - \lambda(b, c]| + \lambda(\mu(x), b] + \lambda(c, \mu(y)] < 6\varepsilon. \end{aligned}$$

Thus $\lambda_n \circ \mu_n \to \lambda \circ \mu$, and this completes the proof.

The composition mapping may not be continuous at (λ, μ) with $\mu\{0\} > 0$. To see this let $\mu_n = m + \delta_{a_n}$, where $a_n \downarrow 0$, and let $\mu = m + \delta_0$ and $\lambda_n = \mu^{-1}$. Clearly $\lambda_n \to \mu^{-1}$, $\mu_n \to \mu$ and $D_{\tilde{\mu}} = \phi$. But since $\lambda_n \circ \mu_n(x) = x + 1$ for $a_n \leqq x$, then $\lambda_n \circ \mu_n$ does not converge to $\mu^{-1} \circ \mu = m$. The condition $\mu\{0\}$ is needed here (and in Theorem 2.1) because of the convention of using c.d.f.'s centered at zero.

The assumption $\lambda D_{\tilde{\mu}} = 0$ is also needed in general for continuity of the composition mapping at (λ, μ). Indeed if $\lambda_n = \lambda = m + \delta_1$, and $\mu_n = (m + a\,\delta_{b_n})^{-1}$ where $b_n \downarrow 1$, then $\mu_n \to \mu = (m + a\,\delta_1)^{-1}$, $\lambda_n \to \lambda$ and $\lambda D_{\tilde{\mu}} = \lambda\{1\} = 1$. But

$$\lambda_n \circ \mu_n = \mu_n + \delta_{1+a} \to \mu + \delta_{1+a} \neq \lambda \circ \mu = \mu + \delta_1.$$

Note that $D_{\tilde{\mu}} = D_{\mu^{-1}}$ and that

$$\lambda D_{\tilde{\mu}} = 0 \Leftrightarrow D_\lambda \cap D_{\mu^{-1}} = \phi \Leftrightarrow \mu^{-1} D_\lambda = 0.$$

This follows since D_μ is a countable set for each $\mu \in \mathscr{M}$. Also note that $\lambda D_{\bar\mu}=0$ when $\mu(x)$ is strictly increasing or $\lambda(x)$ is continuous. This follows since for any $\mu \in \mathscr{M}$, $D_{\bar\mu}=\phi \Leftrightarrow \mu(x)$ is strictly increasing $\Leftrightarrow \mu^{-1}(x)$ is continuous.

(2.4) **Corollary.** *Suppose $\mu_n \to \mu$ in $\mathscr{M}$, and $a_n \leqq b_n$ in R are such that $a_n \to a$ and $b_n \to b$, where a and b are in C_μ. Then $\mu_n(a_n, b_n] \to \mu(a, b]$ and $\mu_n\{a_n\} \to 0$.*

Proof. Fix $c<d$ in R and pick $\nu_n \in \mathscr{M}$ such that $\nu_n(c)=a_n$ and $\nu_n(d)=b_n$, and $\nu_n \to \nu$ in $\mathscr{M}$ where $D_\nu=\phi$. Clearly c and d are in $C_{\mu\circ\nu}$. Then by Theorem 2.2, $\mu_n(a_n, b_n]= \mu_n \circ \nu_n(c, d] \to \mu \circ \nu(c, d]=\mu(a, b]$. To prove $\mu_n\{a_n\} \to 0$, pick δ and $\varepsilon>0$ such that $\mu(a-\delta, a+\delta)<\varepsilon$. Pick $c<d$ in C_μ such that $a-\delta<c<d<a+\delta$. Then for sufficiently large n

$$\mu_n\{a_n\} \leqq \mu_n(c, d) \to \mu(c, d)<\varepsilon,$$

and so $\mu_n\{a_n\} \to 0$.

3. Convergence of Inverses and Compositions of Random Measures

A random measure ξ on R is a measurable mapping from a probability space into $(\mathscr{M}, \mathscr{B}(\mathscr{M}))$. If each of the random variables ξA, for $A \in \mathscr{B}(R)$, is integer-valued then ξ is a point process. A sequence ξ_n of random measures converges in distribution to a random measure ξ, written $\xi_n \xrightarrow{d} \xi$, if the distribution of ξ_n converges weakly to the distribution of ξ, i.e. $Eh(\xi_n) \to Eh(\xi)$ for each bounded continuous function h on $\mathscr{M}$. The following are equivalent statements:

(i) $\xi_n \xrightarrow{d} \xi$,

(ii) $\int f(x)\, d\xi_n(x) \xrightarrow{d} \int f(x)\, d\xi(x)$ for each $f \in \mathscr{C}_c$, and

(iii) $(\xi_n A_1, \ldots, \xi_n A_k) \xrightarrow{d} (\xi A_1, \ldots, \xi A_k)$ for all $A_1, \ldots, A_k$ in $\mathscr{B}(R)$ satisfying $\xi\, \partial A_1=\cdots=\xi\, \partial A_k=0$ a.s. These and other basic properties of random measures are discussed in [8] and [12].

Throughout the remainder of this paper we let ξ, η and ζ (with or without subscripts) denote random measures on R. For simplicity we assume that they take values in $\mathscr{M}_\infty$. This means, for example, that $\xi(\pm\infty)=\pm\infty$ a.s. Our first result concerns inverses.

(3.1) **Theorem.** *If $\xi_n \xrightarrow{d} \xi$ and $\xi\{0\}=0$ a.s., then $\xi_n^{-1} \xrightarrow{d} \xi^{-1}$. If $\xi_n^{-1} \xrightarrow{d} \xi^{-1}$ and $\xi^{-1}\{0\}=0$ a.s., then $\xi_n \xrightarrow{d} \xi$.*

Proof. The first statement follows from Theorem 2.1 and the continuous mapping theorem [3, Sect. 5]. Under the hypothesis of the second statement, it follows from Corollary 2.4 and the first statement that $((\xi_n^{-1})^{-1}, \bar\xi_n(0)) \xrightarrow{d} ((\xi^{-1})^{-1}, 0)$. Using this, along with $\xi_n A=(\xi_n^{-1})^{-1}\{A-\bar\xi_n(0)\}$ and Corollary 2.4, it follows that $\xi_n \xrightarrow{d} \xi$.

For the next result we assume that $\xi_n=\eta_n \circ \zeta_n$.

(3.2) **Theorem.** (i) *If $(\eta_n, \zeta_n) \xrightarrow{d} (\eta, \zeta)$ where $\eta D_{\bar\zeta}=0$ a.s. and $\zeta\{0\}=0$ a.s., then $\xi_n \xrightarrow{d} \eta \circ \zeta$.*

(ii) *If $(\xi_n, \zeta_n) \xrightarrow{d} (\xi, \zeta)$ where $\xi D_\zeta=0$ a.s. and $\zeta\{0\}=\zeta^{-1}\{0\}=0$ a.s., then $\eta_n \xrightarrow{d} \xi \circ \zeta^{-1}$.*

(iii) *If $(\xi_n, \eta_n) \xrightarrow{d} (\xi, \eta)$ where $\eta^{-1} D_{\bar\xi}=0$ a.s. and $\xi\{0\}=\eta\{0\}=0$ a.s., then $\zeta_n \xrightarrow{d} \eta^{-1} \circ \xi$.*

Proof. Part (i) follows by Theorem 2.2 and the continuous mapping theorem. To prove (ii) we use the representation

$$\eta_n=\xi_n\circ\zeta_n^{-1}+(\eta_n-\xi_n\circ\zeta_n^{-1}). \tag{1}$$

By Theorem 2.1 and the continuous mapping theorem we have $(\xi_n,\zeta_n^{-1})\xrightarrow{d}(\xi,\zeta^{-1})$, and so by part (i)

$$\xi_n\circ\zeta_n^{-1}\xrightarrow{d}\xi\circ\zeta^{-1}. \tag{2}$$

Let $r_n=\eta_n(a,b]-\xi_n\circ\zeta_n^{-1}(a,b]$ for $a<b$ in $C_{\xi\circ\zeta^{-1}}$ a.s. Using the relations

$$\tilde{\mu}(\mu(x)-)\leqq x\leqq\tilde{\mu}(\mu(x)) \quad \text{for } x\in R \text{ and } \mu\in\mathcal{M}, \tag{3}$$

we have

$$(\zeta_n(\tilde{\zeta}_n(a)),\zeta_n(\tilde{\zeta}_n(b)-)]\subset(a,b]\subset(\zeta_n(\zeta_n(a)-),\zeta_n(\tilde{\zeta}_n(b))].$$

This and Theorems 2.1–2.3 yield

$$r_n\leqq\xi_n[\tilde{\zeta}_n(a),\tilde{\zeta}_n(b)]-\xi_n\circ\zeta_n^{-1}(a,b]\xrightarrow{d}0.$$

Similarly

$$r_n\geqq\xi_n(\tilde{\zeta}_n(a),\tilde{\zeta}_n(b))-\xi_n\circ\zeta_n^{-1}(a,b]\xrightarrow{d}0.$$

Then

$$r_n\xrightarrow{d}0 \quad \text{for } a<b \text{ in } C_{\xi\circ\zeta^{-1}} \text{ a.s.} \tag{4}$$

Now pick $a_1<b_1\leqq\cdots\leqq a_k<b_k$ in $C_{\xi\circ\zeta^{-1}}$ a.s. From (1), (2) and (4) it follows that

$$(\eta_n(a_1,b_1],\ldots,\eta_n(a_k,b_k])\xrightarrow{d}(\xi\circ\zeta^{-1}(a_1,b_1],\ldots,\xi\circ\zeta^{-1}(a_k,b_k]). \tag{5}$$

Thus $\eta_n\xrightarrow{d}\xi\circ\zeta^{-1}$.

To prove (iii) we use the representation

$$\zeta_n=\eta_n^{-1}\circ\xi_n+(\zeta_n-\eta_n^{-1}\circ\xi_n). \tag{6}$$

Similar to (2) we have

$$\eta_n^{-1}\circ\xi_n\xrightarrow{d}\eta^{-1}\circ\xi. \tag{7}$$

Let $r_n=\zeta_n(a_n,b_n]-\eta_n^{-1}\circ\xi_n(a,b]$ for $a<b$ in $C_{\eta^{-1}\circ\xi}$ a.s. Using (3) we have

$$\begin{aligned} r_n&=\zeta_n(b)-\zeta_n(a)-\tilde{\eta}_n(\eta_n(\zeta_n(b)))+\tilde{\eta}_n(\xi_n(a))\\ &\leqq\tilde{\eta}_n(\xi_n(a))-\tilde{\eta}_n(\xi(a)-)\leqq\eta_n^{-1}\circ\xi_n\{a\}\xrightarrow{d}0. \end{aligned}$$

Similarly,

$$r_n\geqq\tilde{\eta}_n(\xi_n(b)-)-\tilde{\eta}_n(\xi_n(b))\geqq-\eta_n^{-1}\circ\xi_n\{b\}\xrightarrow{d}0.$$

Then

$$r_n\xrightarrow{d}0 \quad \text{for } a<b \text{ in } C_{\eta^{-1}\circ\xi}. \tag{8}$$

From (6)–(8) it follows, similar to (5), that $\zeta_n\xrightarrow{d}\eta^{-1}\circ\xi$.

We end this section with some comments on the preceding theorems. First note that these results hold for normalized random measures such as

$$c_n^{-1}\,\xi_n\circ a_n=(c_n^{-1}\,\eta_n\circ b_n)\circ(b_n^{-1}\,\zeta_n\circ a_n)\quad\text{where}\quad \mu\circ a=\mu\circ a\,m\quad\text{for } \mu\in\mathcal{M}$$

and $a\in R$. Such measures appear in the next section. The a_n, b_n and c_n could be constants or random variables that converge jointly with the measures.

Many joint convergence statements follow directly from Theorems 3.1 and 3.2 and the continuous mapping theorem. For example, suppose

$$(\eta_n,\zeta_n)\xrightarrow{d}(\eta,\zeta),\qquad \eta D_{\xi}=0,\quad\text{and}\quad \eta\{0\}=\zeta\{0\}=\eta^{-1}\{0\}=0\quad\text{a.s.}$$

Then

$$(\xi_n,\eta_n,\zeta_n,\xi_n^{-1},\eta_n^{-1},\zeta_n^{-1})\xrightarrow{d}(\eta\circ\zeta,\eta,\zeta,\zeta^{-1}\circ\eta^{-1},\eta^{-1},\zeta^{-1}).$$

The condition $\eta D_{\zeta}=0$ a.s. holds if η and ζ are independent and either η or ζ^{-1} is stochastically continuous. Indeed if η is stochastically continuous (i.e. $\eta\{x\}=0$ a.s. for each $x\in R$) then $\eta A=0$ a.s. for any countable set $A\in\mathcal{B}(R)$, and so

$$P(\eta D_{\zeta}=0)=E(P(\eta D_{\zeta}=0\,|\,\zeta))=1.$$

Similarly, if ζ^{-1} is stochastically continuous, then $\zeta^{-1}D_{\eta}=0$ a.s., and so $\eta D_{\zeta}=0$ a.s.

Finally, note that Theorem 3.2 describes the convergence of a subclass of semi-stationary random measures which are of the form $\xi_n=\eta_n\circ\zeta_n$, where (η_n,ζ_n^{-1}) are jointly (strict-sense) stationary random measures and $\zeta_n(x)$ is continuous and strictly increasing. These are the analogues of semi-stationary processes of the form $X(t)=Y(\tau_t)$ as in Corollary 2.3 of [24].

4. Convergence of Thinnings of Random Measures

In this section we discuss how certain thinnings of random measures can be characterized by compositions of random measures, and then we present some corollaries to Theorem 3.2 that apply to thinnings. Our focus here is on thinnings on the real line. In the next section we indicate how some of our results carry over to more general spaces.

The thinnings of point processes on R that have been studied so far are describable as follows. Consider a point process ζ on R. For simplicity assume $\zeta(\pm\infty)=\pm\infty$ a.s. We can write $\zeta=\sum_n \delta_{S_n}$, where the random variables $S_n=\zeta(n)$ are the locations of points (unit masses) of ζ. Clearly $\cdots\leqq S_{-1}\leqq 0<S_0\leqq S_1\leqq\cdots$. Let Y_n $(n=0,\pm1,\ldots)$ be random variables on the same probability space as ζ, which take the values 0 or 1. (We make no independence assumptions.) Thin the process ζ according to the rule: delete or retain the unit mass at S_n according as $Y_n=0$ or 1. Then the process of retained points ξ has the three representations (recall property (vi) of compositions)

$$\xi=\sum_n Y_n\,\delta_{S_n}=\eta\circ\zeta=\sum_n \delta_{T_n} \tag{1}$$

where $\eta=\sum Y_n\,\delta_{n+1}$ and $T_n=\tilde{\zeta}(\tilde{\eta}(n))$. (Again for simplicity we assume $\eta(\pm\infty)=\pm\infty$ a.s.) The theorems in the references for such thinnings are basically convergence theorems for various compositions $\xi_n=\eta_n\circ\zeta_n$ which we have abstracted in Theorem 3.2. We shall return to this shortly.

This thinning of point processes also makes sense for random measures. To see this, consider a composition $\xi=\eta\circ\zeta$ of random measures η and ζ on R. This can be viewed as follows. Mass is randomly deposited on R, starting from 0 and proceeding forward such that for each $x>0$ a mass $\zeta(x)$ is placed in $(0,x]$ and this is in turn replaced by the amount $\eta(\zeta(x))$. Similarly, starting from 0 and proceeding backward, for each $x\leqq 0$ a mass $\zeta(x)$ is deposited in $(x,0]$ and this in turn is replaced by $\eta(\zeta(x))$. In other words, if a mass $y=\zeta(a,b]$ is deposited in $(a,b]$, then it is subsequently replaced by $\eta(\zeta(a),\zeta(a)+y]=\eta(\zeta(b)-y,\zeta(b)]$. The resulting mass is thus described by $\xi=\eta\circ\zeta$. One can think of ξ as an η-replacement of ζ. If $\eta(a,b]\leqq b-a$ for each $a<b$, then $\xi A\leqq\zeta A$ a.s. for each Borel set A, and we call the replacement a thinning.

Note that this replacement procedure is ordered and centered about 0 in the sense that a replacement at a given location is a random function of the mass deposited prior to this location starting from 0. Accordingly we shall call this an ordered replacement or thinning. Two important examples are (i) $\xi=\eta\circ\zeta=\sum Y_n\,\delta_{S_n}$ where the Y_n's are independent with a common distribution and are independent of the S_n's, and (ii) $\xi=\eta\circ\zeta$ where η and ζ are independent and $\eta(x)$ has stationary independent increments. In these cases $\eta(\zeta(a),\zeta(a)+y]\stackrel{d}{=}\eta(y)$. That is, the dependency on $\zeta(a)$ disappears. We shall call these independent homogeneous replacements or thinnings. More comments on thinnings follow in the next section.

The rest of this section is devoted to convergence theorems for special compositions that arise in thinnings. As in Section 3, we let ξ, η and ζ (with or without subscripts) denote random elements of $\mathcal{M}_\infty$. Our first result contains the major result in [9] which in turn contains many classical results on thinning. For this we assume that $\xi_n=\eta_n\circ\zeta$, and that a_n and c are in R with $a_n\to\infty$.

(4.1) **Theorem.** *If* $x^{-1}\zeta(x)\xrightarrow{d}c$ *as* $|x|\to\infty$, *then* $\xi_n\circ a_n\xrightarrow{d}\eta\circ c$ *if and only if* $\eta_n\circ a_n\xrightarrow{d}\eta$.

If $(\xi_n\circ a_n,\eta_n\circ a_n)\xrightarrow{d}(\eta,\eta\circ c)$, *where* $\eta(x)$ *is strictly increasing with* $\eta\{0\}=0$ *a.s., and* $a_n/a_{n+1}\to 1$, *then* $x^{-1}\zeta(x)\xrightarrow{d}c$ *as* $|x|\to\infty$.

Proof. Clearly $\xi_n\circ a_n=(\eta_n\circ a_n)\circ(a_n^{-1}\zeta\circ a_n)$. Note that $x^{-1}\zeta(x)\xrightarrow{d}c$ as $|x|\to\infty$ implies $a_n^{-1}\zeta\circ a_n\xrightarrow{d}c\,m$. Thus the first statement follows from Theorem 3.2(i) and (ii).

To prove the second statement, first note that by Theorem 3.2(iii),

$$a_n^{-1}\zeta\circ a_n\xrightarrow{d}\eta^{-1}\circ\eta\circ c=c\,m.$$

In particular, $a_n^{-1}\zeta(a_n)\xrightarrow{d}c$. This and $a_n/a_{n+1}\to 1$ yield $a_n^{-1}(\zeta(a_{n+1})-\zeta(a_n))\xrightarrow{d}0$. Now let $v(x)=\inf\{n\colon a_n>x\}$. Since $a_n/a_{n+1}\to 1$, then

$$x^{-1}a_{v(x)}=a_{v(x)}/(x-a_{v(x)}+a_{v(x)})\to 1.$$

It follows that as $|x|\to\infty$,

$$x^{-1}\zeta(x)=x^{-1}a_{v(x)}\{(\zeta(x)-\zeta(a_{v(x)}))/a_{v(x)}+\zeta(a_{v(x)})/a_{v(x)}\}\xrightarrow{d}c.$$

(4.2) **Example.** (Thinning of a renewal process.) Let $\zeta=\sum_n \delta_{S_n}$ be a renewal process and set $c^{-1}=E(S_n-S_{n-1})$. Let $\eta_n=\sum_k \delta_{T_{nk}}$ be a renewal process in which the T_{nk}'s are integer-valued and assume $a_n=EW_n<\infty$, where $W_n \overset{d}{=} T_{nk}-T_{n,k-1}$. Let $\eta=\sum_n \delta_{T_n}$ be a renewal process with $W \overset{d}{=} T_n-T_{n-1}$. It is well-known that $\eta_n \circ a_n \xrightarrow{d} \eta$ if and only if $a_n^{-1} W_n \xrightarrow{d} W$. Consequently by Theorem 4.1 we have $\xi_n \circ a_n \xrightarrow{d} \eta \circ c$ if and only if $a_n^{-1} W_n \xrightarrow{d} W$. Results like this are discussed in [14, 20, 21, 17, 18, 25] and [26]. In the latter four articles, $\eta_n=\nu_n \circ \cdots \circ \nu_1$, where $\nu_1, \nu_2, \ldots$ are n successive independent copies of a single thinning operation.

Continuing our analysis of $\xi_n=\eta_n \circ \zeta$, we now assume that $\eta_n=\sum Y_{nk}\,\delta_k$ where for each n the Y_{nk} $(k=0, \pm 1, \ldots)$ are independent with a common distribution. Let η be such that $\eta(x)$ has stationary independent increments with

$$E(e^{-s\eta(1)})=\exp-\int_0^\infty (1-e^{-sy})\,d\alpha(y),$$

where α is a (Levy) measure on $(0, \infty)$ satisfying $\int_0^\infty \min\{1, y\}\,d\alpha(y)<\infty$. We write X^ε for the random variable X truncated at $\varepsilon>0$.

(4.3) **Corollary.** *If $x^{-1}\zeta(x) \xrightarrow{d} c$ as $|x|\to\infty$, then $\xi_n \circ a_n \xrightarrow{d} \eta \circ c$ if and only if*

(i) $a_n P(Y_{n1} \leqq y) \to \alpha(y)$ *for each* $y \in C_\alpha$, *and*
(ii) $\lim_{\varepsilon\to 0} \underline{\lim}_{n\to\infty} a_n E(Y_{n1}^\varepsilon)=\lim_{\varepsilon\to 0} \overline{\lim}_{n\to\infty} a_n E(Y_{n1}^\varepsilon)=0$.

Proof. Conditions (i) and (ii) are necessary and sufficient for $\sum_{k=1}^{a_n} Y_{nk} \xrightarrow{d} \eta(1)$; see [12] which is a generalization of [4, p. 564]. The latter convergence is equivalent to $\eta_n \circ a_n \xrightarrow{d} \eta$; see [5, p. 480]. Thus the assertions follow by Theorem 4.1.

(4.4) **Example.** (Rényi's result.) Suppose $\zeta=\sum_n \delta_{S_n}$ is a renewal process with $c^{-1}=E(S_n-S_{n-1})$, and that

$$p_n=P(Y_{nk}=1) \quad \text{and} \quad P(Y_{nk}=0)=1-p_n,$$

where $p_n \to 0$. Then by Corollary 4.3, $\xi_n \circ p_n^{-1} \xrightarrow{d} \eta \circ c$ where η is a Poisson process with unit intensity. Other examples of Corollary 4.3 appear in [9, 11, 12] and [15].

Results similar to the above hold for compositions $\xi_n=\eta \circ \zeta_n$. The analog to Theorem 4.1 is as follows. Here $b_n \to \infty$ in R.

(4.5) **Theorem.** *If $x^{-1}\eta(x) \xrightarrow{d} c$ as $|x|\to\infty$, and $\nu\{0\}=0$ a.s., then $b_n^{-1}\xi_n \circ a_n \xrightarrow{d} c\zeta$ if and only if $b_n^{-1}\zeta_n \circ a_n \xrightarrow{d} \zeta$.*

If $(b_n^{-1}\xi_n \circ a_n, b_n^{-1}\zeta_n \circ a_n) \xrightarrow{d} (c\zeta, \zeta)$, where ζ is strictly increasing with $\zeta\{0\}=\zeta^{-1}\{0\}=0$ a.s., and $b_n/b_{n+1} \to 1$, then $x^{-1}\eta(x) \xrightarrow{d} c$.

For our last result we consider the atomic random measures $\xi_n=\sum_k Y_{nk}\,\delta_{S_{nk}}$ where $\ldots S_{n,-1}<0 \leqq S_{n0} \leqq \ldots$ for each n. Let $Z_{nk}=S_{nk}-S_{n,k-1}$. We assume that the Y_{nk}'s are independent of the Z_{nk}'s, and that $P(Y_{nk} \leqq x)$ and $P(Z_{nk} \leqq x)$ are independent of k for each n. As before $a_n \to \infty$ and $b_n \to \infty$ in R.

(4.6) **Theorem.** *Suppose the* Y_{nk} *satisfy* (i) *and* (ii) *in Corollary* 4.3, *and that the* $b_n^{-1} Z_{nk}$ *also satisfy these conditions with* α *replaced by* β. *Then* $\xi_n \circ b_n \xrightarrow{d} \eta \circ \zeta^{-1}$, *where* η *and* ζ *are independent measures with stationary independent increments with Levy measures* α *and* β, *respectively.*

Proof. Clearly $\xi_n \circ b_n = (\eta_n \circ a_n) \circ (a_n^{-1} \zeta_n \circ b_n)$, where $\eta_n = \sum Y_{nk} \delta_k$ and $\zeta_n = \sum_k \delta_{S_{nk}}$. As in Corollary 4.2 we have $\eta_n \circ a_n \xrightarrow{d} \eta$ and

$$b_n^{-1} \zeta_n^{-1} \circ a_n = \sum b_n^{-1} Z_{nk} \delta_k \circ a_n \xrightarrow{d} \zeta.$$

Furthermore $\zeta\{0\} = 0$ a.s., and so by Theorem 3.1,

$$a_n^{-1} \zeta_n \circ b_n = (b_n^{-1} \zeta_n^{-1} \circ a_n)^{-1} \xrightarrow{d} \zeta^{-1}.$$

Thus Theorem 3.2(i) yields $\xi_n \circ b_n \xrightarrow{d} \eta \circ \zeta^{-1}$.

Theorem 4.6 is similar to the results in [11] and [12] for compound point processes on more general spaces. Note that $\eta(\zeta^{-1}(x))$ is a process with conditional stationary independent increments [23].

5. Extensions

In this section we point out some rather immediate extensions of the above results. We first indicate how homogeneous independent thinnings on general spaces can be characterized by multidimensional compositions of measures.

Let $\zeta_n = \sum_k \delta_{S_{nk}}$ $(n \geqq 1)$ be a sequence of point processes on a locally compact second countable Hausdorff space X. Let $\eta_n = \sum_k Y_{nk} \; \delta_{k+1}$ $(n \geqq 1)$ be random measures on R. Similar to Section 4, we define an η_n-replacement of ζ_n as $\xi_n = \sum_k Y_{nk} \delta_{S_{nk}}$. We call this an independent homogeneous η_n-replacement if η_n is independent of ζ_n and $Y_{n1}, Y_{n2}, \ldots$ are independent with a common distribution for each $n \geqq 1$. In this case we can write

$$(\xi_n A_1, \ldots, \xi_n A_k) \stackrel{d}{=} (\eta_{n1}(\zeta_n A_1), \ldots, \eta_{nk}(\zeta_n A_k)) \tag{1}$$

for any disjoint Borel sets $A_1, \ldots, A_k$ in X, where $\eta_{n1}, \ldots, \eta_{nk}$ are independent copies of η_n. Directly from (1) and Corollary 2.4 one can obtain results such as the following.

(5.1) **Theorem.** *Suppose the* Y_{nk}*'s satisfy conditions* (i) *and* (ii) *in Corollary* 4.3 *and* $a_n^{-1} \zeta_n \xrightarrow{d} \zeta$. *Then* $\xi_n \xrightarrow{d} \xi$, *where* ξ *is defined by*

$$(\xi A_1, \ldots, \xi A_k) \stackrel{d}{=} (\eta_1(\zeta A_1), \ldots, \eta_k(\zeta A_k)) \tag{2}$$

for disjoint Borel sets $A_1, \ldots, A_k$ *in* X *and* $\eta_1, \ldots, \eta_k$ *are independent copies of a random measure* η *with stationary independent increments and Levy measure* α.

Results similar to the above appear in [11, 12] and [16], which are proved via Laplace transforms and the fact that the probability distribution of ξ_n is a mixture of probability distributions. The random measure version of the above is as follows. Let ζ_n $(n \geqq 1)$ be a random measure on X. For each Borel set A of X, replace the mass $\zeta_n A$ by a random mass that is equal in distribution to $\eta_n(\zeta_n A)$ where η_n is a random measure on R with stationary independent increments and is independent of ζ_n. Do this replacement so that masses replaced in disjoint sets $A_1, \ldots, A_k$ are conditionally independent given $\zeta A_1, \ldots, \zeta A_k$. The resulting random measure ξ_n on X is characterized by (1). Similar to Theorem 5.1 it follows that if $\eta_n \circ a_n \xrightarrow{d} \eta$ and $a_n^{-1} \zeta_n \xrightarrow{d} \zeta$, then $\xi_n \xrightarrow{d} \xi$ where ξ is defined as in (2).

To generalize the notion of ordered thinning to the space X, one needs an implicit ordering for depositing mass and/or thinning it. For example, one can deposit mass on X by ζ, and then thin the mass via increasing Borel sets $A_t \uparrow X$ as $t \to \infty$, where A_0 is the empty set, such that the mass ζA_t is replaced by $\xi A_t = \eta(\zeta A_t)$ for each $t \geqq 0$. This defines ξ on the smallest σ-field containing the A_t's. The convergence of a sequence ξ_n of such measures can be studied in terms of the measure $\xi_n^*(t) = \xi_n A_t = \eta_n(\zeta_n A_t)$ on R.

A more general analogue of ordered thinning on X is as follows. Think of $R_+ = [0, \infty)$ as a slab of mass that is deposited on X according to a random function ϕ from R_+ to X such that the t^{th} bit of R_+ is deposited at the location $\phi(t)$. Thin the deposited mass by a random measure η on R_+ such that the mass $\phi^{-1} A = \{t: \phi(t) \in A\}$ deposited in a Borel set A in X is replaced by an amount $\xi A = \eta \phi^{-1} A$. That is, the resulting measure ξ is an image of η under the mapping ϕ. Note that if $X = R$, then $\xi = \eta \phi^{-1} = \eta \circ \zeta$ where $\phi(t) = \tilde{\zeta}(t-) = \inf\{s: \zeta(s) \geqq t\}$.

Our results can also be extended in an obvious way to multidimensional compositions. One application is to the deletion of jumps in a continuous time integer-valued stochastic process, as in [27]. Here the jumps are deleted in such a way that the resulting process can be described by $\xi_n = (\eta_n^1 \circ \zeta_n^1, \eta_n^2 \circ \zeta_n^2, \ldots)$, which converges in distribution as $n \to \infty$ to a step process whose jump times form a Poisson process, and whose successive states are independent and identically distributed.

Independent homogeneous thinning and ordered thinnings are the only thinnings (like Renyi's) that have been discussed in the literature. There are, of course, a wide variety of other thinnings that one could define, based on various dependencies between the initial mass and the thinning measure, that cannot be characterized by compositions of measures. The term thinning is sometimes used in describing (i) rare events in stochastic processes (e.g. high level crossings) and (ii) interactions of renewal processes (or inhibitory thinning) as studied by Ten Hoopen and Reuver and others, as referenced in [21]. These thinnings are quite different from Renyi's and cannot generally be viewed as compositions.

References

1. Bauer, H.: Probability Theory and Elements of Measure Theory. New York: Holt, Rinehart and Winston 1972
2. Belyaev, Yu K.: Limit theorems for dissipative flows. Theor. Probability Appl. **8**, 165–173 (1963)
3. Billingsley, P.: Convergence of Probability Measures. New York: John Wiley 1968
4. Feller, W.: An Introduction to Probability Theory and Its Applications. Vol. II, 2nd Ed. New York: John Wiley 1971
5. Gikhman, I., Skorokhod, A.: Introduction to the Theory of Random Processes. Philadelphia: Saunders
6. Goldman, J.R.: Stochastic Point Processes: limit theorems. Ann. Math. Statist. **38**, 771–779 (1967)

7. Gnedenko, B. V., Fraier, B.: A few remarks on a result by I. N. Kovalenko. Litovsk. Math. SB. **9**, 463–470 (In Russian) (1969)
8. Jagers, P.: Aspects of random measures and point processes. In: Advances in Probability and Related Topics III, 179–239. New York: Marcel Decker 1974
9. Jagers, P., Lindvall, T.: Thinning and rare events in point processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. **28**, 89–98 (1974)
10. Kallenberg, O.: Characterization and convergence of random measures and point processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. **27**, 9–21 (1973)
11. Kallenberg, O.: Limits of compound and thinned point process. J. Appl. Probability **12**, 269–278 (1975)
12. Kallenberg, O.: Random Measures. Schriftenreihe des Zentralinstitutes für Mathematik und Mechanik der AdW der DDR. Berlin: Akademie-Verlag (1975)
13. Kerstan, J., Matthes, K., Mecke, J.: Unbegrenzt teilbare Punktprozesse. Berlin: Akademie-Verlag, 1974
14. Kovalenko, I. N.: On the class of limit distributions for thinning of homogeneous events. Litovsk. Mat. Sb. **5**, 569–573. (English translation in Selected Transl. Math. Statist. and Probab. **9**, 75–81, 1970) (1965)
15. Lindvall, T.: An invariance principle for thinned random measures. Technical report, Chalmers University of Technology and the University of Goteborg (1974)
16. Mecke, J.: Eine charakteristische Eigenschaft der doppelt stochastischen Poissonschen Prozesse. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11**, 74–81 (1968)
17. Mogyoródi, J.: On the rarefaction of renewal processes I, II. Studia Sci. Math. Hungar. **7**, 258–291, 293–305 (1972)
18. Mogyoródi, J.: On the rarefaction of renewal processes III, IV, V, VI. Studia Sci. Math. Hungar. **8**, 21–28, 29–38, 193–198, 199–209 (1973)
19. Nawrotzki, K.: Ein Grenzwertsatz für homogene zufällige Punktfolgen (Verallgemeinerung eines Satzes von A. Rényi). Math. Nachr. **24**, 201–217 (1962)
20. Råde, L.: Limit theorems for thinning of renewal point processes. J. Appl. Probability **9**, 847–851 (1972a)
21. Råde, L.: Thinning of Renewal Point Processes: A Flow Graph Study. Mathematisk Statistik AB, Goteborg Sweden (1972b)
22. Rényi, A.: A characterization of the Poisson process. Magyar Tud. Akad. Mat. Kutato' Int. Kozl. **1**, 519–527 (Hungarian) (1956)
23. Serfozo, R.: Processes with conditional stationary independent increments. J. Appl. Probability **9**, 303–315 (1972)
24. Serfozo, R.: Semi-stationary processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. **23**, 125–132 (1972)
25. Szantai, T.: On limiting distributions for the sums of random variables concerning the rarefaction of recurrent processes. Studia Sci. Math. Hungar. **6**, 443–452 (1971a)
26. Szantai, T.: On an invariance problem related to different rarefactions of recurrent processes. Studia Sci. Math. Hungar. **6**, 453–456 (1971b)
27. Tomko, J.: On the rarefaction of multivariate point processes. Colloquia Mathematica Societatis Janos Bolyai, 9 Progress in Statistics, Vol. **II**. 843–868 (1974)

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