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Intersections of Markov Random Sets

John Hawkes
Department of Statistics, University College of Swansea, Singleton Park, Swansea SA2 8 PP, Wales, U.K.

Suppose that $\{E(n)\}$ is a recurrent event, in the sense of Feller [3], and that Φ is the set of renewal epochs. Then

$$u_n = P[E(n)] = P[n \in \Phi] \tag{1}$$

is the corresponding renewal sequence. If Φ_i $i=1,2$ are independent renewal epochs, with renewal sequences u_n^i $i=1,2$, so is $\Phi_1 \cap \Phi_2$. Also $P(n \in \Phi_1 \cap \Phi_2) = u_n^1 u_n^2$. Thus the mapping

$$(\Phi_1, \Phi_2) \to \Phi_1 \cap \Phi_2 \tag{2}$$

is explained by the map

$$(u_n^1, u_n^2) \to u_n^1 u_n^2. \tag{3}$$

This fact seems to have been first noted by Lamperti ([12]).

The theory of Markov random sets is a generalization of renewal theory to continuous time. If Φ_i $i=1,2$ are independent Markov random sets such that $\Phi_1 \cap \Phi_2 \neq \phi$ almost surely then $\Phi_1 \cap \Phi_2$ is also a Markov random set and one can again ask about the structure of the operation (2). Two problems arise here. Firstly in continuous time the occurrences can be so rare that the sets fail to intersect. Secondly even when they do intersect the probabilities in (1) are often zero so, with this interpretation, (3) is a trivial statement. However u_n is also the potential kernel density of the increasing random walk associated with $\{E(n)\}$ so if we interpret (3) as a multiplication of potential kernel densities we might hope to make some progress by looking at the potential kernel density of the subordinator which corresponds to Φ. An obvious difficulty here is that the potential kernel of this subordinator need not have a density.

The principal object of this paper is to examine when Markov random sets do intersect and the nature of the intersection when it is non empty. In the course of this we extend Chung's results on the representation of hitting probabilities to some processes whose kernels do not necessarily have densities. We also obtain results on the regular behaviour of kernels and how this is related to the behaviour of the Lévy measure.

The theory of p-functions is a special case of the theory of Markov random sets where the probabilities in (1) are always positive. In this situation the structure of (2) has been studied in great detail (see [8] and [9]).

1. Definitions

For our purposes a subordinator is an increasing process, X_t, whose Laplace transform is given by

$$E\exp(-\theta X_t)=\exp[-tg(\theta)]$$

where

$$g(\theta)=\delta\theta+\int_{(0,\infty]}[1-\exp(-\theta x)]\,\mu(dx).$$

Here $\delta\geqq 0$ is the drift of the process and μ, the Lévy measure, has

$$\int_{(0,\infty]}\min(x,1)\,\mu(dx)$$

finite. Thus μ is allowed to have a finite atom at infinity, this atom corresponding to an exponential killing of the usual subordinator. The distribution of X_t is continuous if and only if μ is infinite, an assumption we now make.

If $\alpha\geqq 0$ the measure U^α, defined for Borel sets A by

$$U^\alpha(A)=E\int_0^\infty\exp(-\alpha t)\,I_A(X_t)\,dt,$$

is finite on bounded sets. Related to U^α is the potential kernel

$$U^\alpha(x,A)=U^\alpha(A-x)$$

and its dual

$$\tilde{U}^\alpha(y,B)=U^\alpha(y-B).$$

The Laplace transform of U^α satisfies

$$\int_0^\infty\exp(-\theta t)\,dU^\alpha(t)=[\alpha+g(\theta)]^{-1}.$$

The density of U^α, when it exists, is denoted by u^α (we write u for u^0) and is called the potential kernel density of X. When u^α exists we can always find a version such that $x\to u^\alpha(y-x)$ is α excessive. The simplest conditions sufficient for the existence of u are

i) that $\delta>0$, in which case $\delta u(t)$ is a p-function and is uniformly continuous

ii) that if the measure μ is absolutely continuous then u exists.

It is easy to give examples where u does not exist.

There are various equivalent definitions of a Markov random set (see [6, 7, 11] and [12]). They are all equivalent to defining it as the range of a subordinator, X_t, that is

$$\Phi=\{x\colon x=X_t \text{ for some } t>0\}.$$

2. Potential Theory

A function $g: R_1 \times R_1 \to R_+$ is called a kernel density. The capacity of a bounded set B is defined by

$$C^g(B) = \sup\{m(B): \int_B g(x,y)\,dm(y) \leqq 1\}$$

where the supremum of the empty set is zero. A kernel with $g(x,y) = k(y-x)$ is called a difference kernel. If $k(z) > 0$ if and only if $z > 0$ we call k one sided.

A one sided monotone kernel k satisfies the energy principle, namely $C^k(B) > 0$ if and only if for some non atomic measure m

$$\int_B \int_B k(y-x)\,dm(y)\,dm(x) < \infty.$$

If X_t is a subordinator and B a Borel set the last exit time is defined by

$$\gamma_B = \inf\{s: t > s \text{ implies } X_t \notin B\}.$$

Hunt's results on hitting probabilities (see [1], pp. 283–285) show that if U is absolutely continuous with density u there is a measure π_B supported by $\bar{B}$ such that

$$P^x(\gamma_B > 0) = \int u(y-x)\,\pi_B(dy).$$

The disadvantage here is the need to assume the existence of a density u. Chung's argument in [2] can easily be modified to prove:

(2.1) **Theorem.** *Let X be a subordinator whose Lévy measure has infinite mass. Then if A is an open set, and $C \subset \bar{B}$*

$$\int_A P^x\{X(\gamma_B -) \in C\}\,dx = \int_C \tilde{U}(y, A)\,\pi_B(dy)$$

where π_B is a measure supported by B.

3. Intersections

Now we suppose that Φ_1 and Φ_2 are independent Markov random sets and let X^1 and X^2 denote the corresponding processes. We suppose that X^1 has a potential kernel density u_1 and let U_2 be the potential measure of X^2. First we prove

(3.1) **Theorem.** *The following are equivalent:*

i) *$\Phi_1 \cap \Phi_2 \neq \phi$ almost surely;*
ii) *$C^{u_1}(\Phi_2) > 0$ almost surely.*

Proof. This is an immediate consequence of the Hunt results on hitting probabilities.

(3.2) **Theorem.** *If $\Phi_1 \cap \Phi_2 \neq \phi$ almost surely then $\int_0^1 u_1(t)\,dU_2(t)$ is finite. Furthermore if u_1 is monotone the converse holds.*

Proof. Suppose that $\Phi_1 \cap \Phi_2 \neq \phi$ almost surely and let I be a closed subinterval of $(0, \infty)$ of length $|I|$ such that $\Phi_1 \cap \Phi_2 \cap I \neq \phi$ with positive probability. Let

$$\psi(x, y) = \int_I u_1(z-x)\, U_2(y, dz),$$

$$T = \inf\{t\colon X_t^1 \in R_2 \cap I\},$$

and

$$S = \inf\{s\colon X_s^2 \in R_1 \cap I\}.$$

Now, since $x \to \psi(x, y)$ is excessive with respect to X_t^1,

$$t \to \psi(X_t^1, y)$$

is a non negative supermartingale. Thus by the optional sampling theorem

$$\psi(0, y) \geqq E\psi(X_T^1, y).$$

Also

$$s \to E\psi(X_T^1, X_s^2)$$

is also a non negative supermartingale so again we have

$$\psi(0, 0) \geqq E\psi(X_T^1, X_S^2).$$

(Problems regarding the joint measurability can easily be dealt with.) Now $X_T^1 = X_S^2$ almost surely on $(T < \infty)$ so that

$$\begin{aligned} E\psi(X_T^1, X_S^2) &= \int_I P(X_T^1 \in dy) \{\int_I u_1(z-y)\, U_2(y, dz)\} \\ &= \int_I P(X_T^1 \in dy) \left\{ \int_0^{|I|-y} u_1(z)\, dU_2(z) \right\}. \end{aligned}$$

Since this is less than $\psi(0, 0)$ we have $\int_{0+} u_1(t)\, dU_2(t) < \infty$.

Now suppose that u_1 is monotone and $\int_0^1 u_1(t)\, dU_2(t) < \infty$ so that, if $\alpha > 0$, $\int_0^\infty u_1(t)\, dU_2^\alpha(t) < \infty$. Then if $x \geqq 0$

$$\int_0^\infty u_1(z-x)\, dU_2^\alpha(z) \leqq \int_0^\infty u_1(z)\, dU_2^\alpha(z)$$

so that

$$\int_0^\infty \int_0^\infty u_1(z-x)\, dU_2^\alpha(z)\, dU_2^\alpha(x) < \infty.$$

Now let m be the image under X^2 of the measure with density $\alpha e^{-\alpha t}$ on $(0, \infty)$. Thus

$$E \iint_{\Phi_2 \times \Phi_2} u_1(y-x)\, dm(y)\, dm(x) < \infty$$

and

$$\iint_{\Phi_2 \times \Phi_2} u_1(y-x)\, dm(y)\, dm(x) < \infty \qquad \text{almost surely.}$$

Since u_1 is a one sided monotone kernel and m is supported by $\bar{\Phi}_2$ this implies that $C^{u_1}(\bar{\Phi}_2) > 0$ almost surely. Thus $C^{u_1}(\Phi_2) > 0$ almost surely and, by Theorem 3.1, $\Phi_1 \cap \Phi_2 \neq \phi$ almost surely.

Before we examine the structure of the intersection we need to introduce some new concepts. If X is a topological measure space, $\mathscr{B}(X)$ denotes the bounded Borel functions on X.

Definition. Let X and Y be two topological measure spaces and $V(x, B)$ a kernel defined for $x \in X$ and $B \subset Y$. A kernel $\tilde{V}(y, A)$ defined for $y \in Y$ and $A \subset X$ is said to be (X, Y) dual to V if

$$(f, Vg) = (g, \tilde{V}f) \qquad \textit{whenever } f \in \mathscr{B}(X) \textit{ and } g \in \mathscr{B}(Y).$$

If $x \in R_1$ and $B \subset R_2$ we define

$$W(x, B) = [U_1(x, .) \times U_2(x, .)](B)$$

and let $\tilde{W}$ be the (R_1, R_2) dual of W, whenever the latter is defined. If U_1 is absolutely continuous $\tilde{W}$ exists and a version of $\tilde{W}$ is given by

$$\tilde{W}(\underline{z}, A) = \int_A u_1(z_1 - x)\, \tilde{U}(z_2, dx)$$

where $\underline{z} = (z_1, z_2)$. Next we let

$$C = \{f \colon f \text{ continuous with compact support in } (-\infty, 0)\}$$

and

$$C^+ = \{f \colon f \in C,\ f \geqq 0 \text{ and } f \not\equiv 0\}.$$

We say that $\tilde{W}$ is regular if whenever $f \in C^+$ there is a neighbourhood D_f of the positive diagonal such that $\tilde{W}f$ is continuous and positive on D_f.

(3.3) **Theorem.** *Suppose that $\Phi_1 \cap \Phi_2 \neq \phi$ almost surely and let U_{12} be the potential measure of $\Phi_1 \cap \Phi_2$. Then, if $\tilde{W}$ exists and is regular and $\bar{y} = (y, y)$, $\tilde{W}(\bar{y}, .)$ is proportional to $\tilde{U}_{12}(y, .)$ the kernel dual to U_{12}.*

Proof. Suppose that B is an open interval strictly contained in $(0, \infty)$ and let $M = \sup\{x \colon x \in \Phi_1 \cap \Phi_2 \cap B\}$. It follows from Theorem 2.1 that if $f \in C^+$ and $L(x, dy) = P^x(M \in dy)$ then

$$\frac{\int f(x)\, L(x, dy)\, dx}{\int f(x)\, \tilde{U}_{12}(y, dx)} = \pi_B(dy)$$

where π_B is independent of f.

Let

$$\gamma_1 = \inf\{s \colon t > s \text{ implies } X_t^1 \notin \Phi_2 \cap B\}$$

and

$$\gamma_2 = \inf\{s \colon t > s \text{ implies } X_t^2 \notin \Phi_1 \cap B\}.$$

Then $X^i(\gamma_i -)=M$ and so $[X^1(\gamma_1 -), X^2(\gamma_2 -)]=(M, M)$. The idea of the proof is to consider the process $(t,s)\to(X_t^1, X_s^2)$ to get a second representation for $\int f(x)\,L(x,dy)\,dx$ in terms of $\tilde{W}$. We first introduce some extra notation to cope with this new process. Let $\bar{x}=(x,x)$,

$$\underline{t}=(t,s),\quad \underline{X}_{\underline{t}}=(X_t^1, X_s^2),\quad \underline{\gamma}=(\gamma_1,\gamma_2),\quad P^{(x,y)}=P_1^x\times P_2^y$$

and

$$[\underline{t},\underline{t}+\underline{\varepsilon})=[t,t+\varepsilon)\times[s,s+\varepsilon).$$

Now let $f\in C^+$ and let g be continuous on D_f. Then if $\varepsilon>0$ we evaluate the limit, as $\varepsilon\to 0$, of the integral

$$\frac{1}{\varepsilon^2}\int f(x)\{\iint_{D_f} E^{\bar{x}}[g(\underline{X}_{\underline{t}}): \underline{\gamma}\in(\underline{t},\underline{t}+\underline{\varepsilon}]]\,d\underline{t}\}\,dx$$

in two ways. First note that from the Markov property:

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int f(x)\{\iint_{D_f}[\int_{\underline{\gamma}\in(\underline{t},\underline{t}+\underline{\varepsilon}]} g(\underline{X}_{\underline{t}})\,dP^{\bar{x}}]\,d\underline{t}\}\,dx\\ &\quad=\frac{1}{\varepsilon^2}\int f(x)\{\iint_{D_f} E^{\bar{x}}[g(\underline{X}_{\underline{t}})\,P^{\underline{X}_{\underline{t}}}(\underline{\gamma}\in(0,\underline{\varepsilon}]]\,d\underline{t}\}\,dx.\end{aligned}\tag{4}$$

Chung's argument shows that the left hand side converges to

$$\int f(x)\,E^{\bar{x}}\,g(\bar{M})\,dx=\int f(x)\{\int_B L(x,d\bar{y})\,g(\bar{y})\}\,dx.$$

Let $\underline{z}=(z_1,z_2)$ and $\psi_\varepsilon(\underline{z})=\frac{1}{\varepsilon^2}P^{\underline{z}}(\underline{\gamma}\in(0,\underline{\varepsilon}])$. Then the right hand side of (4) can be rewritten as

$$\begin{aligned}(f, Wg\psi_\varepsilon)&=(f, W\psi_\varepsilon g)\\ &=(\psi_\varepsilon g, \tilde{W}f)\\ &=(\psi_\varepsilon, g\tilde{W}f).\end{aligned}$$

In particular the latter tends to a limit as $\varepsilon\to 0$. Now let h be any continuous function with compact support contained in D_f. Then choose $g=h/\tilde{W}f$ so that g is continuous with compact support. Thus

$$\lim_{\varepsilon\to 0}(\psi_\varepsilon, h)\qquad \text{exists for each } h.$$

Thus $\psi_\varepsilon(\underline{z})\,d\underline{z}$ converges weakly to the measure π_2 on the diagonal given by

$$\frac{\int f(x)\,L(x,d\bar{y})\,dx}{(\tilde{W}f)(\bar{y})}=\pi_2(d\bar{y}).$$

This measure is independent of f (all the domains D_f contain the diagonal).

Now, if we identify π_2 with a measure on the half line, we have

$$\frac{d\pi_1}{d\pi_2}(y)=\frac{\int f(x)\,\tilde{U}_{12}(y,dx)}{\int f(x)\,\tilde{W}(\bar{y},dx)}\qquad \forall f\in C^+.$$

If we replace f by a translate of f we see that $\frac{d\pi_1}{d\pi_2}$ is constant on B and hence for some constant C_B

$$\int f(x)\,\tilde{U}_{12}(y, dx) = C_B \int f(x)\,\tilde{W}(\bar{y}, dx)$$

for $f \in C^+$ and $y \in B$. Finally we let B increase to $(0, \infty)$ to see that C_B is independent of B. It follows that there is a constant c such that

$$\tilde{U}_{12}(y, .) = c\,\tilde{W}(\bar{y}, .)$$

for all y. The theorem is thus proved.

Corollary 1. *If u_1 exists and is continuous and monotone one has*

$$\Phi_1 \cap \Phi_2 \neq \phi \qquad \textit{almost surely}$$

if and only if

$$u_1(x)\,dU_2(x) \qquad \textit{is locally finite,}$$

in which case this measure is proportional to the kernel measure of $\Phi_1 \cap \Phi_2$.

Proof. Theorem 3.2 gives the first part of the corollary. The second part follows by observing that under the given hypotheses

$$\tilde{W}f = \int f(x)\,u_1(z_1 - x)\,\tilde{U}_2(z_2, dx)$$

satisfies the conditions on Theorem 3.3. Hence

$$\tilde{U}_{12}(y, A) = \int_A u_1(y - x)\,U_2(y, dx)$$

is proportional to the dual of the kernel of $\Phi_1 \cap \Phi_2$. The corollary follows.

The above theorems can similarly be applied to prove the following corollary.

Corollary 2. *If u_1 exists and is continuous and bounded then $\Phi_1 \cap \Phi_2 \neq \phi$ almost surely, and $u_1(t)\,dU_2(t)$ is proportional to $dU_{12}(t)$, the kernel measure of $\Phi_1 \cap \Phi_2$. If, in addition, U_2 has a bounded continuous density u_2 then so has U_{12} and u_{12} is proportional to $u_1 u_2$ and $\frac{u_{12}(x)}{u_{12}(0)} = \frac{u_1(x)}{u_1(0)}\frac{u_2(x)}{u_2(0)}$.*

Note. If u_1 is bounded $u_1(t)/u_1(0)$ is a p-function so the corollary shows that the product of two p-functions is again a p-function.

4. Monotone Kernels

In Section 3 we needed to suppose that certain kernels were monotone. Here we examine conditions which are sufficient to ensure this. Let $H(u) = \mu\{(u, \infty]\}$ be the tail of the Lévy measure, so that $g(\theta) = \delta\theta + \theta \int_0 \exp(-\theta u)\,H(u)\,du$.

(4.1) **Theorem.** *If $H(u)$ is* log *convex, that is $HH'' \geq (H')^2$, then $dU(t)$ has a density $u(t)\,dt$ such that u is monotone.*

Proof. This is Theorem 2.1 of [4].

(4.2) **Theorem.** *If u is a non negative, locally integrable, and* log *convex then u is the kernel density of some subordinator.*

Proof. Let $\phi = \log u$ then ϕ is convex. If $M > 1$ we let ϕ_M be the greatest convex function less than $\log M$ and ϕ. Then, if $u_M = \exp(\phi_M)$, u_M/M is a p-function so that

$$\int_0^\infty \exp(-\theta t)\, u_M(t)\, dt = \left[\theta/M + \theta \int_0^\infty \exp(-\theta u)\, H_M(u)\, du\right]^{-1} = [g_M(\theta)]^{-1}.$$

Clearly

$$\int_0^\infty \exp(-\theta t)\, u(t)\, dt = \lim_{M\to\infty} \int_0^\infty \exp(-\theta t)\, u_M(t)\, dt$$

so that $\lim_{M\to\infty} \int_0^\infty \exp(-\theta u)\, H_M(u)\, du$ exists for each θ. Let $H(u)\, du$ be a weak limit of the measures $\{H_M(u)\, du\}$ then we have

$$\int_0^\infty \exp(-\theta t)\, u(t)\, dt = \left[\theta \int_0^\infty \exp(-\theta u)\, H(u)\, du\right]$$

so that u is a kernel density.

If u is completely monotonic, non negative and locally integrable u is log convex and so u is a potential kernel density. The following theorem characterizes such kernels.

(4.3) **Theorem.** *A kernel density u is completely monotonic if and only if the tail of the Lévy measure is completely monotonic.*

Proof. In [10] Kingman proves this in the case where $\delta = 1$. His argument works in the case where $\delta > 0$. It remains to show that one can let δ tend to zero. As the proof that this is possible involves no new ideas we shall omit the details.

If $u_i\; i = 1, 2$ are log convex kernels $u_1 u_2$ is again log convex and so is a kernel if and only if $u_1 u_2$ is locally integrable. This provides one of the few examples where it is analytically obvious when the product of two kernels is again a kernel.

5. Examples

We now give some illustrations of the application of these ideas.

Example 1. A stable subordinator of index α has a potential kernel density $u_\alpha(x)$ proportional to $x^{\alpha-1}$. Thus $u_\alpha u_\beta$ is proportional to $x^{(\alpha+\beta-1)-1}$, which is locally integrable if and only if $\alpha + \beta > 1$. Thus the ranges of independent stable subordinators intersect if and only if $\alpha + \beta > 1$, in which case the intersection is stochastically equivalent to the range of a stable subordinator of index $\alpha + \beta - 1$.

Example 2. Let $Z_t=(X_t, Y_t)$ be a two dimensional brownian motion, where X_t and Y_t are independent one dimensional brownian motions. Then $Z_1=\{t: X_t=0\}$ and $Z_2=\{t: Y_t=0\}$ are independent Markov random sets with corresponding kernel densities $u_i(t)$ proportional to $t^{-\frac{1}{2}}$. Since $u_1 u_2$ is not locally integrable $Z_1 \cap Z_2=\phi$ almost surely and we arrive at the conclusion that $\{t: Z_t=0\}$ is empty. Thus we have an alternative proof that two dimensional brownian motion does not have zeros.

Theorem 5 of [5] contains a further application of this type.

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