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Stability of the Classification of Stopping Times

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Introduction

Let $(\mathscr{F}_t)$, $0 \leqq t \leqq \infty$ be an increasing family of σ-algebras which is right continuous, with $\mathscr{F}_t$ complete and each $\mathscr{F}_t \subseteqq \mathscr{F}$, on a probability space $(\Omega, \mathscr{F}, P)$. If T, S are $(\mathscr{F}_t)$ stopping times, then so are $T \wedge S$, $T \vee S$ and $T+S$. Dubins [4] has characterized all the deterministic functions of n stopping times $(n \geqq 1)$ such that the image is a stopping time.

In "the general theory of processes" one classifies stopping times into previsible (or predictable), accessible, and totally inaccessible stopping times. These stopping times are intimately connected with the previsible and well-measurable σ-algebras that are used, for example, in Markov process theory and stochastic integration. It is elementary that if T, S belong to one of the three classes, so also do $T \wedge S$ and $T \vee S$. This paper determines which of the deterministic functions that carry stopping times into stopping times ("preserve" stopping times) also preserve the classifications.

The main Theorems are (3.3) and (4.3) which show that all Borel functions ϕ which preserve stopping times also preserve previsibility; Corollaries (3.4) and (4.4) show that all such ϕ also preserve accessibility; Theorems (3.6) and (4.9) reveal a more complicated situation for the class of totally inaccessible times. For ϕ to preserve total inaccessibility there must be a partition of the domain such that ϕ is a lattice operation in n variables on each set in the partition, except for a small set. The structure of this exceptional set is determined.

2. Preliminaries and Notation

A standard reference for the classification of stopping times is the book by Dellacherie [2]. We recall some of the definitions and elementary properties. A stopping time T is *previsible* if there exists an increasing sequence (T^n) of stopping times with limit T such that $T^n < T$ on $\{T > 0\}$ for each n. Such a sequence *announces* T. A time T is *accessible* if there is a sequence (T^n) of previsible times such that $\bigcup \{T^n = T < \infty\} = \{T < \infty\}$ a.s. A time T is *totally inaccessible* if $P\{T = S < \infty\} = 0$ whenever S is previsible. It is elementary that all three classifications are preserved under the operations min($\wedge$) and max($\vee$); for example, if T, S are accessible, $T \wedge S$ and $T \vee S$ are also accessible. If $A \in \mathscr{F}$, then T_A is the random variable equal to T on A and $+\infty$ on A', the complement of A. Also T_A is again a stopping time if and only if $A \in \mathscr{F}_T$. If T is previsible, then T_A is again previsible if and only if $A \in \mathscr{F}_{T-}$. If S, T are stopping times, then $\{S \leqq T\}$ is in both $\mathscr{F}_S$ and $\mathscr{F}_T$. If S is previsible, then $\{S \leqq T\} \in \mathscr{F}_{T-}$. We also recall that T is $\mathscr{F}_{T-}$ measurable.

We let $\mathbf{t} = (t_1, \dots, t_n)$, where the dimension should be clear from the context; and $\mathbf{t}^*, \mathbf{t}_*$ will denote $t_1 \vee t_2 \vee \cdots \vee t_n$ and $t_1 \wedge t_2 \wedge \cdots \wedge t_n$ respectively. The letters R, S, T will always be stopping times, $\mathbf{T}$ a vector of stopping times, etc. If $\mathbf{t}$ is an n-dimensional vector ("n-vector"), the quantity $\mathbf{t}_{(i)} = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ is an $(n-1)$-vector. Similarly $\mathbf{t}_{(i,j)}$ is an $(n-2)$-vector, etc. If Λ is a set in $\mathbb{R}^n_+$, then $\Lambda(\mathbf{t}_{(i)})$ will denote the section of Λ holding $\mathbf{t}_{(i)}$ fixed. The letter ϕ is reserved for functions which preserve stopping times, though its domain may vary.

Dubins [4] has shown that for Borel $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ to preserve stopping times it is necessary and sufficient that there exist a (possibly infinite) constant c such that

$$(2.1)\quad \begin{aligned} \phi(t) &\geqq t && \text{if } 0 \leqq t \leqq c \\ &= c && \text{if } c < t < \infty. \end{aligned}$$

Dubins considers finite-valued stopping times, but the extension to infinite-valued stopping times follows from it: let $f: \bar{\mathbb{R}}_+ \to I$ be given by $f(x) = x/(1+x)$, $0 \leqq x \leqq \infty$ (where $\bar{\mathbb{R}}_+ = [0, \infty]$, and $I = [0, 1]$). Let $g: I \to \bar{\mathbb{R}}_+$ be given by $g(y) = y/(1-y)$. Let $\psi: \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$, and let $\phi: I \to I$ be given by $\phi = f \circ \psi \circ g$. Then letting $s = f(t)$, $\mathscr{G}_s = \mathscr{F}_t$, (2.1) for ϕ becomes

$$(2.2)\quad \begin{aligned} \phi(t) &\geqq t && 0 \leqq t \leqq c \\ &= c && c < t \leqq 1. \end{aligned}$$

One easily checks that ϕ preserves $\mathscr{G}$ stopping times if and only if ψ preserves $\mathscr{F}$ stopping times. Thus $\psi = g \circ \phi \circ f$ preserves stopping times if and only if it is of the form (for some constant c):

$$(2.3)\quad \begin{aligned} \psi(s) &\geqq s && 0 \leqq s \leqq c \\ &= c && c < s \leqq \infty. \end{aligned}$$

For a Borel function $\phi: \mathbb{R}^n_+ \to \mathbb{R}_+$, Dubins showed that it is necessary and sufficient that

$$(2.4)\quad t \to \phi(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n)$$

satisfy (2.1) for each $\mathbf{t}_{(i)}$, $1 \leqq i \leqq n$, in order for ϕ to preserve stopping times. For ϕ to carry n-vectors of possibly infinite stopping times into stopping times, by letting $T_j = t_j$, $j \neq i$, it is clearly necessary for the function of t in (2.4) to satisfy (2.3) for each vector $\mathbf{t}_{(i)} \in \bar{\mathbb{R}}^{n-1}_+$. Let $F(\mathbf{x}) = (x_1/(1+x_1), \dots, x_n/(1+x_n))$, and $G(\mathbf{y}) = (y_1/(1-y_1), \dots, y_n/(1-y_n))$, where $F: \bar{\mathbb{R}}^n_+ \to I^n$ and $G: I^n \to \bar{\mathbb{R}}^n_+$. Also let $\psi: \bar{\mathbb{R}}^n_+ \to \bar{\mathbb{R}}_+$; $\phi: I^n \to I$ where $\phi = f \circ \psi \circ G$; $s = f(t)$; and $\mathscr{G}_s = \mathscr{F}_t$. Then we have:

(2.5) **Lemma.** *ψ preserves $\mathscr{F}$ stopping times if ϕ preserves $\mathscr{G}$ stopping times.*

Proof. Let $\mathbf{T}$ be an n-vector of $\mathscr{F}$ stopping times. Then $\mathbf{R}=F(\mathbf{T})$ is an n-vector of $\mathscr{G}$ stopping times. Then

$$\begin{aligned}\{\psi(\mathbf{T})\leqq t\}&=\{\psi\circ G\circ F(\mathbf{T})\leqq t\}\\&=\{\psi\circ G(\mathbf{R})\leqq t\}\\&=\{\phi(\mathbf{R})\leqq f(t)\}\\&=\{\phi(\mathbf{R})\leqq s\}\in\mathscr{G}_s=\mathscr{F}_t.\end{aligned}$$

Therefore $\psi(\mathbf{T})$ is an $\mathscr{F}$ stopping time, which establishes Lemma (2.5).

If ψ is of the form (2.3), then ϕ is of the form (2.2), and Dubins' result assures us ϕ preserves stopping times. Lemma (2.5) then shows that it is also sufficient for ϕ to preserve stopping times that each function as in (2.4) satisfy (2.3) for each vector $\mathbf{t}_{(i)}\in\bar{\mathbb{R}}_+^{n-1}$.

Note that the constant $c_i=c_i(\mathbf{t}_{(i)})$ which is called for in (2.3) is merely

$$\phi(t_1,\ldots,t_{i-1},\infty,t_{i+1},\ldots,t_n).$$

Hence c_i: $\bar{\mathbb{R}}_+^{n-1}\to\bar{\mathbb{R}}_+$ is also Borel. We call c_i the i-th *function of constants.*

3. One Variable Results

In this section ϕ: $\bar{\mathbb{R}}_+\to\bar{\mathbb{R}}_+$ is Borel and preserves stopping times; that is, ϕ satisfies (2.3) and $\phi(\infty)=c$. Always R, S and T will denote arbitrary stopping times.

(3.1) **Lemma.** *If $\phi(t)>t$ for all t, then $\phi(T)$ is previsible.*

Proof. Let $\phi^n(t)=(n-1/n)\,[\phi(t)-t]+t$. Then $\phi^n(T)$ announces $\phi(T)$.

(3.2) **Lemma.** *Let $\phi(t)\geqq t$ for all t, $\Lambda=\{t\colon\phi(t)=t\}$, $A=T^{-1}(\Lambda)$. If T_A is previsible, then $\phi(T)$ is previsible.*

Proof. Note that $A\in\mathscr{F}_{T-}$. Since $\phi(T)=\phi(T_A)\wedge\phi(T_{A'})=T_A\wedge\phi(T_{A'})$, it suffices to show $\phi(T_{A'})$ is previsible. Let $\psi(t)=\phi(t)\,1_{\Lambda'}+2t\,1_\Lambda$, where 1_Λ is the indicator function of the set Λ. Then $\psi(T_{A'})=\phi(T_{A'})$ is previsible by Lemma (3.1).

(3.3) **Theorem.** *All ϕ preserve previsibility.*

Proof. Let T be a previsible time. If $\phi(t)\geqq t$ all t, then $\phi(T)$ is previsible by Lemma (3.2). Suppose there exists $c<\infty$ such that $\phi(t)=c$ for $t>c$. Then

$$\phi(T)=\phi(T\wedge c)\wedge c_B,$$

where $B=\{c<T\}$. Since T is previsible, we have $B\in\mathscr{F}_{c-}$, hence c_B is previsible. Let $\psi(t)=\phi(t)\,1_{[0,c]}+2t\,1_{(c,\infty)}$. Then $\psi(T\wedge c)=\phi(T\wedge c)$ is previsible by Lemma (3.2).

(3.4) **Corollary.** *All ϕ preserve accessibility.*

Proof. Let T be an accessible time. Let (T^n) be previsible such that

$$\bigcup_{n=1}^{\infty}\{T_n=T<\infty\}=\{T<\infty\}\quad\text{a.s.}$$

Then $\bigcup_{n=1}^{\infty} \{\phi(T_n)=\phi(T)<\infty\}=\{\phi(T)<\infty\}$ a.s., and each $\phi(T_n)$ is previsible by Theorem (3.3).

If $\phi(t)<t$ for some t, then $\phi(t)=c$ for all $t>c$ for some finite c. So a necessary condition for ϕ to preserve total inaccessibility is that $\phi(t)\geqq t$ for all t.

(3.5) **Theorem.** *Let $\phi(t)\geqq t$ for all t and T be totally inaccessible. Let*

$$\Lambda=\{t\colon t<\phi(t)<\infty\}$$

and $A=T^{-1}(\Lambda)$. Then $\phi(T)$ is totally inaccessible if and only if $P(A)=0$.

Proof (Necessity). Suppose $\phi(T)$ is totally inaccessible. Then $\phi(T_A)$ is previsible by Lemma (3.2), so $P\{\phi(T_A)=\phi(T)<\infty\}=0$, hence $P(A)=0$.

(Sufficiency). Suppose $P(A)=0$. Then $\phi(T)=\phi(T_A)\wedge\phi(T_{A'})$. Since $A\in\mathscr{F}_T$, we know T_A, $T_{A'}$ are also totally inaccessible. Let $\psi(t)=\phi(t)\,1_{\Lambda'}+t\,1_{\Lambda}$. Then when S is previsible, we have $R=T_{A'}$ totally inaccessible and

$$P\{S=\psi(R)<\infty\}\leqq P\{S=R<\infty\}=0.$$

So $\phi(T_{A'})=\psi(T_{A'})$ is totally inaccessible, and $\phi(T_A)=\infty$ a.s.

(3.6). **Theorem.** *ϕ preserves total inaccessibility if and only if $\phi(t)\geqq t$ and*

$$\Lambda=\{t\colon t<\phi(t)<\infty\}$$

is countable.

Proof (Sufficiency). Let T be totally inaccessible, and S be an arbitrary previsible time. Then

$$\begin{aligned} P\{\phi(T)=S<\infty\} &\leqq P\{\phi(T)=S;\ T\in\Lambda\}+P\{\phi(T)=S<\infty;\ T\notin\Lambda\}\\ &\leqq P\{T=S\}+P\{T\in\Lambda\}\\ &\leqq \sum_{\alpha\in\Lambda} P\{T=\alpha\}=0.\end{aligned}$$

(Necessity). Suppose Λ is uncountable. Since Λ is Borel, by a theorem of Alexandrov-Hausdorff (see Dellacherie [2], p. 25), Λ contains an uncountable compact set. It is well known that a closed set may be written as the union of a perfect set and a countable set, so in particular Λ contains a nonempty perfect set. For any perfect set there exists a bounded continuous nondecreasing function which induces a diffuse probability measure whose support is that set. So we may find a diffuse probability P on $\mathbb{R}_+$ which gives mass to Λ. Take $\Omega=\mathbb{R}_+$, $S(t)=t$, $\mathscr{F}$ the Borel sets, $\mathscr{F}_t^\circ=\sigma(S\wedge t)$, and $(\mathscr{F}_t)$ the completed family $\mathscr{F}_t^\circ$ made right continuous. Dellacherie [3] has studied this example, and shown that P diffuse guarantees S is totally inaccessible. Since $P\{S\in\Lambda\}>0$, $\phi(S)$ cannot be totally inaccessible by Lemma (3.5).

(3.7) **Theorem.** *Suppose $\phi(T)$ is totally inaccessible. Then $R=T_A$ is totally inaccessible, where $A=\{\omega\colon \phi(T(\omega))<\infty\}$.*

Proof. Note $A=T^{-1}\{t\colon \phi(t)<\infty\}$, so $A\in\mathscr{F}_T$ and R is a stopping time. If R is not totally inaccessible, we can find a set B such that $P(B)>0$, $R_B<\infty$ on B, and R_B

is accessible. By Corollary (3.4), $\phi(R_B)$ is accessible. But

$$P\{\phi(T)=\phi(R_B)<\infty\}\geqq P(B)>0,$$

a contradiction.

4. *n* Variables Results

Throughout this section $\phi\colon \bar{\mathbb{R}}_+^n \to \bar{\mathbb{R}}_+$ is Borel, and preserves stopping times; that is, for each i the function in (2.4) satisfies condition (2.3) for all fixed $\mathbf{t}_{(i)}\in\bar{\mathbb{R}}_+^{n-1}$. We let $c_i=c_i(\mathbf{t}_{(i)})=\phi(t_1, \dots, t_{i-1}, \infty, t_{i+1}, \dots, t_n)$ denote the i-th function of constants. For ease of reference we note the following obvious but fundamental result.

(4.1) **Theorem.** *Each c_i preserves stopping times, $1\leqq i\leqq n$.*

Proof. If $(T_1, \dots, T_{n-1})$ is an $(n-1)$-vector of stopping times, then

$$c_i(T_1, \dots, T_{n-1})=\phi(T_1, \dots, \infty, \dots, T_n)$$

is a stopping time, since ϕ preserves stopping times.

(4.2) **Lemma.** *Suppose $\phi(\mathbf{t})>\mathbf{t}^*$ for all $\mathbf{t}$. Then $\phi(\mathbf{T})$ is previsible for all $\mathbf{T}$.*

Proof. Let $\phi_n(\mathbf{t})=(n-1/n)\,[\phi(\mathbf{t})-\mathbf{t}^*]+\mathbf{t}^*$. Then $\phi^n(\mathbf{T})$ announces $\phi(\mathbf{T})$.

(4.3) **Lemma.** *Suppose $\phi(\mathbf{t})\geqq\mathbf{t}^*$. Then ϕ preserves previsibility.*

Proof. Let $\Lambda=\{\mathbf{t}\colon \phi(\mathbf{t})>\mathbf{t}^*\}$, $A=\mathbf{T}^{-1}(\Lambda)$, and $\mathbf{T}^*=\sup_{1\leqq i\leqq n}(T_i)$. Then $R=\mathbf{T}^*_{A'}$ is previsible, since $A'\in\mathscr{F}_{R-}$. Since $\phi(\mathbf{T})=\phi(T_A^1, \dots, T_A^n)\wedge R$, it suffices to show $\phi(T_A^1, \dots, T_A^n)=\phi(\mathbf{T}_A)$ is previsible. Let $\psi(\mathbf{t})=\phi(\mathbf{t})\,1_\Lambda+2\mathbf{t}^*\,1_{\Lambda'}$. Then $\psi(\mathbf{T}_A)=\phi(\mathbf{T}_A)$ is previsible by Lemma (4.1).

(4.4) **Theorem.** *All ϕ preserve previsibility.*

Proof. (Induction on the number of variables.) Theorem (3.3) establishes the theorem for $n=1$. Let $\mathbf{T}$ be an n-vector of previsible times. Recall that c_i, $1\leqq i\leqq n$, are the functions of constants for ϕ. By Theorem (4.1) and the inductive hypothesis each c_i preserves previsibility. Let $R_i=c_i(\mathbf{T}_{(i)})$, and $A[i]=\{R_i<T_i\}$, $1\leqq i\leqq n$. Since T_i is previsible, we have $A[i]\in\mathscr{F}_{R_i-}$. We let R_i' denote the restriction of R_i to the set $A[i]$; that is, $R_i'=R_i$ *on* $A[i]$ and $+\infty$ otherwise. Each R_i' is previsible. Since

$$\phi(\mathbf{T})=\phi(T_1\wedge R_1, \dots, T_n\wedge R_n)\wedge\mathbf{R}'_*,$$

$\phi(\mathbf{T})$ is previsible if $\phi(T_1\wedge R_1, \dots, T_n\wedge R_n)$ is. Let $\psi(\mathbf{t})=\phi(\mathbf{t})\,1_\Delta+\mathbf{t}^*\,1_{\Delta'}$, where $\Delta=\{\mathbf{t}\colon t_i\leqq c_i(\mathbf{t}_{(i)})$ for each $i\}$. Then $\phi(T_1\wedge R_1, \dots, T_n\wedge R_n)=\psi(T_1\wedge R_1, \dots, T_n\wedge R_n)$ is previsible by Lemma (4.3).

(4.5) **Corollary.** *All ϕ preserve accessibility.*

Proof. Proceed as in Corollary (3.4).

The next lemma reveals some of the structure of ϕ which will be used to prove the results on total inaccessibility.

(4.6) **Lemma.** *Suppose there exists* $\mathbf{t}$ *such that* $\phi(\mathbf{t})=\alpha<\mathbf{t}_*$. *If* $\boldsymbol{\mu}_*>\alpha$, *then* $\phi(\boldsymbol{\mu})=\alpha$.

Proof. This is trivial if ϕ is a function of one variable. Let $\mathbf{t}$ be an n-dimensional vector with $\phi(\mathbf{t})=\alpha<\mathbf{t}_*$. Then $\phi(\mathbf{t})=c_1(\mathbf{t}_{(1)})=\alpha<\mathbf{t}_{(1)*}$. By the inductive hypothesis, if $\boldsymbol{\mu}_*>\alpha$ and hence $\boldsymbol{\mu}_{(1)*}>\alpha$, then $c_1(\boldsymbol{\mu}_{(1)})=\alpha<\boldsymbol{\mu}_{(1)*}$. Since also $\mu_1>\alpha$, we have $\phi(\boldsymbol{\mu})=c_1(\boldsymbol{\mu}_{(1)})=\alpha$.

(4.7) **Corollary.** *A necessary condition for* ϕ *to preserve total inaccessibility is that* $\phi(\mathbf{t})\geqq\mathbf{t}_*$ *for all* $\mathbf{t}$.

Proof. Suppose $\alpha=\phi(\mathbf{t})<\mathbf{t}_*$ for some $\mathbf{t}$. Let $\psi(t)=\phi(t,\dots,t)$. Then ψ must preserve total inaccessibility. But for any T, we have $\psi(T)=\alpha$ on $\{T>\alpha\}$ by Lemma (4.6). Let T be the first jump time of a Poisson process, which is totally inaccessible. Then $P\{\psi(T)=\alpha\}>0$, an impossibility.

For an n-vector $\mathbf{t}$, there are $h(n)$ possible lattice operations. We shall denote these by $L_k(\mathbf{t})$, $1\leqq k\leqq h(n)$.

(4.8) **Theorem.** *Let* $\phi(\mathbf{t})\geqq\mathbf{t}_*$ *and let* $\Delta=\{\mathbf{t}\colon \phi(\mathbf{t})<\infty$ *and* $\phi(\mathbf{t})\neq L_k(\mathbf{t})$, *any* $k\}$. *Let* $\mathbf{T}$ *be a vector of totally inaccessible times. Then* $\phi(\mathbf{T})$ *is totally inaccessible if and only if* $P\{\mathbf{T}\in\Delta\}=0$.

Proof (Sufficiency). It is elementary that the lattice operations preserve total inaccessibility. If S is an arbitrary previsible time, then

$$\begin{aligned}P\{\phi(\mathbf{T})=S<\infty\}&\leqq P\{\mathbf{T}\in\Delta\}+\sum_{k=1}^{h(n)}P\{S=L_k(\mathbf{T})<\infty\}\\&=0.\end{aligned}$$

(Necessity). Let $\Lambda_0=\{\mathbf{t}\colon \phi(\mathbf{t})>\mathbf{t}^*\}$. Let $\psi_0(\mathbf{t})=\phi(\mathbf{t})\,1_{\Lambda_0}+2\,\mathbf{t}^*\,1_{\Lambda_0'}$. Then $\psi(\mathbf{T})$ is previsible (Lemma (4.2)), and so $P\{\mathbf{T}\in\Lambda_0\}=0$.

Let $\Lambda[i]=\{\mathbf{t}\colon \mathbf{t}^*_{(i)}<\phi(\mathbf{t})<t_i\}$, $1\leqq i\leqq n$. Then on $\Lambda[i]$, $\phi(\mathbf{t})=c_i(\mathbf{t}_{(i)})>\mathbf{t}^*_{(i)}$. Defining $\psi_i(\mathbf{t}_{(i)})=c_i(\mathbf{t}_{(i)})\,1_{\Lambda[i]}+2\,\mathbf{t}^*_{(i)}\,1_{\Lambda[i]'}$, we have $\psi_i(\mathbf{T}_{(i)})$ previsible, so $P\{\mathbf{T}\in\Lambda[i]\}=0$.

Let $\mathbf{t}_{(i_1,\dots,i_k)}$ be the $(n-k)$-vector with the $i_1,\dots,i_k$ coordinates deleted. Let $c_{i_1,\dots,i_k}\colon \mathbb{R}_+^{n-k}\to\mathbb{R}_+$ be the i_k function of constants for the stopping time preserving function $c_{i_1,\dots,i_{k-1}}$, where c_{i_1} is the i_1 function of constants for ϕ. Suppose

$$\mathbf{t}^*_{(i_1,\dots,i_k)}<\phi(\mathbf{t})<\min(t_{i_1},\dots,t_{i_k}).$$

Then $\phi(\mathbf{t})=c_{i_1}(\mathbf{t}_{(i_1)})=c_{i_1,i_2}(\mathbf{t}_{(i_1,i_2)})<t_{i_2}$; letting

$$\Lambda[i_1,\dots,i_k]=\{\mathbf{t}\colon \mathbf{t}^*_{(i_1,\dots,i_k)}<\phi(\mathbf{t})<\min(t_{i_1},t_{i_2},\dots,t_{i_k})\},$$

we have that $\phi(\mathbf{t})=c_{i_1,\dots,i_k}(\mathbf{t}_{(i_1,\dots,i_k)})>\mathbf{t}^*_{(i_1,\dots,i_k)}$ on $\Lambda[i_1,\dots,i_k]$. So on

$$\{\mathbf{T}\in\Lambda[i_1,\dots,i_k]\},$$

we have $\phi(\mathbf{T})=\psi_{i_1,\dots,i_k}(\mathbf{T}_{(i_1,\dots,i_k)})$, a previsible time. Hence $P\{\Lambda[i_1,\dots,i_k]\}=0$. Since $\Delta=\bigcup\limits_{\substack{\text{all } i_1,\dots,i_k\\ 1\leqq k\leqq n}}\Lambda[i_1,\dots,i_k]$, we have $P\{\mathbf{T}\in\Delta\}=0$. QED

Suppose $\Lambda\in 2^{\mathbb{R}_+^n}$, and $\mathbf{t}=(t_1,\dots,t_{n-1})$ is a vector in $\mathbb{R}_+^{n-1}$. We will denote the $\mathbf{t}_{(n)}$ section of Λ by $\Lambda(\mathbf{t}_{(n)})=\{t_n\colon (\mathbf{t}_{(n)},t_n)\in\Lambda\}$. We let

$$\rho(\Lambda)=\{(t_1,\dots,t_{n-1})\colon \Lambda(t_1,\dots,t_{n-1}) \text{ is uncountable}\}$$

denote the essential projection of Λ onto $\mathbb{R}_+^{n-1}$. We will now give a lemma that we will use in the proof of Theorem (4.10). It generalizes a result given in Dellacherie [2, p. 135]. If $\mathscr{B}$ are the Borel sets in $\mathbb{R}_+^n$, we let $\mathscr{B}^*$ be the universally measurable sets; that is, $\mathscr{B}^* = \bigcap_{\mu \text{ finite}} \mathscr{B}^\mu$, where $\mathscr{B}^\mu$ is the completion of $\mathscr{B}$ by the measure μ.

(4.9) **Lemma.** *Suppose $\Lambda \in \mathscr{B}^*$. Then $\rho(\Lambda)$ is universally measurable as a subset of $\mathbb{R}_+^{n-1}$.*

Proof. We know that $H \in \mathscr{B}$ implies $\rho(H)$ is universally measurable as a subset of $\mathbb{R}_+^{n-1}$ (cf. Dellacherie [2, p. 135]). Let ν be any finite measure on $\mathscr{B}(\mathbb{R}_+^{n-1})$, the Borel sets of $\mathbb{R}_+^{n-1}$, μ a finite measure on $\mathscr{B}(\mathbb{R}_+)$, and $\tau = \mu \times \nu$. Then since $\Lambda \in \mathscr{B}^*$, there exist $B_2, B_1 \in \mathscr{B}$ such that $B_2 \supseteqq \Lambda \supseteqq B_1$ and $\tau(B_2 - B_1) = 0$. Let ω denote a point in $\mathbb{R}_+^{n-1}$, $\Gamma = B_2 - B_1$, and

$$\Delta = \{\omega : \int 1_{\Gamma(\omega)}(x)\, \mu(dx) > 0\}.$$

Since $\tau(B_2 - B_1) = 0$, we have $\nu(\Delta) = 0$. But $\Delta \supseteqq \rho(B_2) - \rho(B_1)$, so $\nu[\rho(B_2) - \rho(B_1)] = 0$. Since $\rho(B_2)$, $\rho(B_1)$ are universally measurable and $\rho(B_2) \supseteqq \rho(\Lambda) \supseteqq \rho(B_1)$, we have $\rho(\Lambda) \in \mathscr{B}(\mathbb{R}_+^{n-1})^\nu$.

(4.10) **Theorem.** *Suppose $\phi(\mathbf{t}) \geqq \mathbf{t}_*$. Let $\Sigma = \{\mathbf{t} : \phi(\mathbf{t}) = L_k(\mathbf{t})$, some k, $1 \leqq k \leqq h(n)\}$. Let $\Lambda = \mathbb{R}_+^n \setminus \Sigma$. Let $(a_1, \ldots, a_n)$ be the j-th permutation of the labelling of the n axes. We define the following sets, where Γ_{n-k}^j is a subset of $\mathbb{R}_+^{n-k}$:*

$$\Gamma_{n-1}^j = \left\{ \mathbf{t}_{(a_n)} : \Lambda(\mathbf{t}_{(a_n)}) \cap \prod_{k=1}^{n-1} [t_{a_k}, \infty) \text{ is uncountable} \right\}$$

$$\Gamma_{n-2}^j = \left\{ \mathbf{t}_{(a_n, a_{n-1})} : \Gamma_{n-1}^j(\mathbf{t}_{(a_n, a_{n-1})}) \cap \prod_{k=1}^{n-2} [t_{a_k}, \infty) \text{ is uncountable} \right\}$$

$$\vdots$$

$$\Gamma_1^j = \{t_{a_1} : \Gamma_2^j(t_{a_1}) \cap [t_{a_1}, \infty) \text{ is uncountable}\}.$$

Then ϕ preserves total inaccessibility if and only if Γ_1^j is countable for each j, $1 \leqq j \leqq n!$.

Proof. By rearrangement if necessary, we take $L_k(\mathbf{t})$ to be the lattice operation on the coordinates of $\mathbf{t}$ that exhibits the $(n-k)$-th largest coordinate. Let

$$\Delta(j) = \{T_{a_1} \leqq T_{a_2} \leqq \cdots \leqq T_{a_n}\}, \qquad 1 \leqq j \leqq n!$$

where $(a_1, \ldots, a_n)$ is the j-th permutation of the labelling of the axes. Let

$$\phi_j(\mathbf{t}) = \phi(L_{a_1}(\mathbf{t}), \ldots, L_{a_n}(\mathbf{t})).$$

Then

$$\phi(\mathbf{T}) = \min_{1 \leqq j \leqq n!} \{\phi_j(\mathbf{T})_{\Delta(j)}\}.$$

Note that $\Delta(j) \in \mathscr{F}_{T_*}$, and $\phi(\mathbf{t}) \geqq \mathbf{t}_*$, so $\phi_j(\mathbf{T})_{\Delta(j)}$ is a stopping time. Therefore it suffices to consider only the case where the stopping times are totally ordered.

We also note that if $0 \leqq T_1 \leqq \cdots \leqq T_n$ are random variables and $(\mathscr{F}_t^\circ)$ is the minimal filtration making them stopping times then $(\mathscr{F}_t^\circ)$ is already right continuous. Let $\mathscr{F}_t$ be the completed σ-fields. Let N_t be the counting process for (T_i),

$1 \leqq i \leqq n$. The dual previsible projection $\hat{N}_t$ (or "compensator") of N_t has been calculated by Chou and Meyer [1, p. 234]. It is continuous if and only if the distribution F_1 of T_1 and all the conditional distributions

$$F_i(t_1, \dots, t_{i-1}; t_i) = P(T_i \leqq t_i;\ T_1 = t_1, \dots, T_{i-1} = t_{i-1})$$

are continuous. It is well known (cf. Dellacherie [2, p. 111]) that $\hat{N}_t$ is continuous if and only if the jumps of N_t are totally inaccessible. So $T_1, \dots, T_n$ are totally inaccessible times if and only if $t \to F_1(t)$ and $t \to F_i(t_1, t_2, \dots, t_{i-1}; t)$ are continuous for each fixed $(t_1, \dots, t_{i-1})$, $2 \leqq i \leqq n$.

(Proof of Sufficiency). Suppose each Γ_1^j is countable. By symmetry it suffices to show $\phi_j(\mathbf{T})_{\Lambda(j)}$ is totally inaccessible, assuming $\mathbf{T}$ is. So we may assume the stopping times are totally ordered. We fix a permutation $(a_1, \dots, a_n)$ and by abuse of notation we identify a_i with i. By Theorem (4.8) it suffices to show $P\{\mathbf{T} \in \Lambda\} = 0$. We have

$$P\{\mathbf{T} \in \Lambda\} = \int 1_\Lambda(t_1, \dots, t_n)\, F_n(t_1, \dots, t_{n-1}; dt_n) \dots F_2(t_1; dt_2)\, F_1(dt_1).$$

Since $\mathbf{T}$ is totally inaccessible, we must have $t \to F_i(t_1, \dots, t_{i-1}; t)$ continuous, $1 \leqq i \leqq n$, regardless of the filtration. So the conditional distributions give rise to diffuse kernels. Therefore

$$\int 1_\Lambda(\mathbf{t})\, F_n(t_1, \dots, t_{n-1}; dt_n) \dots F_{n-i}(t_1, \dots, t_{n-i-1}; dt_{n-i}) = 0$$

unless $(t_1, \dots, t_{n-i-1}) \in \Gamma_{n-i-1}^j$, where the j corresponds to the permutation we chose. Finally, we have

$$\int 1_\Lambda(\mathbf{t})\, F_n(t_1, \dots, t_{n-1}; dt_n) \dots F_2(t_1; dt_2) = 0$$

unless $t_1 \in \Gamma_1^j$. But Γ_1^j is countable, so $F_1(dt_1)$ does not charge it. Hence $P\{\mathbf{T} \in \Lambda\} = 0$.

(Necessity). Suppose ϕ preserves total inaccessibility and $\mathbf{T}$ is an n-vector of totally inaccessible times. We choose an arbitrary j and we will show Γ_1^j is countable. By our previous discussion, we may assume $\mathbf{T}$ is totally ordered, i.e. that $\Lambda(j) = \Omega$. We now suppress j and identify $(a_1, \dots, a_n)$, our j-th permutation, with $(1, \dots, n)$; that is, we assume $0 \leqq T_1 \leqq T_2 \leqq \dots \leqq T_n$.

Since Λ is Borel, Γ_{n-1} is universally measurable. By Lemma (4.9), Γ_k is universally measurable in $\mathbb{R}_+^k$, $1 \leqq k \leqq n-2$.

Suppose that Γ_1 is uncountable. Suppose also that there is no diffuse measure on $\mathbb{R}_+$ that gives mass to Γ_1. Let ν be an arbitrary diffuse measure and $\varepsilon > 0$. Then there exists a compact set $\Sigma_{\varepsilon,\nu} \supseteqq \Gamma_1$ such that $\nu(\Sigma_{\varepsilon,\nu}) < \varepsilon$. Let

$$\Sigma_\varepsilon = \bigcap_{\text{all diffuse } \nu} \Sigma_{\varepsilon,\nu}.$$

Then $\Sigma_\varepsilon \supseteqq \Gamma_1$ is compact and $\nu(\Sigma_\varepsilon) < \varepsilon$, for all ν. Let ε_n decrease to 0, $\Sigma_{(n)} = \Sigma_{\varepsilon_1} \cdots \Sigma_{\varepsilon_n}$, and $\Sigma = \bigcap_n \Sigma_{(n)}$. Then $\nu(\Sigma) = 0$ for all diffuse ν, $\Sigma \supseteqq \Gamma_1$, and Σ is compact. Since Σ is uncountable, we know Σ contains a non-empty perfect set, hence there exists a diffuse measure which charges Σ, a contradiction. So Γ_1 uncountable implies there exists a diffuse measure which charges it; take it to be $F_1(dt_1)$.

Now suppose there exist diffuse kernels $F_i(t_1, \ldots, t_{i-1}; dt_i)$, $1 \leqq i \leqq k-1$, such that

$$\int 1_{\Gamma_{k-1}} F_{k-1}(t_1, \ldots, t_{k-2}; dt_{k-1}) \ldots F_1(dt_1) > 0.$$

Then there must exist a diffuse kernel F_k such that

$$(4.11) \quad \nu(\Gamma_k) = \int 1_{\Gamma_k}(t_1, \ldots, t_k) F_k(t_1, \ldots, t_{k-1}; dt_k) \ldots F_1(dt_1) > 0.$$

Suppose not. Then as we have shown above, we can find a compact set $\Sigma \supseteqq \Gamma_k$ such that $\nu(\Sigma)=0$ for all such ν. We know (see Dellacherie [2, p. 136]) that we can put a measurable set K in Σ such that $K(t_1, \ldots, t_{k-1})$ is perfect or empty, and the projection of K onto $\mathbb{R}_+^{k-1}$ equals the essential projection of Σ onto $\mathbb{R}_+^{k-1}$. A result of Dellacherie [2, p. 140] tells us there exists a continuous bounded kernel $F_k(t_1, \ldots, t_{k-1}; dt_k)$ whose support is $K \subseteqq \Sigma$. This is a contradiction. Hence there must exist a diffuse kernel F_k such that (4.11) holds.

Taking the kernels F_i, $1 \leqq i \leqq n$ to be the conditional distributions of (T_i), $1 \leqq i \leqq n$, we have shown $P\{(T_1, \ldots, T_n) \in \Lambda\} > 0$. Taking the filtration $\mathscr{F}_t$ to be the completion under P of the minimal filtration $\mathscr{F}_t^\circ$ making all T_i, $1 \leqq i \leqq n$, stopping times, the continuity of the F_i assures us the (T_i) are totally inaccessible. An application of Theorem (4.8) completes the proof.

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