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Martin-Dynkin Boundary of Mixed Poisson Processes

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§ 0. Introduction

Recently Dynkin [2–4] gave a direct stochastic approach to the construction of the Martin boundary in the case of Markov processes which has proven quite fruitful when adapted to specified stochastic fields (Föllmer [5], Miyamoto [12]). In particular one can derive general integral representations of stochastic fields as mixtures of phases even if the Markov property is dropped. Our purpose here is to characterize the class of mixed Poisson processes with infinite intensity measures as canonical Gibbs states in the sense of Georgii [6], so that Dobrushin's theorem [1], which characterizes the mixed Poisson process as the limiting process of random translations, may be viewed as a special case for the present discussion about time development of infinite particle systems which have Gibbs states as limiting processes. Since canonical specifications fit into the general setting of specified stochastic fields which have been considered by Föllmer [5] and Preston [14] we can apply Dynkin's method, and this gives us some new results for mixed Poisson processes. Our main results (Theorem 1 and Theorem 2) may be viewed as a generalization of de Finetti's theorem on exchangeable 0–1 variables [17] to point processes with a general state space.

§ 1. Stochastic Fields and Martin-Dynkin Boundary

We first recall the definition of a stochastic field. (cf. Föllmer [5]) Let $(\Omega, \mathscr{F})$ be a standard Borel space, $\mathscr{V}$ an index-set ordered by a relation $\subseteq$, and $(\hat{\mathscr{F}}_V)_{V \in \mathscr{V}}$ a decreasing family of sub-σ-fields of $\mathscr{F}$. For each $V \in \mathscr{V}$ let Π_V be a probability kernel on $(\Omega, \mathscr{F})$ such that

$$\Pi_V(\cdot, A) \quad \text{is } \hat{\mathscr{F}}_V\text{-measurable } (A \in \mathscr{F}); \tag{1.1}$$

$$\Pi_V(\cdot, A) = \mathbb{1}_A \quad \text{if } A \in \hat{\mathscr{F}}_V; \tag{1.2}$$

suppose further that the collection $\Pi = (\Pi_V)_V$ satisfies the consistency condition

$$\Pi_W \Pi_V = \Pi_W \quad (V, W \in \mathscr{V}, V \subseteq W). \tag{1.3}$$

Now call any probability measure P on $(\Omega, \mathscr{F})$ which is compatible with Π in the sense

$$E_P(A \mid \hat{\mathscr{F}}_V) = \Pi_V(., A) \qquad P\text{-a.s.}\ (A \in \mathscr{F}, V \in \mathscr{V}) \tag{1.4}$$

a *stochastic field with local characteristic* Π. The set $\mathfrak{G} = \mathfrak{G}(\Pi)$ of all such fields is convex: its extremal points are called *phases of* Π. If there is more than one element in $\mathfrak{G}(\Pi)$ then the stochastic field P is not uniquely determined by its local characteristics. This case will be called a *phase transition.*

Given a collection $\Pi = (\Pi_V)_{V \in \mathscr{V}}$ of local characteristics over $(\Omega, \mathscr{F})$ one constructs the Martin-Dynkin boundary in the following way. Suppose that $\mathscr{V}$ has a countable base; fix a polish topology on Ω compatible with $\mathscr{F}$, and thereby a polish topology on the set $\mathfrak{P}\Omega$ of all probability measures on $(\Omega, \mathscr{F})$. Let $\mathfrak{G}_\infty = \mathfrak{G}_\infty(\Pi)$ be the set of all limits

$$\lim_n \Pi_{V_n}(\omega_n, .)$$

where (V_n) is some countable base of $\mathscr{V}$ and (ω_n) some sequence in Ω. $\mathfrak{G}_\infty$ is complete in $\mathfrak{P}\Omega$, and thus a polish space whose Borel field we denote by $\mathscr{G}_\infty$. The measurable space $(\mathfrak{G}_\infty, \mathscr{G}_\infty)$ is called *Martin-Dynkin boundary of* Π.

§ 2. Mixed Poisson Processes

We now give an example of a specified stochastic field which turns out to be the class of mixed Poisson processes. The idea of the construction is that this field is locally a Poisson point process given the particle number of the realizations of the process. To be more precise, such a process is defined as follows.

Let X be a locally compact second countable Hausdorff topological space and $\mathscr{B}$ the σ-field of Borel sets in X, i.e. the σ-field generated by the open sets in X. A subset of X is called bounded if its closure is compact. Denote by $\mathscr{B}_0$ the set of all bounded Borel subsets of X. Let $\mathfrak{M}$ be the set of all non-negative Radon-measures in X and $\mathscr{F}$ the smallest σ-field of subsets of $\mathfrak{M}$ making all functions $N(., B): \mathfrak{M} \to \mathbb{R}$, $N(\mu, B) = \mu(B)$ $(B \in \mathscr{B}_0)$ measurable functions of μ and, for $G \in \mathscr{B}_0$, $\mathscr{F}_G$ will denote the sub-σ-field generated by all $N(., G')$ with $G' \subseteq G$, i.e. the sub-σ-field of 'events occurring in G'. $\mathfrak{M}$ can be made into a polish space in such a way that $\mathscr{F}$ coincides with the σ-field generated by the open subsets of $\mathfrak{M}$. For details see [13]. If ρ is a Radon measure on $(X, \mathscr{B})$, then the *Poisson process with intensity* ρ is the probability measure in $(\mathfrak{M}, \mathscr{F})$ such that, if $G_1, \ldots, G_n$ are pairwise disjoint bounded Borel sets, then the random variables $N(., G_1), \ldots, N(., G_n)$ are independent and each $N(., G_i)$ has a Poisson distribution with parameter $\rho(G_i)$. We denote this probability measure by P_ρ. Let $\mathfrak{M}^{\cdot\cdot}$ be the set of all non-negative integer-valued Radon measures on X and $\mathscr{F}^{\cdot\cdot} = \mathfrak{M}^{\cdot\cdot} \cap \mathscr{F}$. It is well known that P_ρ is a *point process*, that is P_ρ is concentrated on $\mathfrak{M}^{\cdot\cdot}$, and that P_ρ is a *simple point process* (i.e. concentrated on all Radon point measures on X) if and only if the intensity measure ρ is diffuse (for details see Krickeberg [10]). Let T_G: $\mathfrak{M} \to \mathfrak{M}$ be the 'projection' $T_G \mu = \mathrm{Res}_G \mu$ $(G \in \mathscr{B}_0)$ and call

$$P_{\rho, G}(\varphi) = P_\rho(\varphi \circ T_G) \qquad (\varphi \in L_1(P_\rho)) \tag{2.1}$$

the Poisson process with intensity ρ restricted to G. It is well known (see for instance Krickenberg [9]) that

$$P_{\rho,G}(\varphi)=e^{-\rho(G)}\cdot\sum_{n\geqq 0}\frac{1}{n!}\int_G\cdots\int_G\varphi(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n})\,\rho(dx_1)\ldots\rho(dx_n) \tag{2.2}$$

for any bounded measurable function φ on $\mathfrak{M}$.

By the Fourier transform of a probability measure P on $\mathfrak{M}$ we mean the mapping

$$\hat{P}(f)=\int_{\mathfrak{M}}\exp(i\cdot v(f))\,P(dv)$$

of the set $\mathscr{K}(X)$ of all continuous f with compact support into the complex plane. If $f\in\mathscr{K}(X)$, $f\geqq 0$, the Fourier transform

$$\hat{P}(if)=\int_{\mathfrak{M}}\exp(-v(f))\,P(dv)$$

is well defined, and is called the Laplace transform of P evaluated for the function f. It is well known that the Laplace transform of P determines P and that the Laplace transform of the Poisson process P_ρ is given by

$$\hat{P}_\rho(if)=\exp(\rho(\exp(-f)-1), \qquad f\in\mathscr{K}^+(X)$$

(see Krickeberg [10]).

We fix now an *infinite* measure $\rho\in\mathfrak{M}$ and define

$$\Pi_G(\mu, A_1\cap A_2)=\mathbb{1}_{A_2}(\mu)\cdot P_{\rho,G}(A_1|N(.,G)=\mu(G)), \tag{2.3}$$

where $\mu\in\mathfrak{M}^{\cdot\cdot}$, $G\in\mathscr{B}_0$, $A_1\in\mathscr{F}_G^{\cdot\cdot}$, $A_2\in\mathscr{F}_{X\setminus G}^{\cdot\cdot}$. Π_G can be extended in the usual way to a Markovian kernel on $(\mathfrak{M}^{\cdot\cdot}\,\mathscr{F}^{\cdot\cdot})$. It holds in general for any bounded $\mathscr{F}^{\cdot\cdot}$-measurable function φ on $\mathfrak{M}^{\cdot\cdot}$ that

$$\Pi_G(\mu,\varphi)=\int_{\mathfrak{M}_G^{\cdot\cdot}}\varphi(\xi+\mu_{X\setminus G})\,P_{\rho,G}(d\xi\,|\,N(.,G)=\mu(G)).$$

Here $\mathfrak{M}_G^{\cdot\cdot}$ denotes the set of all $\mu\in\mathfrak{M}^{\cdot\cdot}$ with support in G.

Let $\hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot}$ be the σ-field generated by all sets of the form

$$\{\mu\in\mathfrak{M}^{\cdot\cdot}\colon N(\mu,G)=n\}\cap A \qquad (A\in\mathscr{F}_{X\setminus G}^{\cdot\cdot},\ n\in\mathbb{N}).$$

It is not difficult to see that the collection $\Pi=(\Pi_G)_{G\in\mathscr{B}_0}$ satisfies the conditions (1.1), (1.2) and (1.3), that is

$$\Pi_G(.,A) \text{ is } \hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot}\text{-measurable } (A\in\mathscr{F}^{\cdot\cdot}); \qquad \Pi_G(.,A)=\mathbb{1}_A \qquad (A\in\hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot}); \tag{2.4}$$

$$\Pi_G(.,A)=\mathbb{1}_A \qquad (A\in\hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot}); \tag{2.5}$$

$$\Pi_{G'}\,\Pi_G=\Pi_{G'} \quad (G,G'\in\mathscr{B}_0,\ G\subseteq G'). \tag{2.6}$$

Thus using the explicit formula (2.2) for $P_{\rho,G}$ a stochastic field P with local characteristic Π is given by

$$\begin{aligned} &P(A_1A_2\,|\,\hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot})(\mu)\\ &=\mathbb{1}_{A_2}(\mu)\cdot\frac{1}{\rho(G)^{\mu(G)}}\cdot\int_G\cdots\int_G\mathbb{1}_{A_1}(\varepsilon_{x_1}+\cdots+\varepsilon_{x_{\mu(G)}})\,\rho(dx_1)\ldots\rho(dx_{\mu(G)})\end{aligned} \tag{2.7}$$

P-a.s. in μ for each $G\in\mathscr{B}_0$, $A_1\in\mathscr{F}_G^{\cdot\cdot}$, $A_2\in\mathscr{F}_{X\setminus G}^{\cdot\cdot}$. In the terminology of Georgii [6] P is called a *canonical Gibbs state for the local characteristic Π.*

Let $\mathfrak{N}$ be the set of all measures $\mu\in\mathfrak{M}^{\cdot\cdot}$ which are *well-distributed with parameter $Y(\mu)$* with respect to ρ in the sense of Goldman [8]. Thus $\mu\in\mathfrak{N}$ iff for every expanding sequence $\{G_n\}$ of bounded Borel subsets of X with $G_1\subseteq \mathring{G}_2\subseteq G_2\subseteq\cdots$ exausting X, we have

$$\lim\frac{N(\mu, G_n)}{\rho(G_n)}=Y(\mu).$$

Recall that a point process P in X is called well-distributed with respect to ρ if its sample points are well-distributed with probability one (i.e. $P\mathfrak{N}=1$). Note that $\mathfrak{N}\in\mathscr{F}^{\cdot\cdot}$.

Remarks. (1) It is clear from the definition of Π that the boundary of Π includes the set of all Poisson processes $P_{z\cdot\rho}$, $z\geqq 0$, according to the well known theorem of von Waldenfels [16] which states that $P_{\rho,G_n}(.\,|\,N(.,G_n)=\mu(G_n))$ converges weakly towards $P_{Y(\mu)\cdot\rho}$ as $n\to+\infty$, $\mu\in\mathfrak{N}$.

(2) To see that $\mathfrak{G}(\Pi)$ is non empty, note that $P_{z\rho}\in\mathfrak{G}(\Pi)$ for any $z\geqq 0$. For this we have to check that

$$P_{z\rho}(A_1A_2B)=\int_B \mathbb{1}_{A_2}(\mu)\,P_{\rho,G}(A_1\,|\,N(.,G)=\mu(G))\,P_{z\rho}(d\mu)$$

for any $A_1\in\mathscr{F}_G^{\cdot\cdot}$, $A_2\in\mathscr{F}_{X\setminus G}^{\cdot\cdot}$, $B\in\hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot}$. It is sufficient to show this for sets B of the form $B=\{N(.,G)=n\}\cap C$ with $C\in\mathscr{F}_{X\setminus G}^{\cdot\cdot}$. Using well known properties of the Poisson process the assertion follows at once.

We now briefly follow Dynkin's construction of the boundary of Π, in the form elaborated by Föllmer [5], combined with the theorem of von Waldenfels mentioned above, in order to see how it works in our particular case.

Fix a sequence of bounded Borel subsets G_n of X with $G_1\subseteq\mathring{G}_2\subseteq G_2\subseteq\cdots$ exausting X and let $P\in\mathfrak{G}(\Pi)$. Then we have for any $\varphi\in L_1(P)$

$$P(\varphi\,|\,\hat{\mathscr{F}}_\infty^{\cdot\cdot})=\lim P(\varphi\,|\,\hat{\mathscr{F}}_{X\setminus G_n}^{\cdot\cdot})=\lim \Pi_{G_n}\varphi \qquad P\text{-a.s.,} \tag{2.8}$$

where $\hat{\mathscr{F}}_\infty^{\cdot\cdot}$ denotes the tail-field $\bigcap_n \hat{\mathscr{F}}_{X\setminus G_n}^{\cdot\cdot}=\bigcap_{B\in\mathscr{B}_0}\hat{\mathscr{F}}_{X\setminus G}^{\cdot\cdot}$. This implies, P-a.s. in μ, the existence of

$$\rho(\mu)=\lim \Pi_{G_n}(\mu,.)\quad\text{and}\quad \rho(\mu)\in\mathfrak{G}_\infty(\Pi).$$

We now give the explicit form of the limits $\rho(\mu)$ by simulating the proof of von Waldenfels' theorem ([16], Satz 4).

Proposition 1. *Every canonical Gibbs state P with local characteristic Π is well-distributed and*

$$\lim \Pi_{G_n}(\mu,.)=P_{Y(\mu)\cdot\rho}\qquad P\text{-a.s. in } \mu. \tag{2.9}$$

Proof. Let N be the set of all $\mu\in\mathfrak{M}^{\cdot\cdot}$ for which $\lim \Pi_{G_n}(\mu,.)$ exists, and fix a continuous function f on X with compact support such that $\int_X(\exp(-f)-1)\,d\rho=$

$a \neq 0$. Then the Laplace transform of $\Pi_{G_n}(\mu, .)$, $\mu \in \mathfrak{M}^{\cdot\cdot}$, evaluated for f is

$$\widehat{\Pi_{G_n}(\mu, .)}(if) = \int_{\mathfrak{M}} \exp(-\nu(f))\, \Pi_{G_n}(\mu, d\nu)$$
$$= \frac{1}{\rho(G_n)^{\mu(G_n)}} \cdot \int_{G_n} \cdots \int_{G_n} \exp\left(-\sum_{j=1}^{\mu(G_n)} f(x_j) - \mu_{X \setminus G_n}(f)\right) \rho(dx_1) \ldots \rho(dx_{\mu(G_n)})$$
$$= \exp(-\mu_{X \setminus G_n}(f)) \cdot \left[\frac{1}{\rho(G_n)} \cdot \int_{G_n} \exp(-f)\, d\rho\right]^{\mu(G_n)}.$$

If n is large, G_n contains the support of f, and then the integral $\int_{G_n} \exp(-f) d\rho$ does not depend on n and $\mu_{X \setminus G_n}(f) = 0$. Therefore

$$\begin{aligned} \widehat{\Pi_{G_n}(\mu, .)}(if) &= \left[1 + \frac{1}{\rho(G_n)} \cdot \int_X (\exp(-f) - 1)\, d\rho\right]^{\mu(G_n)} \\ &= \left[\left(1 + \frac{a}{\rho(G_n)}\right)^{\rho(G_n)}\right]^{\frac{\mu(G_n)}{\rho(G_n)}}. \end{aligned} \tag{2.10}$$

Now let $\mu \in N$. Since $\Pi_{G_n}(\mu, .)$ converges weakly, the left hand side of (2.10) converges. On the other hand $\left(1 + \frac{a}{\rho(G_n)}\right)^{\rho(G_n)}$ converges against $\exp a \neq 1$. Therefore

$$\lim \frac{\mu(G_n)}{\rho(G_n)} = Y(\mu) \tag{2.11}$$

exists and does not depend on the choice of the sequence $\{G_n\}$. So we have proved $N \subseteq \mathfrak{N}$; in particular P is well-distributed.

In view of (2.11), combined with (2.10), we have for *any* continuous function f on X with compact support and $\mu \in N$

$$\widehat{\rho(\mu)}(if) = \widehat{\lim \Pi_{G_n}(\mu, .)}(if) = \exp(Y(\mu) \cdot \rho(\exp(-f) - 1)). \tag{2.12}$$

The right hand side of (2.12) is the Laplace transform of $P_{Y(\mu) \cdot \rho}$ evaluated for f, so that $\rho(\mu) = P_{Y(\mu) \cdot \rho}$. ┘

Combining (2.8) and (2.9) we have

$$P(\varphi \,|\, \hat{\mathscr{F}}_\infty^{\cdot\cdot}) = P_{Y(\cdot) \cdot \rho}(\varphi) \qquad P\text{-a.s.} \tag{2.13}$$

We thus have constructed a general conditional probability distribution relative to $\hat{\mathscr{F}}_\infty^{\cdot\cdot}$ for all $P \in \mathfrak{G}(\Pi)$. Since (2.13) implies that for any bounded, $\hat{\mathscr{F}}_\infty^{\cdot\cdot}$-measurable $\hat{\varphi}$

$$P(\hat{\varphi} \cdot f(Y)) = P(\hat{\varphi} \cdot P_{Y \cdot \rho}(f(Y)))$$

for any bounded measurable function f on $\mathbb{R}_+$, we have

$$P_{Y(\cdot) \cdot \rho}(Y = Y(.)) = 1 \qquad P\text{-a.s.} \tag{2.14}$$

Let us call

$$\Delta_\rho = \{z \cdot \rho \,|\, z \geqq 0,\ P_{z \cdot \rho}(Y = z) = 1\} \tag{2.15}$$

the *essential part of the boundary of* Π.[1] If we define a probability measure V_P on Δ_ρ by

$$V_P(A)=P(Y\cdot\rho\in A) \qquad (A\in\Delta_\rho\cap\mathscr{F}^{\cdot\cdot}), \tag{2.16}$$

then it has been shown above that $V_P(\Delta_\rho)=1$. Thus (2.13) implies that for any $\varphi\in L_1(P)$

$$P(\varphi)=P(P_{Y\cdot\rho}(\varphi))=\int_{\Delta_\rho} P_\delta(\varphi)\,V_P(d\delta). \tag{2.17}$$

It is not difficult to see that, conversely, any probability measure V on Δ_ρ induces a canonical Gibbs state for Π, which we denote by P_V for short.

Proposition 2. *The essential part of the boundary of* Π *consists of all measures* $z\cdot\rho$, $z\geqq 0$, *i.e.*

$$\Delta_\rho=\{z\cdot\rho;\ z\geqq 0\}.$$

Proof. According to Remark (2) any Poisson process $P_{z\rho}$, $z\geqq 0$, belongs to $\mathfrak{G}(\Pi)$. Therefore

$$P_{z\rho}=\int_{\Delta_\rho} P_\delta\, V_{P_{z\rho}}(d\delta)=P_{V_{P_{z\rho}}}.$$

As one can prove in the theory of Cox processes (see Knopsmeier [18]) this implies $1_{\Delta_\rho} V_{P_{z\rho}}=\varepsilon_{z\rho}$. This is only possible if $z\cdot\rho\in\Delta_\rho$. ⌟

We recall that a *mixed Poisson process* is rougly a Poisson process not constructed on a fixed measure ρ but rather on a random measure which is concentrated on the set of all measures $z\cdot\rho$, $z\geqq 0$: first a realization of the random measure is determined and then a Poisson process is constructed, having this realization as intensity. More precisely: if V is a probability measure concentrated on $\Delta_\rho=\{z\cdot\rho\colon z\geqq 0\}$, then a mixed Poisson process built on V is given by

$$P(\varphi)=\int P_\delta(\varphi)\,V(d\delta) \qquad (\varphi\in L_1(P)).$$

To summarize, we have shown the following:

Theorem 1. *Let* P *be a point process in* X *and* ρ *an infinite Radon measure on* X. *Then the following conditions are equivalent:*

(1) P *is a canonical Gibbs state for the local characteristic* Π.

(2) P *is a mixed Poisson process built on a probability measure* V *on* Δ_ρ.

Specializing Corollary (3.13) of Föllmer [5] or Lemma 2.1 of Dynkin [3] we have

Theorem 2. *Let* ρ *be an infinite Radon measure on* X *and* P *a mixed Poisson process built on a probability measure* V *on* Δ_ρ. *Then the following conditions are equivalent:*

(1) P *is an extreme point of* $\mathfrak{G}(\Pi)$.

(2) $P=P_{Y(.)\rho}$ P-a.s.

(3) P *is a Poisson process with intensity measure* $z\cdot\rho$, $z\geqq 0$.

(4) $P(A)\in\{0,1\}$ $(A\in\hat{\mathscr{F}}^{\cdot\cdot}_\infty)$; *i.e.* P *is ergodic with respect to the tail field.*

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[1] In fact here we identify $z\cdot\rho$ with $P_{z\cdot\rho}$

Corollaries. *Let ρ, ρ_1, ρ_2 be infinite Radon measures on X. Then we have:*

(1) $P_{z\rho}$ *is ergodic with respect to* $\mathscr{F}_\infty^{\cdot\cdot} = \bigcap_{G\in\mathscr{B}_0} \mathscr{F}_{X\setminus G}^{\cdot\cdot}$.

(2) $P_{z\rho}(Y=z)=1$ *for any* $z\geqq 0$.

(3) $P_{z_1\rho}\perp P_{z_2\rho}$ *for* $z_1, z_2\geqq 0$ *with* $z_1 \neq z_2$.

(4) $P_{\rho_1}\perp P_{\rho_2}$ *if either* $\dfrac{\rho_1(G_n)}{\rho_2(G_n)}$ *does not converge or converges towards a real positive* $z\neq 1$.

Proof. For (1) note that $\mathscr{F}_\infty^{\cdot\cdot}\subseteq\hat{\mathscr{F}}_\infty^{\cdot\cdot}$. (2) and (3) follow easily from the equivalence (2)$\Leftrightarrow$(3) of Theorem 2. To see (4) note that P_{ρ_1} and P_{ρ_2} are concentrated on the disjoint sets $\left\{\mu\colon \lim \dfrac{\mu(G_n)}{\rho_i(G_n)}=1\right\}$, $i=1,2$. $\lrcorner$

Remarks. (1) Canonical Gibbs states are well suited to serve as a statistical model in the following sense: in this model the mapping Y is an *exact estimator for the particle density of the underlying process,* that is (see (2.14))

$$P_{Y(.)\rho}(Y=Y(.))=1 \qquad P\text{-a.s.}$$

This means roughly that you can estimate the particle density of the underlying process from a *single* realization of the process. If $P=P_{z\rho}$ (i.e. the underlying process is not a mixture but a pure state) then Y is an exact estimator in the usual sense.

(2) Corollary (4) can be used to generalize our considerations in § 2 in the following way: Let M be a set of infinite Radon measures on X with the property

$$\left(\rho_1, \rho_2\in M, \rho_1 \neq \rho_2 \Rightarrow \frac{\rho_1(G_n)}{\rho_2(G_n)} \text{ does not converge or converges towards a real positive } z\neq 1\right). \tag{2.18}$$

Thus for $\rho_1, \rho_2\in M$, $\rho_1\neq\rho_2$ we have $P_{\rho_1}\perp P_{\rho_2}$. Let $\mathfrak{N}_\rho$ be the set of all point measures μ with $\dfrac{\mu(G_n)}{\rho(G_n)}\to 1$, where G_n is again an expanding sequence of bounded subsets of X with $G_1\subseteq\mathring{G}_2\subseteq G_2\subseteq\ldots$ exausting X, and let $\Omega=\bigcup_{\rho\in M}\mathfrak{N}_\rho$. In view of (2.18) this union is disjoint, so that we can define a mapping $\chi\colon \Omega\to M$ by $\chi(\mu)=\rho$, if $\mu\in\mathfrak{N}_\rho$. Now we define the local characteristic exactly as before:

$$\Pi_G(\mu, A)=\frac{1}{\rho(G)^{\mu(G)}}\int_G\cdots\int_G \mathbb{1}_A(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n}+\mu_{X\setminus G})\,\rho(dx_1)\ldots\rho(dx_{\mu(G)}), \tag{2.19}$$

where $G\in\mathscr{B}_0$, $A\in\mathscr{F}^{\cdot\cdot}$, $\mu\in\Omega$ and $\rho=\chi(\mu)$. The conditions (2.4), (2.5) and (2.6) are satisfied again, so that we can consider $\mathfrak{G}(\Pi)$. The same procedure as above leads us to the following results: The essential part of the boundary of Π is M and each $P\in\mathfrak{G}(\Pi)$ has the integral representation

$$P=\int_M P_\rho\, V(d\rho). \tag{2.20}$$

That is, P is a Cox process built on a probability measure V on M (for details see Krickeberg [10]). We conjecture that (2.20) is valid for wider classes of M having the property $(\rho_1, \rho_2\in M, \rho_1\neq\rho_2 \Rightarrow P_{\rho_1}\perp P_{\rho_2})$.

§ 3. The Stationary Case

Now let $\mathbf{G}$ be a locally compact group which acts continuously on X. In this case the mappings $g: x \to gx$ ($g \in \mathbf{G}$) are homeomorphisms of X. Since $f \circ g$ is an element of $\mathscr{K}(X)$ if $f \in \mathscr{K}(X)$, $g \in \mathbf{G}$,

$$(g\mu)(f) = \mu(f \circ g) \qquad (f \in \mathscr{K}(X),\ \mu \in \mathfrak{M}) \tag{3.1}$$

defines for any $g \in \mathbf{G}$ a homeomorphism of $\mathfrak{M}$ with respect to the vague topology in $\mathfrak{M}$. Assume that

$$\text{there exists a nontrivial Radon measure } \lambda \text{ on } X \text{ invariant under } \mathbf{G}. \tag{3.2}$$

The mapping $\mu \to g\mu$ with $g\mu$ defined by (3.1), which we denote again by g, induces in the same way a mapping g on the set of all probability measures P on $\mathfrak{M}$:

$$(gP)(\varphi) = P(\varphi \circ g). \tag{3.3}$$

We call P *stationary* if $P = gP$ for any $g \in \mathbf{G}$.

If the local characteristic Π is built on λ (i.e. if we take λ instead of ρ in (2.3)) then Theorem 1 implies immediately that $\mathfrak{G}(\Pi)$ coincides with the set $\mathfrak{G}_0(\Pi)$ of all stationary canonical Gibbs states for Π. Assume now that

$$\text{for every } G, G' \in \mathscr{B}_0 \text{ there exists } g \in \mathbf{G} \text{ such that } G \cap g(G') = \emptyset. \tag{3.4}$$

Then it is well known (see Preston [14]) that $P \in \mathfrak{G}(\Pi)$ is an extreme point if and only if P is ergodic with respect to the σ-field $\mathscr{J}$ of $\mathbf{G}$-invariant elements of $\mathscr{F}^{\cdot\cdot}$. If we assume finally that

$$\lambda \text{ is in an infinite Radon measure on } X, \tag{3.5}$$

then we have as a corollary of Theorem 2:

Theorem 3. *Let P be a stationary mixed Poisson process in X. The following conditions are equivalent:*

(1) *P is an extreme point of $\mathfrak{G}(\Pi)$.*

(2) *P is a Poisson process with intensity measure $z \cdot \lambda$ for some $z \geqq 0$.*

(3) *P is ergodic with respect to $\mathscr{J}$.*

In particular we have proved that every stationary Poisson process $P_{z\lambda}$ in X is ergodic. In many cases of interest the assumptions (3.2), (3.4) and (3.5) are satisfied:

Examples. A. *(Lattice models.)* Let $X = Z^r$ and $\mathbf{G}$ be the group of translations. The measure λ defined through $\lambda\{x\} = 1$ for any $x \in X$ is the (unique) $\mathbf{G}$-invariant measure in X.

B. *(Continuous models.)* Let $X = \mathbb{R}^r$ and $\mathbf{G}$ the group of translations. Here λ is the r-dimensional Lebesgue measure.

C. *(Line processes.)* Here the underlying space X is the set of all oriented lines in the plane. We choose for a natural reference system a fixed line ℓ_0 and a fixed point 0 on ℓ_0. The normal coordinates of an oriented line are (τ, p), where τ is the angle it makes with ℓ_0 and p its signed distance from 0: $X = [0, 2\pi) \times \mathbb{R}$. Let $\mathbf{G}$ be the group of transformations in X which are induced by the Euclidean motions of $\mathbb{R}^2$. It is well known [15] that $d\lambda = d\tau\, dp$ is the (unique) measure invariant under $\mathbf{G}$.

Remark. It should be mentioned that the class of mixed Poisson processes with infinite intensity measures models infinite particle systems in a general state space in which there is *no interaction* between the particles. Canonical Gibbs states *with interaction* have been studied in the cases of lattice and continuum systems by Georgii [6, 7].

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Note Added in Proof. 1. After having submitted this paper for publication we became aware of a paper of Nawrotzki (cf. Nawrotzki, K.: Ein Grenzwertsatz für homogene zufällige Punktfolgen (Verallgemeinerung eines Satzes von A. Renyi), Math. Nachr. **24**, 201–217 (1962)) where Theorem 1 can be found in the case of the real line. We became also aware of the book Kerstan, J., Matthes, K., Mecke, J.: Unbegrenzt teilbare Punktprozesse, Berlin: Akademie-Verlag (1974), where Theorem 1 and the equivalence (2)–(3) of Theorem 2 can be found in the same generality as below. We think, however, that the mixed Poisson processes are more easily and more completely treated by the methods below.

2. Stimulated by a discussion with H.-O. Georgii and in order to be more complete we remark that our considerations are valid also in the case when the underlying intensity measure ρ is finite. (In fact they are more easily in this case.) Theorem 1 then has the following form: If P is a point process in X and ρ a *finite* Radon measure on X, then P is a canonical Gibbs measure with local characteristic Π if and only if P satisfies

$$P = \sum_{n=0}^{\infty} P(N(X)=n) \cdot \chi_n \left(\frac{\rho^n}{\rho(X)^n} \right),$$

where $\chi_n(\tau)$ denotes the image of τ with respect to $\chi_n: (x_1, \ldots, x_n) \mapsto \varepsilon_{x_1} + \cdots + \varepsilon_{x_n}$ $(x_j \in X)$; this means that P is a mixture of the measures $\chi_n \left(\frac{\rho^n}{\rho(X)^n} \right)$, $n = 1, 2, \ldots$. Theorem 2 characterizes in the finite case the extremal points of $\mathfrak{G}(\Pi)$ as the measures $\chi_n \left(\frac{\rho^n}{\rho(X)^n} \right)$, $n = 1, 2, \ldots$.