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On the Approximate Local Growth of Multidimensional Random Fields*

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We compute the approximate, local growth rate for a (nondifferentiable) random process $X(t)$, $t=(t_1, \ldots, t_N) \in \mathbb{R}^N$, with values in $\mathbb{R}^d$ which satisfies a condition on the distribution of $\|X(s)-X(t)\|$, namely: for some $0 \leqq k < N$ and Lebesgue almost every $t \in \mathbb{R}^N$, the function $\eta_k(s,t) = \sup_{\varepsilon>0} \varepsilon^{-d} P\{\|X(t_1, \ldots, t_k, s) - X(t)\| \leqq \varepsilon\}$ is locally integrable (ds) over $\mathbb{R}^{N-k}$. Then, with $r=\frac{N-k}{d}$ and with probability one, the approximate limit as $s \to t$ of $\|X(s)-X(t)\|/\|s-t\|^r$ is infinite for almost every $t \in \mathbb{R}^N$, which means (for t fixed) that for every $Q>0$, the (Lebesgue) proportion of s with $\|s-t\|<\varepsilon$ and $\|X(s)-X(t)\| \leqq Q\|s-t\|^r$ is asymptotically (as $\varepsilon \downarrow 0$) equal to zero. When $X=(X_1, \ldots, X_d)$ is Gaussian, the largest $k<N$ for which η_k is integrable is computed in various special cases. For example, for i.i.d. components, $EX_i(t) \equiv 0$, $E(X_i(t)-X_i(s))^2 = \|s-t\|^\alpha$, $0<\alpha<2$, η_k is integrable if and only if $k<N-\frac{\alpha d}{2}$.

§1

Let $X(t)$, $t \in T^N = [0,1]^N$, be a random process with values in $\mathbb{R}^d$. We write $B_m(t,\varepsilon)$ for the closed ball in $\mathbb{R}^m$ with center t and radius ε, relative to the usual Euclidean norm $\|\cdot\|$, and $\lambda^m(dt)$ for Lebesgue measure on $\mathbb{R}^m$. We are interested in results of the form

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda^N\left\{s \in \mathrm{B}_N(\mathrm{t}, \varepsilon): \frac{\|X(s)-X(t)\|^d}{\|s-t\|^{N-k}} \leqq Q\right\}}{\lambda^N\{B_N(t,\varepsilon)\}} = 0 \qquad \forall Q>0, \tag{1}$$

where $0 \leqq k < N$ depends on the law of X and (1) is to hold, with probability 1, at

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λ^N-a.e. $t \in T^N$. When (1) holds at a particular t, it is customary to write

$$\operatorname{ap}\lim_{s\to t} \frac{\|X(s)-X(t)\|^d}{\|s-t\|^{N-k}} = \infty. \tag{2}$$

Here, "ap lim" stands for *approximate limit.*

Actually, we are going to consider processes with a more general range, namely a metric space (Y,ρ). Any reformulation of (2) depends on how we measure the Borel subsets of Y. To this end, let $\phi(dy)$ be a measure on $\mathscr{Y}$, the Borel σ-field in Y, and let $B_\rho(y,\varepsilon)$ be the closed ball with center y and radius ε. *We make the following assumptions.*

$$\begin{aligned}
&\text{(a) } \mathscr{Y} \text{ is separable}\\
&\text{(b) } \phi(A)<\infty \text{ for every bounded } A\in\mathscr{Y}\\
&\text{(c) the } \phi\text{-measure of } B_\rho(y,\varepsilon) \text{ is independent of } y\\
&\text{(d) } \xi(\varepsilon)\equiv\phi(B_\rho(y,\varepsilon)) \text{ is strictly increasing on } [0,\infty)\\
&\text{(e) } \varepsilon^{-1}\xi(\varepsilon) \text{ is continuous and non-decreasing on } (0,\infty)\\
&(f)\ \overline{\lim_{\varepsilon\downarrow 0}}\ \xi(5^N\varepsilon)/\xi(\varepsilon)<\infty.
\end{aligned} \tag{3}$$

(The reasons for (*d*), (*e*), and (*f*) involve the existence of certain Vitali relations and will be discussed in the course of the proofs.) The analogue of (2) is

$$\operatorname{ap}\lim_{s\to t} \frac{\xi(\rho(X(s),X(t)))}{\|s-t\|^{N-k}} = \infty. \tag{4}$$

(That is, (1) holds with $\|X(s)-X(t)\|^d$ replaced by $\xi(\rho(X(s),X(t)))$.)

Let $(\Omega,\mathscr{F},P)$ be the probability space carrying $X(t,\omega)$, and let $\mathscr{B}^m(\mathscr{B}^m(T))$ denote the Borel sets in $\mathbb{R}^m$ (resp. T^m). We assume $X(t,\omega)$ is separable and measurable, $\mathscr{B}^N(T)\otimes\mathscr{F}\to\mathscr{Y}$. We now state Theorem A, one of our two main results. The other is Theorem B, upon which Theorem A is largely based, and which gives conditions for a *non-random* function $X\colon T^N\to Y$ to satisfy (4) at λ_N-a.e. $t\in T^N$. These conditions involve the "local time" of X. As far as we know, it was Berman [1] who first saw the close relationship between local times and approximate limits, and thereby introduced the latter into the analysis of random functions. (See the introduction to [4] and the references therein.) The proof of Theorem A and the statement and proof of Theorem B are given in § 2, and § 3 contains the details of the examples and illustrations mentioned after Theorem A.

Theorem A. *Suppose there exists a* $0\leqq k<N$ *such that for* λ^N*-a.e.* $t=(t_1,\ldots,t_N)\in T^N$,

$$\int_{T^{N-k}} \sup_{\varepsilon>0} \frac{1}{\xi(\varepsilon)} P\{\rho(X(t_1,\ldots,t_k,s),X(t))\leqq\varepsilon\}\,\lambda^{N-k}(ds)<\infty. \tag{5}$$

Then (4) *holds at* $\lambda^N\times P$*-a.e.* (t,ω).

The proof is based on the existence, and suitable regularity, of an "occupation density" for X. When (5) holds for $k=0$, the occupation density exists and is continuous as a measure on $\mathscr{B}^N(T)$, which means that with probability 1:

$\phi(A)=0 \Rightarrow \lambda^N\{t\in T^N\colon X(t,\omega)\in A\}=0$, $A\in\mathscr{Y}$, and a version $\gamma(y,B,\omega)$ of the Radon-Nikodym derivative of the measure $\lambda^N\{t\in B\colon X(t,\omega)\in dy\}$ with respect to $\phi(dy)$ may be chosen such that $\gamma(y,\{t\},\omega)=0$ $\forall\, t\in T^N$, $y\in Y$, $\omega\in\Omega$. (Here, $\gamma(y,\{t\},\omega)$ is the mass placed on t by the measure $\gamma(y,\cdot,\omega)$.) When (5) holds for $0<k<N$, then γ exists and the measure $B\to\gamma(y,B,\omega)$ has a k-dimensional "marginal" distribution which is *absolutely continuous* with respect to λ^k, i.e.

$$\gamma(y, ds\,dt, \omega)=g(y,s,dt,\omega)\,\lambda^k(ds),$$

all with probability one. (Also, see the note after Lemma 1.)

To fix the ideas and to compare our results with those in [4] and [6], we take $Y=\mathbb{R}^d$, $\phi=\lambda^d$, etc. for the remainder of this section. Obviously (3) holds. Now (2) is equivalent to the existence of a set $A_t\in\mathscr{B}^N(T)$ for which t is a point of (metric) density 1, i.e.

$$\lim_{\varepsilon\downarrow 0}\frac{\lambda^N\{B_N(t,\varepsilon)\cap A_t\}}{\lambda^N\{B_N(t,\varepsilon)\}}=1,$$

and for which

$$\lim_{\substack{s\to t\\ s\in A_t}}\frac{\|X(t)-X(s)\|^d}{\|s-t\|^{N-k}}=\infty. \tag{6}$$

(Here, of course, t and ω are fixed.) If one removes the restriction "$s\in A_t$" in (6), i.e. considers the true limit, it may happen (depending on N, k, and d) that *no* function $X\colon \mathbb{R}^N\to\mathbb{R}^d$ can satisfy (6) on a set of t's of even *positive* λ^N-measure. For example,

$$\lambda^1\left\{t\in\mathbb{R}\colon \lim_{s\to t}\frac{|X(s)-X(t)|}{|s-t|}=\infty\right\}=0$$

for any function $X\colon \mathbb{R}^1\to\mathbb{R}^1$, a result due to Banach – see [7, p. 270].

For X Gaussian and $N\geqq d$, (5) is widely satisfied for $k=N-d$ when X fails to be differentiable. When $N=d=1$, for example, (5) reduces to the integrability of $s\to(E(X(s)-X(t))^2)^{-1/2}$ over $T^1=[0,1]$ for λ^1-a.e. $t\in T^1$; this and related matters were discussed in [4]. Roughly, the *faster* the growth of the incremental variance $E\|X(s)-X(t)\|^2$ in neighborhoods of its zeros, the larger may be chosen k. As an illustration, take $X=(X_1,\dots,X_d)$ where $X_1,\dots,X_d$ are independent, identically distributed Gaussian fields on T^N and

$$EX_j(t)\equiv 0,\qquad EX_j(t)\,X_j(s)=\|t\|^\alpha+\|s\|^\alpha-\|t-s\|^\alpha,\qquad 0<\alpha<2. \tag{7}$$

As will be seen in §3, (5) holds for k if and only if $k<N-\dfrac{\alpha d}{2}$. Consequently, if $0<N-\dfrac{\alpha d}{2}$,

$$\operatorname{ap}\lim_{s\to t}\frac{\|X(s,\omega)-X(t,\omega)\|}{\|s-t\|^r}=\infty\qquad \lambda^N\text{-a.e., a.s.} \tag{8}$$

where $r=\dfrac{N-k_0}{d}$ and k_0 is the greatest integer *less than* $N-\dfrac{\alpha d}{2}$. When $N\geqq d$, we have $0 \leqq N-d<N-\frac{\alpha d}{2}$ so we can always take $r=1$ in (8). For d-dimensional Brownian motion on T^N, $\alpha=1$ and we can choose $r=1/2+1/d$ for d even and $r=1/2+1/2d$ for d odd. (The conclusion is the same for the "N-parameter Wiener process", i.e.

$$E X_j(t) X_j(s)=\prod_i^N (t_i \wedge s_i), \qquad t=(t_1, \ldots, t_N), \qquad s=(s_1, \ldots, s_N).)$$

Whereas our results are chiefly applicable for processes on T^N when the dimension d of the range is *bounded above* (e.g. $d<\frac{2N}{\alpha}$ as above), the results in [6] are basically for processes on $\mathbb{R}^N$ into *high* dimensional spaces. Extending the work of Dvoretzky and Erdös [2], and others, Kôno [6] considers Gaussian processes $X=(X_1, \ldots, X_d)$ from $\mathbb{R}^N$ to $\mathbb{R}^d$ with i.i.d. components, $\sigma^2(s,t)=E(X_j(s)-X_j(t))^2$, and defines $\mathscr{L}^0(X^d)$ (resp. $\mathscr{U}^0(X^d)$) to be the class of continuous, non-decreasing ϕ: $(0,\infty) \to (0,\infty)$ such that with probability 1 (resp. 0) there is a $\delta(\omega)$ for which

$$0<\|t\|<\delta(\omega) \Rightarrow \|X(t,\omega)-X(0,\omega)\|>\sigma(0,t)\,\phi(\|t\|). \tag{9}$$

Under various conditions on σ, d, and N, Kôno obtains integral tests for $\phi \in \mathscr{L}^0(X^d)$ and $\phi \in \mathscr{U}^0(X^d)$. Thus, for example, Kôno retrieves a result of Dvoretzky and Erdös which in turn implies that for Brownian motion from $\mathbb{R}^1$ to $\mathbb{R}^3$

$$\lim_{s \to t} \frac{\|X(t,\omega)-X(s,\omega)\|}{|t-s|^{1/2}\,|\log|t-s||^{-3}}=\infty \qquad \text{for } \lambda^1\text{-a.e. } t, \text{ a.s.} \tag{10}$$

More generally, for the family of processes described in (7), the conditions of Theorem 1 of [6] are satisfied when $N-\frac{\alpha d}{2}<0$ and one easily checks that $\phi(x)=x^\delta \in \mathscr{L}^0(X^d)$ for any $\delta>0$. It then follows that

$$\lim_{s \to t} \frac{\|X(t,\omega)-X(s,\omega)\|}{\|t-s\|^r}=\infty \qquad \text{for } \lambda^N\text{-a.e. } t, \text{ a.s.} \tag{11}$$

for any $r>\frac{\alpha}{2}$. (Kôno also considers "uniform" upper and lower classes: for σ^2 as in (7), and assuming $2N-\frac{\alpha d}{2}>0$, Theorem 4 of [6] would yield (11) for any $r>\frac{\alpha}{2}+N\left(d-\frac{4N}{\alpha}\right)^{-1}$ with "λ^N-a.e. t" replaced by "every t".) To compare (8) and (11), consider the case $N \approx \frac{\alpha d}{2}$: if $\frac{\alpha d}{2}=N+\varepsilon>N$ then (11) holds whereas (8) doesn't apply; if $\frac{\alpha d}{2}=N-\varepsilon<N$ then (8) holds with $r=\frac{\alpha}{2}+\frac{\varepsilon}{d}$ whereas the results in [6] don't apply.

§ 2

Let $\alpha(m)=\lambda^m(B_m(t,1))$, so $\lambda^m(B_m(t,\varepsilon))=\alpha(m)\,\varepsilon^m$, and $\tau_\omega(\varepsilon,t,Q)=\lambda^N\{s\in B_N(t,\varepsilon)$: $\xi(\rho(X(s,\omega),X(t,\omega)))\leqq\|s-t\|^{N-k}Q\}$, ε, $Q>0$. Under a (mild) additional assumption to (5), it is easy to show that

$$\operatorname{ap}\overline{\lim_{s\to t}}\frac{\xi(\rho(X(s),X(t)))}{\|s-t\|^{N-k}}=\infty \quad \lambda^N\times P\text{-a.e.,} \tag{12}$$

which means that

$$\underline{\lim_{\varepsilon\downarrow 0}}\frac{\tau_\omega(\varepsilon,t,Q)}{\alpha(N)\,\varepsilon^N}<1 \qquad \forall\, Q>0, \qquad \lambda^N\times P\text{-a.e.}$$

Assume that $\forall\, t\ \exists$ constants $M\geqq 0$ and $\eta>0$ such that $s\in B_N(t,\eta)$ implies

$$\begin{aligned}&\sup_{\varepsilon>0}\frac{1}{\xi(\varepsilon)}P\{\rho(X(t),X(s))\leqq\varepsilon\}\\&\quad\leqq M+\sup_{\varepsilon>0}\frac{1}{\xi(\varepsilon)}P\{\rho(X(t_1,\dots,t_k,s_{k+1},\dots,s_N),X(t))\leqq\varepsilon\}.\end{aligned}$$

(For example, this is widely satisfied in the Gaussian case with $M=0$ – see (21) and (22).) Then for any $t\in T^N$ at which (5) holds and for any $Q>0$:

$$\begin{aligned}&E\,\underline{\lim_{\varepsilon\downarrow 0}}\,\varepsilon^{-N}\tau_\omega(\varepsilon,t,Q)\leqq E\,\underline{\lim_{n\to\infty}}\,n^N\tau_\omega(1/n,t,Q)\\
&\quad\leqq\underline{\lim_{n\to\infty}}\,E\Big[n^N\int_{B_N(t,1/n)}I_{[0,\xi^{-1}(Q(1/n)^{N-k})]}(\rho(X(t),X(s)))\,\lambda^N(ds)\Big]\\
&\quad\leqq Q\,\underline{\lim_{n\to\infty}}\,n^k\int_{B_N(t,1/n)}\sup_{\varepsilon>0}\Big[\frac{1}{\xi(\varepsilon)}P\{\rho(X(t),X(s))\leqq\varepsilon\}\Big]\lambda^N(ds)\\
&\quad\leqq Q\,\underline{\lim_{n\to\infty}}\,n^k\int_{B_N(t,1/n)}M\\
&\quad\quad+\sup_{\varepsilon>0}\Big[\frac{1}{\xi(\varepsilon)}P\{\rho(X(t_1,\dots,t_k,s_{k+1},\dots,s_N),X(t))\leqq\varepsilon\Big]\lambda^N(ds)\\
&\quad\leqq Q\,\underline{\lim_{n\to\infty}}\int_D M+\sup_{\varepsilon>0}\Big[\frac{1}{\xi(\varepsilon)}P\{\rho(X(t_1,\dots,t_k,s),X(t))\leqq\varepsilon\}\Big]\lambda^{N-k}(ds)=0,\end{aligned}$$

where

$$D=B_{N-k}((t_{k+1},\dots,t_N),1/n).$$

Thus, for each $Q>0$ and λ^N-a.e. $t\in T^N$, $\underline{\lim}_{\varepsilon\downarrow 0}\,\varepsilon^{-N}\tau_\omega(\varepsilon,t,Q)=0$ a.s., and hence, τ being monotone in Q,

$$\underline{\lim_{\varepsilon\downarrow 0}}\,\varepsilon^{-N}\tau_\omega(\varepsilon,t,Q)=0 \qquad \forall\, Q>0, \qquad \lambda^N\times P\text{-a.e.}$$

We need several lemmas about occupation densities for real functions, a characterization of absolute continuity (Lemma 4) which is implicit in the literature but included here for clarity, and a technical result (Lemma 2) about Vitali relations. The latter will permit us to differentiate indefinite integrals over $Y \times \mathbb{R}^m$ (of $\phi \times \lambda^m$-integrable functions) with respect to coverings which have no "parameter of regularity". When $Y = \mathbb{R}^d$, etc. and the function of $t \in T^N$ in (5) is integrable $\lambda^N(dt)$, Lemma 2 is superfluous. Instead, we can use a classical result of Zygmund [8] about "strong derivatives" – see the Remark at the end of this § and [7, pp. 128–133].

Let $x: T^N \to Y$ be measurable $\mathscr{B}^N(T)$ to $\mathscr{Y}$ and define measures

$$\mu(V; D) = \int_D I_V(x(t))\, \lambda^N(dt), \qquad D \in \mathscr{B}^N(T), \qquad V \in \mathscr{Y},$$

$$\mu(V) = \mu(V; T^N),$$

$$\nu_k(V) = \int_{T^N} I_V(x(t), t_1, \ldots, t_k)\, \lambda^N(dt), \qquad V \in \mathscr{Y} \otimes \mathscr{B}^k(T), \qquad 0 < k < N,$$

$$\mu_{t,k}(V) = \int_{T^{N-k}} I_V(x(t,s))\, \lambda^{N-k}(ds), \qquad V \in \mathscr{Y}, \qquad t \in T^k, \qquad 0 < k < N.$$

If $(G_1, \mathscr{G}_1)$ and $(G_2, \mathscr{G}_2)$ are measurable spaces, by a *kernel* h on $G_1 \times \mathscr{G}_2$ we mean a real function $h(g, B)$, $g \in G_1$, $B \in \mathscr{G}_2$, such that $h(\cdot, B)$ is measurable ($\mathscr{G}_1$ to $\mathscr{B}$) for each $B \in \mathscr{G}_2$ and $h(g, \cdot)$ is a measure on $\mathscr{G}_2$ for each $g \in G_1$.

Lemma 1. *For any* $0 < k < N$, *the following are equivalent:*

(i) $\nu_k \ll \phi \times \lambda^k$,

(ii) $\mu \ll \phi$ *and there exists a kernel* g *on* $Y \times T^k \times \mathscr{B}^{N-k}(T)$ *such that* $\forall y \in Y$, $B \in \mathscr{B}^k(T)$, $A \in \mathscr{B}^{N-k}(T)$,

$$\gamma(y, B \times A) = \int_B g(y, s, A)\, \lambda^k(ds), \tag{13}$$

where $\mu(dy, D) = \gamma(y, D)\, \phi(dy)$.

(iii) $\mu_{t,k} \ll \phi$ *for* λ^k*–a.e.* $t \in T^k$.

Proof. Suppose (i) and define

$$\nu_k(V; A) = \int_A I_V(x(t), t_1, \ldots, t_k)\, \lambda^N(dt), \qquad A \in \mathscr{B}^N(T), \qquad V \in \mathscr{Y} \otimes \mathscr{B}^N(T).$$

Then for each $A \in \mathscr{B}^N(T)$ there exists a measurable function $h(y, t, A)$ on $Y \times T^k$ such that $\nu_k(dy\,dt, A) = h(y, t, A)\, \phi \times \lambda^k(dy\,dt)$. Furthermore, changing h on null sets if necessary, we can assume h is a kernel on $Y \times T^k \times \mathscr{B}^N(T)$; some of the details of such matters are in [5], but the construction of "regular versions" of families of Radon-Nikodym derivatives, and of "regular conditional measures", are well-known. Thus, for $A \in \mathscr{B}^{N-k}(T)$, $B \in \mathscr{B}^k(T)$, $V \in \mathscr{Y}$:

$$\mu(V; B \times A) = \nu_k(V \times B; A \times T^k) = \int_{V \times B} h(y, t, A \times T^k)\, \phi \times \lambda^k(dy\,dt)$$
$$= \int_V \phi(dy) \int_B h(y, t, A \times T^k)\, \lambda^k(dt),$$

and (ii) holds with $g(y, t, A) = h(y, t, A \times T^k)$.

Assuming (ii),

$$\int_B \mu_{t,k}(V)\, \lambda^k(dt) = \mu(V; B \times T^{N-k}) = \int_B \lambda^k(dt) \int_V g(y, t, T^{N-k})\, \phi(dy),$$

for all $B \in \mathscr{B}^k(T)$, $V \in \mathscr{Y}$. As a result, for any $V \in \mathscr{Y}$ there is a λ^k-null set E_V such that

$$\mu_{t,k}(V) = \int_V g(y, t, T^{N-k})\, \phi(dy) \tag{14}$$

for all $t \notin E_V$. Since both sides of (14) are measures on $\mathscr{Y}$ and $\mathscr{Y}$ is separable, (14) holds for *every* $V \in \mathscr{Y}$, for λ^k-a.e. t.

Finally, (iii) implies $\mu_{t,k}(dy) = \alpha(y,t)\, \phi(dy)$ for some $\alpha\colon Y \times T^k \to [0, \infty)$ which can be chosen $\mathscr{Y} \otimes \mathscr{B}^k(T)$ measurable. Integrating $\lambda^k(dt)$ over B yields

$$\nu_k(V \times B) = \int_{V \times B} \alpha(y,t)\, \phi(dy)\, \lambda^k(dt) \quad \forall V \in \mathscr{Y}, \qquad B \in \mathscr{B}^k(T),$$

which extends to $d\nu_k = \alpha\, d(\phi \times \lambda^k)$.

Note. We exclude the case $k = N$ because it corresponds to $\gamma(y, dt) \ll \lambda^N(dt)$, which is impossible; indeed, $\gamma(y, dt) \perp \lambda^N(dt)$ for ϕ-a.e. y since $\lambda^N(M_y) = 0$, $M_y \equiv \{s \in T^N : x(s) = y\}$, except at most for countably many y's, whereas $\gamma(y, M_y^c) = 0$ for ϕ-a.e. y, which follows from

$$\iint f(t,y)\, \mu(dy; dt) = \iint f(t,y)\, \gamma(y, dt)\, \phi(dy) \tag{15}$$

(for any non-negative, measurable f) by choosing $f(t,y) = I_{M_y^c}(t)$.

Lemma 2. *If* $0 < k < N$ *and*

$$V((y,s), \varepsilon) = B_\rho(y, \xi^{-1}(\varepsilon^{N-k})) \times \prod_1^k [s_i - \varepsilon, s_i + \varepsilon],$$
$$(y,s) \in Y \times T^k, \qquad \varepsilon > 0,$$

then

$$\mathscr{K} = \{((y,s), V((y,s), \varepsilon)),\ (y,s) \in Y \times T^k,\ \varepsilon > 0\}$$

is a $\phi \times \lambda^k$*-Vitali relation* [3, p. 151].

Proof. If $V((y,s), \varepsilon)$ is a closed ball in a suitable metric space, we use 2.8.17 and 2.8.8. of [3].

Define δ on $(Y \times \mathbb{R}^k) \times (Y \times \mathbb{R}^k)$ by

$$\delta((y,s),(z,t)) = \max\{\rho(y,z)^{\frac{1}{N-k}}, h(\max_{1 \le i \le k} |s_i - t_i|)\},$$

where $h(u) = (\xi^{-1}(u^{N-k}))^{\frac{1}{N-k}}$, $u \geqq 0$. To verify that δ is a metric it will suffice to check that

$$h(a+b) \leqq h(a) + h(b), \qquad a, b \geqq 0.$$

Now $\frac{\xi(u)}{u} \uparrow$ as $u \uparrow$ implies $\frac{\xi^{-1}(u)}{u} \uparrow$ as $u \downarrow$ implies $\frac{h(u)}{u} \uparrow$ as $u \downarrow$. Consequently, for $0 \leqq a \leqq b$:

$$h(a+b) = \frac{h(a+b)}{a+b}(a+b) \leqq \frac{h(b)}{b}(a+b) = \frac{h(b)}{b} a + h(b) \leqq h(a) + h(b).$$

Let $B_\delta((y,s),\varepsilon)$ be the closed ball contered at (y,s) with radius ε for δ:

$$B_\delta((y,s),\varepsilon)=B_\rho(y,\varepsilon^{N-k})\times\prod_1^k[s_i-h^{-1}(\varepsilon),s_i+h^{-1}(\varepsilon)].$$

Further, let

$$\eta((y,s),\varepsilon)=\phi\times\lambda^k(B_\delta((y,s),\varepsilon))=\xi(\varepsilon^{N-k})(2h^{-1}(\varepsilon))^k.$$

We want to show that $\phi\times\lambda^k$ is "diametrically regular" relative to δ [3, p. 145]; this would imply that

$$\mathscr{K}=\{((y,s),B_\delta((y,s),\varepsilon)),(y,s)\in Y\times\mathbb{R}^k,\varepsilon>0\}$$

is a $\phi\times\lambda^k$-Vitali relation using 2.8.17 of [3]. For diametric regularity we need a $1<\tau<\infty\ni$ for each $(y,s)\exists C=C(y,s)\ni\eta((y,s),(1+2\tau)\varepsilon)<C\eta((y,s),\varepsilon)\,\forall\varepsilon$ small. But,

$$\overline{\lim_{\varepsilon\downarrow 0}}\frac{\eta((y,s),5\varepsilon)}{\eta((y,s),\varepsilon)}=\overline{\lim_{\varepsilon\downarrow 0}}\frac{\eta((y,s),5\varepsilon^{\frac{1}{N-k}})}{\eta((y,s),\varepsilon^{1/N-k})}=\overline{\lim_{\varepsilon\downarrow 0}}\left[\frac{\xi(5^{N-k}\varepsilon)}{\xi(\varepsilon)}\right]^{\frac{N}{N-k}}<\infty.$$

Theorem B. *Suppose for some* $0\leqq k<N$:

$$\begin{aligned}&\mu\ll\phi\\&\gamma(y,ds\,dt)=g(y,s,dt)\,\lambda^k(ds)\qquad\text{(i.e. (13))},\\&\textit{and}\\&g(y,s,\{t\})=0\quad\forall t\in T^{N-k},\qquad\phi\times\lambda^k\text{-a.e.}\end{aligned}\tag{16}$$

Then $\operatorname{ap}\lim_{s\to t}\dfrac{\xi(\rho(x(s),x(t)))}{\|s-t\|^{N-k}}=\infty\qquad\lambda^N$-a.e.

Note. The case $k=0$ refers to $\mu\ll\phi$ and $\gamma(y,dt)$ continuous (i.e. $\gamma(y,\{t\})=0$ $\forall t\in T^N$ for ϕ-a.e. y). We will give the proof only for the case $0<k<N$. However, the proof for $k=0$ is essentially a special case, but to incorporate it would require defining λ^0, ν_0, etc. and is not worth the effort. All that is needed is that

$$\{(y,B_\rho(y,\varepsilon)),y\in Y,\varepsilon>0\}$$

is a ϕ-Vitali relation, which follows immediately from (3) and [3, 2.8.17]. Besides, the case $k=0$ is merely a "higher dimensional" version of what we did in [4] for real functions of one real variable.

Proof. First, we can arrange to have $g(y,s,\{t\})=0\ \forall y,s,t$ and we do.

Next, since $\mathscr{K}$ is a $\phi\times\lambda^k$ Vitali relation, and since $\eta((y,s),h(\varepsilon))=2^{-k}\varepsilon^{-N}$, we know (according to [3, 2.9.8, p. 156]) that for any $f\in L^1(\phi\times\lambda^k)$:

$$f(y,s)=\lim_{\varepsilon\downarrow 0}2^{-k}\varepsilon^{-N}\int_{B_\delta((y,s),h(\varepsilon))}f\,d\phi\times\lambda^k\tag{*}$$

for $\phi\times\lambda^k$-a.e. (y,s).

Let $\mathscr{H}$ be the collection of open rectangles in T^{N-k} with rational vertices, and for each $J\in\mathscr{H}$, set

$$f_J(y,s)=g(y,s,J)\in L^1(\phi\times\lambda^k).$$

Choose Borel sets E_J, $J\in\mathscr{H}$, such that $\phi\times\lambda^k(E_J^c)=0$ and (*) and (17) holds $\forall(y,s)\in E_J$. Consequently, $E\equiv\bigcap_{J\in\mathscr{H}} E_J\in\mathscr{Y}\otimes\mathscr{B}^k(T)$, $\phi\times\lambda^k(E^c)=0$, and hence by Lemma 1:

$$v_k(E^c)=0,\quad \text{i.e. } (x(t),t_1,\dots,t_k)\in E_J\ \forall J\in\mathscr{H}\quad \text{for } \lambda^N\text{-a.e.}$$

$t=(t_1,\dots,t_N)\in T^N$.

Now fix such a $t\in T^N$ and $Q\geqq 1$.

$$\begin{aligned}
&\lambda^N\left\{s\in B_N(t,\varepsilon)\colon \frac{\xi(\rho(x(s),x(t)))}{\|s-t\|^{N-k}}\leqq Q\right\}\\
&\quad\leqq \int_{B_N(t,\varepsilon)} I_{[0,Q\varepsilon^{N-k}]}(\xi(\rho(x(s),x(t)))\,\lambda^N(ds)\\
&\quad\leqq \int_{B_\rho(x(t),\,\xi^{-1}(Q\varepsilon^{N-k}))} \gamma\Big(y,\prod_1^N[t_i-\varepsilon,t_i+\varepsilon]\Big)\,\phi(dy)\\
&\quad= \int_{B_\rho(x(t),\,\xi^{-1}(Q\varepsilon^{N-k}))}\phi(dy)\int_{\prod_1^k[t_i-\varepsilon,t_i+\varepsilon]} g\left(y,s,\prod_{k+1}^N[t_i-\varepsilon,t_i+\varepsilon]\right)\lambda^k(ds)\\
&\quad\leqq \int I_{B_\delta(x(t),t_1,\dots,t_k,h(Q\varepsilon))}\,f_J\,d\phi\times\lambda^k
\end{aligned}$$

for all small ε if $(t_{k+1},\dots,t_N)\in J\in\mathscr{H}$. It then follows from the remarks above that

$$\begin{aligned}
&\overline{\lim_{\varepsilon\downarrow 0}}\ \varepsilon^{-N}\lambda^N\left\{s\in B_N(t,\varepsilon)\colon \frac{\xi(\rho(x(s),x(t)))}{\|s-t\|^{N-k}}\leqq Q\right\}\\
&\quad\leqq 2^{-k}Q^N g(x(t),(t_1,\dots,t_k),J),
\end{aligned}$$

for any $J\in\mathscr{H}$ containing $(t_{k+1},\dots,t_N)$. Letting $J\downarrow(t_{k+1},,t_N)$ completes the proof.

Lemma 3. *Let* $\mu_{t,k}(V;A)=\int_A I_V(x(t,s))\,\lambda^{N-k}(ds)$. *Then* (16) *is equivalent to*

$$\mu_{t,k}\ll\phi\qquad \textit{for } \lambda^k\text{-a.e.}\qquad t\in T^k,$$

and

$$\alpha(y,t,\{s\})=0\ \ \forall s\in T^{N-k},\qquad \phi\times\lambda^k\text{-a.e.} \tag{17}$$

where

$$\mu_{t,k}(dy;ds)=\alpha(y,t,ds)\,\phi(dy).$$

Proof. By Lemma 1, if either g or α exists, then so does the other, in which case we find from

$$\mu(V;B\times A)=\int_B \mu_{t,k}(V;A)\,\lambda^k(dt)$$

that, for any $A\in\mathscr{B}^{N-k}(T)$, $g(y,t,A)$ and $\alpha(y,t,A)$ have the same integrals against $\phi\times\lambda^k$ over rectangles $V\times B$ in $\mathscr{Y}\otimes\mathscr{B}^k(T)$. Since $\alpha(y,t,\cdot)$ and $g(y,t,\cdot)$ are measures on $\mathscr{B}^{N-k}(T)$, which is separable, the results follows. Here, of course, we have assumed that g and α are kernels on $Y\times T^k\times\mathscr{B}^{N-k}(T)$.

Remark. In the proof of Lemma 4 and after that of Theorem A many of the arguments about the measurability of various derivatives will be left aside. As for Lemma 4, these can be readily found in [3] in the section on "Derivates". As for the integrals later on involving dP, etc., more or less all that is needed (beyond what is in [3]) is to remark that the joint measurability of $\rho(y,z)$ in (y,z) implies that of $I_{B_\rho(y,\varepsilon)}(X(t,r,\omega))$ in $(t,r,\omega,y,\varepsilon)$.

Lemma 4. *Let Ψ be a finite measure on $\mathscr{Y}$. Then*

$$\Psi'(y)=\lim_{\varepsilon\downarrow 0}\frac{\Psi(B_\rho(y,\varepsilon))}{\phi(B_\rho(y,\varepsilon))}$$

exists finite or infinite Ψ-a.e. and $\Psi\ll\phi$ if and only if $\Psi'<\infty$ Ψ-a.e.

Proof. Set $L=\{y\in Y: \Psi'(y)=\infty\}$, $F=\{y\in Y: \Psi'(y)<\infty\}$. The assumptions in (3) in conjunction with [3, 2.9.15, p. 160] imply that $\Psi(\cdot\cap L^c)$ is the ϕ-absolutely continuous component of Ψ. Hence

$$\Psi=\Psi_a+\Psi_s,\qquad \Psi_a(\cdot)=\Psi(\cdot\cap L^c),\qquad \Psi_s(\cdot)=\Psi(\cdot\cap L)$$

is the Lebesgue decomposition of Ψ with respect to ϕ. Moreover, $\phi(F^c)=0$ (see [3, 2.9.5, p. 154]) implies $\Psi_a(F^c)=0$, and hence Ψ' exists finite Ψ_a-a.e. and exists at $+\infty$ Ψ_s-a.e. Finally, if $\Psi'<\infty$ Ψ-a.e., then Ψ_s lives on both L and L^c, and hence vanishes.

Proof of Theorem A. As with Theorem B, we are going to omit the case $k=0$, the proof there being obvious from – and easier than – the proof for $0<k<N$.

Defining $\mu(V;D,\omega)$, $\gamma(y,D,\omega)$, $\mu_{t,k}(V,\omega)$, etc. all relative $X(\cdot,\omega)$, we can and do assume these are appropriately measurable in ω, i.e. $\gamma(y,D,\omega)$ is a kernel on $Y\times\mathscr{B}^N(T)\times\Omega$, etc.

According to Lemma 4, for each $t\in T^k$, $\omega\in\Omega$,

$$\lim_{\varepsilon\downarrow 0}\frac{\mu_{t,k}(\mathrm{B}_\rho(y,\varepsilon))}{\xi(\varepsilon)}\ \text{exists (finite or infinite)}$$

$\mu_{t,k}$-a.e. and $\mu_{t,k}\ll\phi$ if and only if the limit is *finite* $\mu_{t,k}$-a.e. In other words,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\xi(\varepsilon)}\int_{T^{N-k}} I_{[0,\varepsilon]}(\rho(X(t,r,\omega),X(t,s,\omega)))\,\lambda^{N-k}(dr) \tag{18}$$

exists for λ^{N-k}-a.e. s, and $\mu_{t,k}\ll\phi$ if and only if (18) is finite λ^{N-k}-a.e. Using Fatou's lemma and Fubini's theorem,

$$\begin{aligned}
&E\,\varliminf_{n\to\infty}\frac{1}{\xi(n^{-1})}\int_{T^{N-k}} I_{[0,n^{-1}]}(\rho(X(t,r,\omega),X(t,s,\omega)))\,\lambda^{N-k}(dr)\\
&\leqq\varliminf_{n\to\infty}\frac{1}{\xi(n^{-1})}\int_{T^{N-k}} P\{\rho(X(t,r,\omega),X(t,s,\omega))\leqq n^{-1}\}\,\lambda^{N-k}(dr)\\
&\leqq\int_{T^{N-k}}\sup_{\varepsilon>0}\left[\frac{1}{\xi(\varepsilon)}P\{\rho(X(t,r,\omega),X(t,s,\omega))\leqq\varepsilon\}\right]\lambda^{N-k}(dr).
\end{aligned}$$

By (5), this is finite for λ^N-a.e. $(t,s)\in T^k\times T^{N-k}$, and hence for each $s\in\Gamma_t\in\mathscr{B}^{N-k}(T)$, $t\in\Delta\in\mathscr{B}^k(T)$, where $\lambda^{N-k}(\Gamma_t^c)=0$ $\forall\,t\in\Delta$, $\lambda^k(\Delta^c)=0$.

Now fix a $t\in\Delta$. For each $s\in\Gamma_t$ there is an ω null set off which (18) is finite, and consequently (using Fubini's theorem) $\mu_{t,k}\ll\phi$ with probability 1, say for

$\omega\in\Omega_0$, $P(\Omega_0)=1$. Moreover, since $\{(y, B_\rho(y,\varepsilon)),\ y\in Y,\ \varepsilon>0\}$ is a ϕ-Vitali relation, for any $A\in\mathscr{B}^{N-k}(T)$,

$$\alpha(y,t,A,\omega)=\lim_{n\to\infty}\frac{1}{\xi(n^{-1})}\,\mu_{t,k}(B_\rho(y,n^{-1});A,\omega)\qquad \phi\text{-a.e.} \tag{19}$$

for $\omega\in\Omega_0$. Since the $\mu_{t,k}$ measure of the exceptional y-set in (19) is zero for each $\omega\in\Omega_0$, we find that

$$\alpha(X(t,s,\omega),t,A,\omega)=\lim_{n\to\infty}\frac{1}{\xi(n^{-1})}\,\mu_{t,k}(B_\rho(X(t,s,\omega),n^{-1});\ A,\omega) \tag{20}$$

for $\lambda^{N-k}\times P$-a.e. (s,ω). As a result, there are sets $\bar{\Gamma}_t\in\mathscr{B}^{N-k}(T)$, $\lambda^{N-k}(\bar{\Gamma}_t^c)=0$, such that for each $s\in\bar{\Gamma}_t$, (20) holds simultaneously for all open rectangles $A\subset T^{N-k}$ with rational vertices with probability 1.

For $t\in\Delta$ and $s\in\Gamma_t\cap\bar{\Gamma}_t$, choose a sequence A_m of such rectangles, $A_m\downarrow\{s\}$. Then

$$\begin{aligned}
E\,\alpha(X(t,s,\omega),t,\{s\},\omega)&=E\lim_m \alpha(X(t,s,\omega),t,A_m,\omega)\\
&=E\lim_m\lim_n\frac{1}{\xi(n^{-1})}\int_{A_m} I_{[0,n^{-1}]}(\rho(X(t,s,\omega),X(t,r,\omega)))\,\lambda^{N-k}(dr)\\
&\leqq\varliminf_m\varliminf_n\int_{A_m}\frac{1}{\xi(n^{-1})}\,P\{\rho(X(t,s,\omega),X(t,r,\omega))\leqq n^{-1}\}\,\lambda^{N-k}(dr)\\
&\leqq\varliminf_m\int_{A_m}\sup_{\varepsilon>0}\left[\frac{1}{\xi(\varepsilon)}\,P\{\rho(X(t,s,\omega),X(t,r,\omega)\leqq\varepsilon\}\right]\lambda^{N-k}(dr)\\
&=0.
\end{aligned}$$

Finally, then, for each $t\in\Delta$,

$$\alpha(X(t,s,\omega),t,\{s\},\omega)=0\qquad \lambda^{N-k}\times P\text{-a.e.}$$

from which it follows with probability 1,

$$0=\int_{T^{N-k}}\alpha(X(t,s,\omega),t,\{s\},\omega)\,\lambda^{N-k}(ds)=\int_Y\sum_{s\in T^{N-k}}\alpha^2(y,t,\{s\},\omega)\,\phi(dy).$$

The second equality is achieved via (15) just as in [4, p. 321].

In summary: for λ^k-a.e. $t\in T^k$, $\mu_{t,k}(\cdot,\omega)\ll\phi$ and $\alpha(y,t,\{s\},\omega)=0$ $\forall\, s\in T^{n-k}$ ϕ-a.e., both with probability 1. Interchanging null sets one last time, we obtain (17) with probability 1. In view of Theorem B and Lemma 3 this concludes the proof.

Remark. Certainly (5) holds if

$$\int_{T^N}\lambda^N(dt)\int_{T^{N-k}}\sup_{\varepsilon>0}\frac{1}{\xi(\varepsilon)}\,P\{\rho(X(t_1,\dots,t_k,s),X(t))\leqq\varepsilon\}\,\lambda^{N-k}(ds)<\infty. \tag{5'}$$

Now (5′) implies that $g(y,t,T^{N-k},\omega)\in L^2(\phi\times\lambda^k)$ with probability one, which in turn implies that for $Y=\mathbb{R}^d$, $\phi=\lambda^d$, etc. we can substitute a well-known result of Zygmund [8] for Lemma 2.

First, here is why (5′) implies g is a.s. square-integrable. We know from the proof that the conditions of Lemma 3 are in force with probability one, and for such ω:

$$\int_Y \phi(dy) \int_{T^k} g^2(y, s, T^{N-k}, \omega)\, \lambda^k(ds)$$

$$= \int_Y \phi(dy) \int_{T^k} g(y, s, T^{N-k}, \omega)\, \gamma(y, ds\, dt, \omega) \qquad \text{(by (13))}$$

$$= \int_{T^N} g(X(t, \omega), t_1, \ldots, t_k, T^{N-k}, \omega)\, \lambda^N(dt) \;\; (\mu(dy, dt) = \gamma(y, dt)\, \phi(dy))$$

$$= \int_{T^k} \lambda^k(dt) \int_{T^{N-k}} \alpha(X(t, s, \omega), t, T^{N-k}, \omega)\, \lambda^{N-k}(ds)$$

$$= \int_{T^k} \lambda^k(dt) \int_{T^{N-k}} \left\{ \lim_n \frac{1}{\xi(n^{-1})} \int_{T^{N-k}} I_{[0, n^{-1}]}(\rho(X(t, s, \omega), X(t, r, \omega))\, \lambda^{N-k}(dr) \right\}$$

$$\cdot\, \lambda^{N-k}(ds),$$

from (20) with $A = T^{N-k}$. It follows that

$$E \int_Y \phi(dy) \int_{T^k} g^2(y, s, T^{N-k}, \omega)\, \lambda^k(ds)$$

$$\leqq \int_{T^k} \lambda^k(dt) \int_{T^{N-k}} \lambda^{N-k}(ds) \int_{T^{N-k}} \sup_{\varepsilon > 0} \frac{1}{\xi(\varepsilon)} P\{\rho(X(t, s), X(t, r)) \leqq \varepsilon\}\, \lambda^{N-k}(dr) < \infty.$$

As for Lemma 2, Zygmund [8] showed that the theorem of Lebesgue that

$$\lim_{\varepsilon \downarrow 0} (\alpha(m))^{-1} \varepsilon^{-m} \int_{B_m(t, \varepsilon)} f(s)\, \lambda^m(ds) = f(t) \;\; \lambda^m\text{-a.e.}$$

for any $f \in L^1(\mathbb{R}^m)$ could be extended by replacing the (closed) balls $B_m(t, \varepsilon)$ by any family of rectangles in $\mathbb{R}^m$ with sides parallel to the axes and contracting to t, *provided* $f \in L^p(\mathbb{R}^m)$ for some $1 < p \leqq \infty$. Since the B_δ's in Theorem B can be enclosed in such a family of rectangles in $\mathbb{R}^{d+k}$, the proof there works if $g \in L^2(\mathbb{R}^{d+k})$ without Lemma 2.

§ 3

First, we will mention some general sufficient conditions for (5), then proceed to specific examples for the case $Y = \mathbb{R}^d$ and X Gaussian.

When Y is a normed linear space, $\rho(y, z) = \|y - z\|$, and when the distribution of $X(s) - X(t)$, $s \neq t$, is absolutely continuous with respect to ϕ, say

$$P(X(s) - X(t) \in dy) = \phi(y; s, t)\, \phi(dy),$$

then a sufficient condition for (5) is

$$\int_{T^{N-k}} \sup_{y \in Y} \phi(y; (t_1, \ldots, t_k, s), t)\, \lambda^{N-k}(ds) < \infty \qquad \lambda^N\text{-a.e.}$$

This can be readily applied in the Gaussian case with $Y = \mathbb{R}^d$. Let $X = (X_1, \ldots, X_d)$ be Gaussian, $E X_i(t) \equiv 0$, and suppose that for each $s \neq t$ (or just λ^{2N}-a.e. (s, t) will do), $\{X_i(t) - X_i(s)\}_{i=1}^d$ has a Lebesgue density $\phi(y; s, t)$, $y \in \mathbb{R}^d$, i.e. $|\Lambda(s, t)| \neq 0$ where $|\Lambda(s, t)|$ is the determinant of the covariance matrix $\Lambda(s, t)$ of

$\{X_i(t)-X_i(s)\}_{i=1}^d$. Then

$$\sup_{y\in\mathbb{R}^d}\phi(y;s,t)=(2\pi)^{-d/2}\,|\Delta(s,t)|^{-1/2}.$$

Consequently, a sufficient condition for (5) is

$$\int_{T^{N-k}} |\Delta((t_1,\ldots,t_k,s),t)|^{-1/2}\,\lambda^{N-k}(ds)<\infty \qquad \lambda^N\text{-a.e.} \tag{21}$$

In fact, (21) is also necessary for (5): since $\phi(y;s,t)$ is continuous at $y=0$, for $s\neq t$,

$$\begin{aligned}\sup_{\varepsilon>0}\varepsilon^{-d}P(\|X(s)-X(t)\|\leqq\varepsilon)&\geqq\lim_{\varepsilon\downarrow 0}\varepsilon^{-d}\int_{B_d(0,\varepsilon)}\phi(y;s,t)\,\lambda^d(dy)\\&=\alpha(d)\,\phi(0;s,t)=\alpha(d)(2\pi)^{-d/2}\,|\Delta(s,t)|^{-1/2}.\end{aligned}$$

When the components $X_1,\ldots,X_d$ are independent, $\sigma_i^2(s,t)=E(X_i(s)-X_i(t))^2$, $i=1,\ldots,d$, then (21) reduces to

$$\int_{T^{N-k}}\left[\prod_{i=1}^d\sigma_i((t_1,\ldots,t_k,s),t)\right]^{-1}\lambda^{N-k}(ds)<\infty \qquad \lambda^N\text{-a.e.} \tag{22}$$

Example 1. Suppose $N\geqq d$ and $\sigma_i(s,t)\geqq\Psi(\|s-t\|)$, $i=1,\ldots,d$, where

$$\int_0^{\sqrt{d}}\frac{dt}{\Psi(t)}<\infty \quad\text{and}\quad \sup_{0<r<\sqrt{d}}\frac{r}{\Psi(r)}=c<\infty.$$

(If Ψ is monotone and $1/\Psi$ is integrable, then $\dfrac{\Psi(r)}{r}\to\infty$ so $c<\infty$.)

Then with $k=N-d$, $r\in T^k$, $t\in T^d$:

$$\begin{aligned}\int_{T^d}\left[\prod_{i=1}^d\sigma_i((r,s),(r,t))\right]^{-1}\lambda^d(ds)&\leqq\int_{T^d}[\Psi(\|s-t\|)]^{-d}\,\lambda^d(ds)\\&\leqq\text{const.}\int_0^{\sqrt{d}}\frac{r^{d-1}\,\lambda^1(dr)}{(\Psi(r))^d}\\&\leqq\text{const.}\,c^{d-1}\int_0^{\sqrt{d}}\frac{dr}{\Psi(r)}<\infty.\end{aligned}$$

(Of course, when $d=1$, we need only assume $\int\dfrac{dt}{\Psi(t)}<\infty$.) Thus

$$\operatorname{ap}\lim_{s\to t}\frac{\|X(s)-X(t)\|}{\|s-t\|}=\infty \qquad \lambda^N\text{-a.e., a.s.} \tag{23}$$

Example 2. Here, $\sigma_i^2(s,t)=\|s-t\|^\alpha$, $0<\alpha<2$, the components again being independent. Changing to polar coordinates in (22), it is easy to check that (22) holds if and only if $0\leqq k<N-\dfrac{\alpha d}{2}$. (Naturally, we then want to choose the largest possible k.)

For example, for d-dimensional ("isotropic") Brownian motion, we have $\alpha=1$ so that if $N>d/2$, the largest integer smaller than $N-d/2$ is $N-1-d/2$ if d

is even and $N-d/2-1/2$ if d is odd, which yields (8) with $r=1/2+1/d$ for d even and $r=1/2+1/2d$ for d odd.

Similarly, for index $\alpha=3/2$, we obtain (8) for $r=1-u/d$ where u is the greatest integer smaller than $d/4$. (Here, of course, we must assume $N>\frac{3}{4}d$.)

Example 3 ("N-parameter Wiener process"). $X=(X_1,\ldots,X_d)$ has independent, identically distributed components,

$$E X_i(t)\equiv 0, \qquad E X_i(t)\, X_i(s)=\prod_{i=1}^{N}(t_i\wedge s_i), \qquad t=(t_1,\ldots,t_N), \qquad s=(s_1,\ldots,s_N).$$

The incremental variance is

$$\sigma_i^2(s,t)=\prod_1^N t_i+\prod_1^N s_i-2\prod_1^N(t_i\wedge s_i),$$

and (22) holds if and only if

$$\int_{T^{N-k}}\left[\prod_1^{N-k} s_i+\prod_{k+1}^{N} t_i-2\prod_1^{N-k}(s_i\wedge t_{i+k})\right]^{-d/2}\lambda^{N-k}(ds)<\infty,$$

$\left(\text{provided }\prod_1^k t_i\neq 0\right)$. Thus we wish to determine those values of $m\geqq 1$ and $\beta>0$ for which

$$\int_{T^m}\left[\prod_1^m s_i+\prod_1^m r_i-2\prod_1^m(r_i\wedge s_i)\right]^{-\beta}\lambda^m(ds)<\infty, \qquad \lambda^m\text{-a.e.} \tag{25}$$

Now the integral in (25) splits into 2^m pieces, and by symmetry a "typical" term is

$$\int_0^{r_1}\cdots\int_0^{r_j}\int_{r_{j+1}}^{1}\cdots\int_{r_m}^{1}\left[\prod_1^m r_i+\prod_1^m s_i-2\prod_1^j s_i\prod_{j+1}^m r_i\right]^{-\beta} ds_1\ldots ds_m, \tag{26}$$

$0\leqq j\leqq m$. Since the only singularity of the integral in (25) occurs when $s=r$, the integral in (26) will converge if and only if

$$\int_0^{r_1}\cdots\int_0^{r_j}\int_{r_{j+1}}^{2r_{j+1}}\cdots\int_{r_m}^{2r_m}\left[\prod_1^m r_i+\prod_1^m s_i-2\prod_1^j s_i\prod_{j+1}^m r_i\right]^{-\beta} ds_1\ldots ds_m \tag{27}$$

converges. When $j=m$, a change of variables transforms (27) into

$$\left(\prod_1^m r_i\right)^{1-\beta}\int_{T^m}\left[1-\prod_1^m s_i\right]^{-\beta}\lambda^m(ds). \tag{28}$$

In fact, for any $0\leqq j\leqq m$, (27) will converge λ^m-a.e. if and only if the integral in (28) converges. For $0\leqq j<m$, make the change of variables $u_k=s_k$, $k=1,\ldots,j$, $u_{j+k}=2r_{j+k}-s_{j+k}$, $k=1,\ldots,m-j$, so (26) becomes

$$\int_0^{r_1}\cdots\int_0^{r_m}\left(\prod_1^m r_i-\prod_1^m u_i+g(r_1,\ldots,r_m,u_1,\ldots,u_m)\right)^{-\beta} du_1\ldots du_m, \tag{29}$$

$$g=\prod_1^m u_i+\prod_1^j u_i\prod_{j+1}^m(2r_i-u_i)-2\prod_1^j u_i\prod_{j+1}^m r_i.$$

But $g \geqq 0$ for $0 \leqq u_i \leqq r_i$, $i=1, \ldots, m$, and hence the integral in (28) dominates the integral in (29), which shows that (25) holds if and only if the integral in (28), call it $Q(m, \beta)$, is finite.

Finally, notice that $Q(m, \beta) \leqq Q(m-1, \beta)$, $m \geqq 2$, and an easy computation shows that $Q(m, \beta) < \infty$ if and only if $Q(m+1, \beta+1) < \infty$. Let $[a]$ be the greatest integer less than or equal to a. We are interested in the case $m=N-k$, $\beta=d/2$. For $0 \leqq k < N - \frac{d}{2}$,

$$\begin{aligned} Q(1, \tfrac{1}{2}) < \infty \Rightarrow Q(2, \tfrac{3}{2}) < \infty \Rightarrow \cdots \Rightarrow Q\left(\left[\frac{d}{2}\right]+1, \left[\frac{d}{2}\right]+\tfrac{1}{2}\right) < \infty \\ \Rightarrow Q\left(N-k, \left[\frac{d}{2}\right]+\tfrac{1}{2}\right) < \infty \\ \Rightarrow Q\left(N-k, \frac{d}{2}\right) < \infty, \end{aligned}$$

since $\frac{d}{2} \leqq \frac{1}{2} + \left[\frac{d}{2}\right]$. For $k \geqq N - \frac{d}{2}$,

$$\begin{aligned} Q(1,1) = \infty \Rightarrow Q(2,2) = \infty \Rightarrow \cdots \Rightarrow Q\left(\left[\frac{d}{2}\right], \left[\frac{d}{2}\right]\right) = \infty \\ \Rightarrow Q\left(N-k, \left[\frac{d}{2}\right]\right) = \infty \\ \Rightarrow Q\left(N-k, \frac{d}{2}\right) = \infty. \end{aligned}$$

Hence, the conclusions here concerning (8) are the same as for the d-dimensional isotropic Brownian motion.

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