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On Certain Self-decomposable Distributions

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1. Introduction

Let ϕ denote a characteristic function (c.f.). If ϕ is such that for every positive $\alpha<1$

$$\phi(t)=\phi(\alpha t)\,\phi_\alpha(t), \qquad -\infty<t<\infty \tag{1}$$

where ϕ_α denotes a c.f., then it is called a self-decomposable c.f. and the corresponding distribution is called a self-decomposable distribution. It is well known that every self-decomposable distribution and the distribution corresponding to ϕ_α in the representation (1) for every self-decomposable c.f. ϕ are infinitely divisible (i.d.) (c.f. Lukacs [4] p. 162).

In the present paper we establish that for every gamma random variable (r.v.) Z, the distribution of $\log Z$ is self-decomposable. In particular, if Z is exponential we obtain that $\log Z$ has a self-decomposable distribution with c.f. ϕ satisfying (1) where ϕ_α denotes the c.f. of $-\alpha\log Y_\alpha$ with Y_α as a stable r.v. having left extremity zero and characteristic exponent α. Thus, if Z is exponential then we have for $0<\alpha<1$

$$\left(\frac{Z}{Y_\alpha}\right)^\alpha \stackrel{d}{=} Z, \tag{2}$$

where Y_α is as defined above and is independent of Z. Using (2), the absolute moments of Y_α and those of any symmetric stable r.v. are given and an extension of Goldie's [3] result that every scale mixture of exponential distribution is i.d. is obtained. In addition, an alternative proof, without involving probability density functions is given for Cressie's [1] result that

$$|X_\alpha|^\alpha \xrightarrow{d} \frac{1}{Z} \qquad \text{as } \alpha\to 0,$$

where X_α and Z respectively denote a strict stable random variable with characteristic exponent α and other parameters independent of α and an exponential random variable.

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From our results several interesting facts are revealed. Among these are self-decomposability of the distributions of $\log|t|$ and $\log F$ and infinite divisibility of the distributions of all powers $\geqq 2$ of t and any symmetric stable random variable where t and F denote Student's t and Snedecor's F respectively.

2. The Main Theorems

In the present section we shall establish three theorems. In these theorems and elsewhere in what follows by an extreme stable random variable we mean a positive stable random variable with Laplace transform $\exp(-t^{\alpha})$ and by gamma and exponential random variables we mean the corresponding random variables with unit scale parameters. Theorem 1 given below establishes the relation (2) mentioned in the introduction. Equivalently it proves that the distribution of the logarithm of an exponential r.v. is self-decomposable with the distribution corresponding to c.f. ϕ_{α} of (1) as that of $-\alpha \log Y_{\alpha}$ where Y_{α} is an extreme stable r.v. with characteristic exponent α. Theorem 2 establishes that the distribution of the logarithm of a gamma variable is self-decomposable and Theorem 3 derives explicitly the absolute moments of symmetric stable r.v.'s in addition to showing that the logarithm of the modulus of such a r.v. is i.d.

Theorem 1. *Let Y_{α} be an extreme stable r.v. with characteristic exponent α and let Z be an exponential r.v. independent of Y_{α}. Then $(Z/Y_{\alpha})^{\alpha}$ is distributed as Z. That is* (2) *holds.*

Proof. For $u>0$

$$P\left(\left(\frac{Z}{Y_{\alpha}}\right)^{\alpha} \geqq u\right) = P(Z \geqq Y_{\alpha} u^{1/\alpha})$$
$$= \int_0^{\infty} e^{-y u^{1/\alpha}} dP(Y_{\alpha} \leqq y)$$
$$= e^{-u} = P(Z \geqq u).$$

Hence we have $(Z/Y_{\alpha})^{\alpha} \stackrel{d}{=} Z$.

Corollary 1. *If Y_{α} is an extreme stable r.v. with characteristic exponent α then*

$$E(Y_{\alpha}^{-r}) = \frac{\Gamma\left(\frac{r}{\alpha}+1\right)}{\Gamma(r+1)} \quad \textit{for } r > -\alpha,$$

or equivalently

$$E(Y_{\alpha}^{r}) = \frac{\Gamma(1-r/\alpha)}{\Gamma(1-r)} \quad \textit{for } r < \alpha.$$

Theorem 2. *If Z has a gamma distribution with index parameter r then* $\log Z$ *has a self-decomposable distribution.*

Proof. Let $0<\alpha<1$ and let ϕ be the c.f. of $\log Z$. Then

$$\phi(t)=\frac{\Gamma(r+it)}{\Gamma(r)}, \qquad -\infty<t<\infty.$$

The c.f. ϕ is self-decomposable if the function f_α given by $f_\alpha(t)=\frac{\phi(t)}{\phi(\alpha t)}$ is a c.f. We have

$$f_\alpha(t)=\frac{\Gamma(r+it)}{\Gamma(r+i\alpha t)}, \qquad -\infty<t<\infty.$$

Using the definition of gamma function (see e.g. [8]) we get

$$\begin{aligned} f_\alpha(t)&=\frac{(r+i\alpha t)}{(r+it)}\, e^{i\gamma t(\alpha-1)} \prod_{n=1}^{\infty}\left[\left(\frac{(1+(r+i\alpha t)/n)}{(1+(r+it)/n)}\right) e^{it(1-\alpha)/n}\right]\\ &=\left(\alpha+\frac{1-\alpha}{(1+it/r)}\right) e^{-i\gamma t(1-\alpha)} \lim_{k\to\infty}\prod_{n=1}^{k}\left[\left(\alpha+\frac{1-\alpha}{(1+it/(r+n))}\right) e^{it(1-\alpha)/n}\right]. \end{aligned}$$

(In the above expressions γ is the Euler's Constant.) Note that, since $0<\alpha<1$, for $p>0$ the function η given by

$$\eta(\mathrm{t})=\alpha+\frac{1-\alpha}{1+it/p}, \qquad -\infty<t<\infty$$

is a mixture c.f. of a conjugate exponential distribution and a degenerate distribution. Hence it follows that f_α is the limit of a sequence of characteristic functions. Since f_α is continuous at zero it must be a c.f. Theorem 2 is thus proved.

Corollary 2. *If X has a normal distribution with mean zero then* $\log|X|$ *has a self-decomposable distribution. Also* $\log|t|$ *and* $\log F$ *have self-decomposable distributions when t and F have respectively Student's t and Snedecor's F distributions.*

Corollary 3. *The distribution functions G_1 and G_2 given by*

$$G_1(x)=1-\exp\{-\exp x\}, \qquad -\infty<x<\infty$$

and

$$G_2(x)=\exp\{-\exp(-x)\}, \qquad -\infty<x<\infty$$

are self-decomposable.

Theorem 3. *If X_α is a symmetric stable r.v. with characteristic exponent $\alpha<2$ and scale* 1 *then* $\log|X_\alpha|$ *is i.d. and*

$$E|X_\alpha|^\delta=\frac{2^\delta\Gamma((1+\delta)/2)\,\Gamma(1-\delta/\alpha)}{\Gamma(1/2)\,\Gamma(1-\delta/2)}$$

for $-1<\delta<\alpha$.

Proof. Let $V_{\alpha/2}$ be an extreme stable r.v. with characteristic exponent $(\alpha/2)$ and let U be a standard normal r.v. independent of $V_{\alpha/2}$. Then it easily follows that

$$X_\alpha\stackrel{d}{=}2^{1/2}\,U\,V_{\alpha/2}^{1/2}, \tag{3}$$

and hence

$$\log|X_\alpha| \stackrel{d}{=} \tfrac{1}{2}\log 2 + \log|U| + \tfrac{1}{2}\log V_{\alpha/2}.$$

Since $\log|U|$ (c.f. Corollary 2) and $\log V_{\alpha/2}$ are i.d. (this latter result follows because of Theorem 1 and the fact that in (1) $\phi_\alpha(t)$ is an i.d. c.f.) it follows that $\log|X_\alpha|$ is also i.d. Next, we have from (3), for $-1<\delta<\alpha$

$$\begin{aligned} E|X_\alpha|^\delta &= 2^{\delta/2}\, E(|U|^\delta)\, E(V_{\alpha/2}^{\delta/2}) \\ &= 2^{\delta/2}\, E((U^2)^{\delta/2})\, E(V_{\alpha/2}^{\delta/2}). \end{aligned}$$

Recalling that $(U^2)/2$ is gamma with index parameter (1/2), we then have the result from Corollary 1.

Using Mellin-Stieltjes Transforms Zolotarev has derived some of these results (see e.g. [9]).

3. An Alternative Proof of a Result of Cressie and DuMouchel

We shall now state the following due to Cressie [1].

Theorem 4. *If Y_α is a strict stable r.v. with characteristic exponent $\alpha<1$ and scale* 1 *then $|Y_\alpha|^\alpha$ converges in distribution to $1/Z$, as $\alpha \to 0$, where Z is exponential.*

This Theorem seems to have been originally proved by DuMouchel [2] but certain steps in his proof were not justified. A rigorous proof of the Theorem has been given recently by Cressie [1] assuming implicitly a corollary of Scheffé's theorem that if the probability density function f_{α_n} of $|Y_{\alpha_n}|^{\alpha_n}$ converges to that of $1/Z$ then $|Y_{\alpha_n}|^{\alpha_n}$ converges in distribution to $1/Z$. In what follows we prove Cressie's theorem as a simple application of our Theorem 1.

Proof of Theorem 4. We shall first prove the result for extreme stable r.v.'s. From Theorem 1 we have

$$(Y_\alpha/Z)^\alpha \stackrel{d}{=} 1/Z. \tag{4}$$

Allowing $\alpha \to 0$ we see that $Y_\alpha^\alpha \xrightarrow{d} \dfrac{1}{Z}$.

Now suppose that Y_α is strictly stable. Then

$$Y_\alpha \stackrel{d}{=} c^{1/\alpha}\, Y_{1\alpha} - (1-c)^{1/\alpha}\, Y_{2\alpha} \tag{5}$$

where $0 \leqq c \leqq 1$ and $Y_{1\alpha}$ and $Y_{2\alpha}$ are independent and are $\left\{\cos\left(\frac{\pi\alpha}{2}\right)\right\}^{-1/\alpha}$ times extreme stable r.v.'s with exponent α. If $c=0$ or 1 in (5) then using (4) we get the result. Suppose $0<c<1$. Define $U_\alpha = \min\{c\,Y_{1\alpha}^\alpha,\ (1-c)\,Y_{2\alpha}^\alpha\}$ and $V_\alpha = \max\{c\,Y_{1\alpha}^\alpha,\ (1-c)\,Y_{2\alpha}^\alpha\}$. Then U_α/V_α converges in distribution to a r.v. Y where $Y<1$ a.s. Now write

$$\begin{aligned} |Y_\alpha|^\alpha &\stackrel{d}{=} |c^{1/\alpha}\, Y_{1\alpha} - (1-c)^{1/\alpha}\, Y_{2\alpha}|^\alpha \\ &= V_\alpha \left|1 - \left(\frac{U_\alpha}{V_\alpha}\right)^{1/\alpha}\right|^\alpha. \end{aligned}$$

It is easy to show that in the last term V_α converges in distribution to $\max\left\{\frac{c}{Z_1}, \frac{1-c}{Z_2}\right\}$ where Z_1 and Z_2 are independent exponential r.v.'s and that $(U_\alpha/V_\alpha)^{1/\alpha}$ converges in probability to zero and hence that

$$|1-(U_\alpha/V_\alpha)^{1/\alpha}|^\alpha \to 1 \quad \text{in probability.}$$

Thus $|Y_\alpha|^\alpha \xrightarrow{d} \max\left\{\frac{c}{Z_1}, \frac{1-c}{Z_2}\right\}$ which is distributed as $1/Z$.

Remarks. The limit distribution obtained in Theorem 4 is i.d. (c.f. Steutel [6] p. 131). Theorem 4 implies Cressie's (2.10).

4. An Extension of Goldie's Result

It has been established by Goldie [3] that if Z is an exponentially distributed r.v. and if W is a non-negative r.v. independent of Z then WZ has an i.d. distribution. It is evident from Theorem 2.3.1 [7] that if Z_r has a gamma distribution with index parameter $0<r\leqq 1$ and W is any r.v. (not necessarily non-negative) independent of Z_r, then WZ_r has an infinitely divisible distribution. (Note that for $r<1$, $Z_r \stackrel{d}{=} V_r Z_1$ where V_r is a r.v. independent of Z_1 and having a beta distribution.) From Theorem 2 it follows that for every $p\geqq 1$ there exists a positive random variable Y_p independent of Z_r such that

$$\frac{Z_r}{Y_p} \stackrel{d}{=} Z_r^p.$$

Considering a r.v. W independent of Z_r and Y_p, we have

$$W\frac{Z_r}{Y_p} \stackrel{d}{=} WZ_r^p,$$

which implies because of the above observation that $W\cdot Z_r^p$ has an infinitely divisible distribution. Hence we have the following.

Theorem 5. *If Z_r is a r.v. distributed according to a gamma distribution with index parameter $0<r\leqq 1$ and W is a r.v. independent of Z_r, then for every $p\geqq 1$ the r.v. $W\cdot Z_r^p$ has an i.d. distribution.*

It may be noted that the distribution of Z_r^p is not i.d. if $0<p<1$. This follows because of Ruegg's [5] result. Consequently, we have that Theorem 5 does not remain valid if we allow p to have any value in $(0, 1)$.

From Theorem 5 we get the following two results.

Corollary 4. *Let $p\geqq 2$ and let X_α and t respectively have a symmetric stable distribution and Student's t distribution. If W_1 and W_2 are any r.v.'s independent respectively of X_α and t then $W_1|X_\alpha|^p$ and $W_2|t|^p$ are i.d.*

Corollary 5. *Every scale mixture of the distribution F given by*

$$F(x)=1-\exp(-\lambda x^\alpha), \quad x>0,\ 0<\alpha\leqq 1,\ \lambda>0$$

is i.d.

It is obvious that $W_1|X_2|^p$ has an i.d. distribution if $p \geqq 2$ and hence it follows that $W_2|t|^p$ has an i.d. distribution if $p \geqq 2$. Because of (3) it further follows that $W_1|X_\alpha|^p$ has an i.d. distribution if $0<\alpha<2$ and $p \geqq 2$. This establishes Corollary 4, while the Corollary 5 is obvious. It may be noted that Corollary 4 does not remain valid if we allow p to have any value in $(0, 2)$. This is because the result of Ruegg [5] gives that for every $0<p<2$ the r.v. $|X_2|^p$ cannot have an i.d. distribution and since there exists a sequence of t distributions converging weakly to a standard normal distribution it follows that given a positive $p<2$ we should have some t variable such that $|t|^p$ is not i.d. Because any symmetric r.v. V has the representation $V \stackrel{d}{=} |V| U$ where U is a symmetric Bernoulli r.v. independent of $|V|$, Corollary 4 remains valid if $|X_\alpha|$, $|t|$ and p are replaced respectively by X_α, t and integer $n \geqq 2$. As an immediate consequence of this we have that the distributions of $t^2, t^3, \ldots$, and $X_\alpha^2, X_\alpha^3, \ldots$ are i.d.

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