Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 97-118 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Self-Decomposable Discrete Distributions and Branching Processes

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1. Introduction and Summary

A random variable (*rv*) X(or its distribution) is said to be self-decomposable (self-dec), if for every $\alpha \in (0,1)$ there is a *rv* X_α such that

$$X \overset{d}{=} \alpha X + X_\alpha, \tag{1.1}$$

where X and X_α are independent ($\overset{d}{=}$means "equal in distribution"). In the special case that $X_\alpha \overset{d}{=} (1-\alpha^\gamma)^{1/\gamma} X$ for some $\gamma > 0$, the *rv* X is called stable (with exponent γ). Self-dec *rv*'s derive their importance from the fact that they are the solutions to a central limit problem: the set of self-dec laws coincides with the set of limit laws of normalized partial sums of independent (in case of stability, also identically distributed) *rv*'s. It is clear from (1.1) that a nondegenerate discrete distribution cannot be self-dec. In fact, the nondegenerate self-dec distributions are known to be absolutely continuous (cf. Fisz and Varadarajan (1963)). For details we refer to Loève (1977).

As an analogue (in distribution) of αX, i.e., of multiplication of a *rv* X by a constant $\alpha \in (0,1)$, in Steutel and Van Harn (1979) an operation $\alpha \odot X$ is introduced for $\mathbb{N}_0$-valued *rv*'s, in such a way that $\alpha \odot X$ is again $\mathbb{N}_0$-valued ($\mathbb{N}_0 = \{0,1,2,\dots\}$). This "multiplication" is then used to define analogues for $\mathbb{N}_0$-valued *rv*'s of the concepts of self-decomposability and stability.

In the present paper a more general operation $\alpha \odot X$ is introduced by defining operators T_α on *PGF*, the set of nonconstant probability generating functions (*pgf*'s), as follows:

$$P_{\alpha \odot X} = T_\alpha P_X, \tag{1.2}$$

where P_Z denotes the *pgf* of Z. It turns out (Sect. 2) that the only operators that can reasonably serve as analogues of multiplication as above are operators of the form

$$T_\alpha P = P \circ F_{-\log \alpha}, \tag{1.3}$$

0044-3719/82/0061/0097/$04.40

where $\circ$ denotes composition of functions and where $F=(F_t)_{t>0}$ is a composition semigroup of *pgf*'s such as occur in continuous-time branching processes. Basic facts about such semigroups are collected in Sect. 3, where also some examples are given, one of which yields the special case considered by Steutel and Van Harn (1979).

In Sect. 4 the concepts of F-self-decomposability and F-stability are discussed. A *pgf* P is called F-self-dec if for every $\alpha\in(0,1)$ there is a *pgf* P_α such that

$$P=(T_\alpha P)P_\alpha,$$

or, equivalently, if for every $t>0$ there is a *pgf* P_t such that

$$P=(P\circ F_t)P_t;$$

F-stability is defined correspondingly.

Sect. 5 treats a natural injection of the classical self-dec distributions on $[0,\infty)$ into the F-self-dec distributions on $\mathbb{N}_0$, which turns out to be a bijection when restricted to the classical stable and F-stable distributions.

In Sects. 6 and 7 canonical representations are derived for F-self-dec and F-stable *pgf*'s generalizing the representations in Steutel and Van Harn (1979), which in turn are close analogues to those in the classical case. Section 6 also indicates a correspondence between F-self-dec distributions and the invariant distributions of branching processes governed by F with immigration, established by Steutel, Vervaat and Wolfe (1980).

In Sect. 8 a central limit problem is considered for sums of independent *rv*'s normalized by means of the "multiplication" introduced in Sect. 2. In the case of identically distributed summands the F-stable distributions appear as limits, but in the general case only a subset of the F-self-dec distributions is obtained, viz. the range of the injection of Sect. 5.

2. Algebraic Considerations

Classical limit results for partial sums S_n of independent realvalued *rv*'s $X_1, X_2, \ldots$ are mostly in terms of "normalizations" $(S_n-b_n)/a_n$ with $a_n>0$. These normalizations also occur in the characterizations of several important classes of limit distributions, such as the self-dec and the stable distributions. Here we shall restrict our attention to limit distributions that occur for nonnegative X_n without translation term: $b_n=0$. In other words, we only consider nonnegative *rv*'s X with normalizations $T_\alpha (0<\alpha\leqq 1)$ given by

$$T_\alpha X=\alpha X$$

(since $a_n\to\infty$ in most limit theorems, the restriction to $\alpha\leqq 1$ is harmless). By abuse of notation we let the transformations T_α also operate on the distribution μ of X. Let $\mathscr{P}$ denote the set of probability measures on $[0,\infty)$, not concentrated at zero, and $\hat{\mu}$ the Laplace Stieltjes (LS) transform of $\mu\in\mathscr{P}$, i.e.,

$$\hat{\mu}(\tau)=Ee^{-\tau X}=\int_{[0,\infty)} e^{-\tau x}\mu(dx)\qquad(\tau\geqq 0).$$

Then the LS transform of $T_\alpha \mu$ is given by

$$(T_\alpha \mu)\hat{}(\tau) = \hat{\mu}(\alpha\tau) \qquad (\tau \geqq 0), \tag{2.1}$$

and $(T_\alpha)_{0<\alpha\leqq 1}$ has the following properties:

$$T_\alpha \text{ maps } \mathscr{P} \text{ into } \mathscr{P}, \tag{2.2a}$$

$$T_\alpha T_\beta = T_{\alpha\beta} \qquad (\text{so } (T_\alpha)_{0<\alpha\leqq 1} \text{ is a semigroup}). \tag{2.2b}$$

$$T_\alpha(\mu * \nu) = (T_\alpha \mu) * (T_\alpha \nu) \qquad (\mu, \nu \in \mathscr{P}), \tag{2.2c}$$

$$T_\alpha(p\mu + (1-p)\nu) = pT_\alpha \mu + (1-p)T_\alpha \nu \qquad (0 \leqq p \leqq 1;\ \mu, \nu \in \mathscr{P}), \tag{2.2d}$$

$$T_\alpha \text{ is continuous,} \tag{2.2e}$$

when $\mathscr{P}$ is endowed with the topology of weak convergence, i.e., convergence in distribution of the corresponding *rv*'s.

In considering analogues of the concepts of self-decomposability and stability for distributions on $\mathbb{N}_0$, Van Harn (1978), § 3.3 and Steutel and Van Harn (1979) were interested in $(T_\alpha)_{0<\alpha\leqq 1}$ satisfying (2.2b through e) that map probability measure on $\mathbb{N}_0$ on probability measures on $\mathbb{N}_0$. Clearly, the restriction of (2.1) to these measures does not have this property. One example of a (T_α) having the desired property, was studied in the above publications. Here we will characterize all such (T_α).

For distributions on $\mathbb{N}_0$ the most convenient transform is the probability generating function. The *pgf* of a distribution $(p_n)_{n\in\mathbb{N}_0}$ will mostly be denoted by the corresponding capital, i.e.,

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \qquad (|z| \leqq 1).$$

In case $p_1 = 1$ we use the notation I, so $I(z) = z$ for $|z| \leqq 1$. The collection of all *pgf*'s P with $P(0) < 1$ will be denoted by PGF, and we define $PGF_+ = \{P \in PGF\colon P(0) > 0\}$.

We now reformulate (2.2) in the present variant, and led T_α operate on PGF rather than on $\mathscr{P}$:

$$T_\alpha \text{ maps } PGF \text{ into } PGF, \tag{2.3a}$$

$$T_\alpha T_\beta = T_{\alpha\beta} \qquad (\text{so } (T_\alpha)_{0<\alpha\leqq 1} \text{ is a semigroup}), \tag{2.3b}$$

$$T_\alpha(PQ) = (T_\alpha P)(T_\alpha Q) \qquad (P, Q \in PGF), \tag{2.3c}$$

$$T_\alpha(pP + (1-p)Q) = pT_\alpha P + (1-p)T_\alpha Q \qquad (0 \leqq p \leqq 1;\ P, Q \in PGF), \tag{2.3d}$$

$$T_\alpha \text{ is continuous,} \tag{2.3e}$$

when PGF is endowed with the topology of pointwise convergence.

Theorem 2.1. *Collections of operators* $(T_\alpha)_{0<\alpha\leqq 1}$ *satisfying* (2.3) *correspond one-to-one to collections* $(F_t)_{t\geqq 0} \subset PGF$ *that are composition semigroups:*

$$F_s \circ F_t = F_{s+t} \qquad (s, t \geqq 0). \tag{2.4}$$

The correspondence is given by

$$T_\alpha P = P \circ F_{-\log\alpha} \qquad (0 < \alpha \leqq 1;\ P \in PGF). \tag{2.5}$$

Proof. First we consider $T=T_\alpha$ (one fixed α) satisfying (2.3a, c, d, e). Set $F=T(I)$, then by (2.3a) $F\in PGF$, and by (2.3c)

$$T(P)=P\circ F, \tag{2.6}$$

for $P=I^n$, $n\in\mathbb{N}_0$. By (2.3d) it follows that (2.6) holds for each polynomial $P\in PGF$, and hence, by (2.3e), for all $P\in PGF$.

Now let $(T_\alpha)_{0<\alpha\leqq 1}$ satisfy (2.3). Set

$$T_\alpha(I)=F_{-\log\alpha} \qquad (0<\alpha\leqq 1), \tag{2.7}$$

then, as has been proved above, $F_t\in PGF$ $(t\geqq 0)$ and (2.5) holds. By (2.3b) it follows that for $s,t\geqq 0$

$$\begin{aligned} F_s\circ F_t &= T_{\exp(-s)}(I)\circ F_t = T_{\exp(-t)}(T_{\exp(-s)}(I)) \\ &= T_{\exp(-s-t)}(I)=F_{s+t}. \end{aligned}$$

The converse correspondence is easily verified. □

Thus, analogues of ordinary multiplication (i.e., of (2.1)) for $\mathbb{N}_0$-valued *rv*'s are characterized by composition semigroups of *pgf*'s. Such semigroups, however, need not be very well-behaved. One could therefore impose (one of) the following additional regularity conditions, all of which, like (2.3), are very natural for analogues of scalar multiplication:

$$\lim_{\alpha\uparrow 1} T_\alpha P=P \qquad (P\in PGF), \tag{2.8}$$

$$\lim_{\alpha\downarrow 0} T_\alpha P\equiv 1 \qquad (P\in PGF), \tag{2.9}$$

$$(T_\alpha P)'\,(1)=\alpha P'(1) \qquad (P\in PGF;\ 0<\alpha\leqq 1), \tag{2.10}$$

or in terms of the corresponding semigroup $(F_t)_{t\geqq 0}$ (cf. (2.5)):

$$\lim_{t\downarrow 0} F_t=I, \tag{2.8$'$}$$

$$\lim_{t\to\infty} F_t\equiv 1, \tag{2.9$'$}$$

$$F_t'(1)=e^{-t} \qquad (t\geqq 0). \tag{2.10$'$}$$

As necessarily $F_0=I$, a semigroup $(F_t)_{t\geqq 0}\subset PGF$ satisfying (2.8′) is continuous at $t=0$, and hence at all $t\geqq 0$. Such *continuous semigroups* are familiar to probabilists; they occur in branching processes. In the next section we summarize some properties of branching processes needed for studying the concepts of self-decomposability and stability for $\mathbb{N}_0$-valued *rv*'s. In Remark 3.1 we return to conditions (2.9′) and (2.10′).

3. Continuous Semigroups and Markov Branching Processes

Let $(F_t)_{t\geqq 0}\subset PGF$ be a continuous (composition) semigroup. As $F_t\in PGF$, we have $F_t\not\equiv 1$ $(t\geqq 0)$, but henceforth we also exclude the trivial case $F_t=I$ $(t>0)$. It can be shown that the continuity requirement (2.8′) implies that $F_t(z)$ is a differentiable function of $t \geqq 0$ with $\left(F_t'(z) = \frac{\partial}{\partial z} F_t(z)\right)$

$$\frac{\partial}{\partial t} F_t(z) = U(F_t(z)) = U(z) F_t'(z) \qquad (|z| \leqq 1;\ t \geqq 0), \tag{3.1}$$

where

$$U(z) = \frac{\partial}{\partial t} F_t(z)|_{t=0} = \lim_{t \downarrow 0} (F_t(z) - z)/t \qquad (|z| \leqq 1) \tag{3.2}$$

is called the (infinitesimal) *generator* of $(F_t)_{t \geqq 0}$. Furthermore, there exist an $a > 0$ and a *pgf* $H(z) = \sum h_n z^n$ with $h_1 = 0$ and satisfying the non-explosion condition:

$$\int_{(1-\varepsilon, 1)} |H(x) - x|^{-1} \, dx = \infty \qquad (\varepsilon > 0), \tag{3.3}$$

such that

$$U(z) = a\{H(z) - z\} \qquad (|z| \leqq 1). \tag{3.4}$$

(Note that (3.3) is satisfied, for instance, if $H'(1) < \infty$.) Conversely, if $a > 0$ and a *pgf* H, with $h_1 = 0$ and satisfying (3.3), are given, then there exists a unique continuous semigroup $(F_t)_{t \geqq 0} \subset PGF$ satisfying (3.1) with U given by (3.4). All these results follow by combining Chapter V of Harris (1963) with Lamperti (1967a) or Silverstein (1968). They are also proved in Vervaat (1980).

Now define the matrix $(p_{ij}(t))_{i,j \in \mathbb{N}_0}$ for $t \geqq 0$ by

$$\sum_{j=0}^{\infty} p_{ij}(t) z^j = \{F_t(z)\}^i \qquad (i \in \mathbb{N}_0;\ t \geqq 0;\ |z| \leqq 1).$$

Then it is easily shown that $(p_{ij}(t))$ is a standard transition matrix, and hence a continuous-time *Markov branching process* $(Z_t)_{t \geqq 0}$ exists such that

$$F_t(z) = \sum_{j=0}^{\infty} \Pr[Z_t = j \mid Z_0 = 1] \, z^j \qquad (t \geqq 0;\ |z| \leqq 1).$$

The process $(Z_t)_{t \geqq 0}$ allows the following "infinitesimal description" in terms of the quantities a and H (cf. Harris (1963) and Athreya and Ney (1972)): each individual in the process has probability $a\Delta t + o(\Delta t)$ of dying in an interval of (small) length Δt; if it dies, it is replaced by n individuals with probability h_n.

Next, consider a continuous semigroup $(F_t)_{t \geqq 0}$ with generator $U(z) = a\{H(z) - z\}$. We give a few results needed later, which can be found in the books mentioned above.

First we note that $H'(1) < \infty$ iff $m = F_1'(1) < \infty$, in which case

$$m = \exp[a\{H'(1) - 1\}], \qquad F_t'(1) = m^t \qquad (t \geqq 0). \tag{3.5}$$

The extinction probability $q = \Pr[\lim_{t \to \infty} Z_t = 0 \mid Z_0 = 1] = \lim_{t \to \infty} F_t(0)$ equals the smallest root in $[0, 1]$ of the equation $H(z) = z$. It satisfies

$$F_t(q) = q \quad (t \geqq 0), \qquad \lim_{t \to \infty} F_t(z) = q \quad (|z| \leqq 1), \tag{3.6}$$

and

$$m \leqq 1 \Rightarrow q=1, \qquad m>1 \Rightarrow q<1. \tag{3.7}$$

Remark 3.1. From (3.6) and (3.7) it follows that the regularity condition (2.9′) is satisfied iff $m \leqq 1$. If $m<1$, then it is no restriction to take $m=e^{-1}$, so $-\log m = 1$ (this can be achieved by a change of time scale, i.e., by considering $(\bar{F}_t)_{t \geqq 0}$ with $\bar{F}_t = F_{t/(-\log m)}$), in which case condition (2.10′) is also satisfied.

We shall mainly be concerned with semigroups $(F_t)_{t \geqq 0}$ with $m \leqq 1$. In this case $q=1$ and

$$U(z)>0 \qquad (0 \leqq z<1), \tag{3.8}$$

so that the first part of (3.1) can be rewritten as

$$\int_{(z,\, F_t(z))} U(x)^{-1}\, dx = t \qquad (t \geqq 0;\ 0 \leqq z<1). \tag{3.9}$$

It follows that the function A, defined by

$$A(z)=\exp\Big[-\int_{(0,z)} U(x)^{-1}\, dx\Big] \qquad (0 \leqq z \leqq 1), \tag{3.10}$$

which is decreasing from 1 to 0 (cf. (3.3)), has the following property:

$$A(F_t(z))=e^{-t} A(z) \qquad (t \geqq 0;\ 0 \leqq z \leqq 1). \tag{3.11}$$

If $m<1$, then

$$B(z)=\lim_{t \to \infty} \frac{F_t(z)-F_t(0)}{1-F_t(0)} \qquad (0 \leqq z \leqq 1) \tag{3.12}$$

exists, and convergence is uniform on $[0,1]$. B is a *pgf* with $B(0)=0$ and is related to A by

$$B(z)=1-A(z)^{-\log m} \qquad (0 \leqq z \leqq 1). \tag{3.13}$$

From (3.12) it follows that

$$1-F_t(z) \sim (1-B(z))\{1-F_t(0)\} \qquad (t \to \infty;\ \text{uniformly on } [0,1]), \tag{3.14}$$

where by the semigroup property

$$1-F_{s+t}(0) \sim m^s \{1-F_t(0)\} \qquad (t \to \infty;\ s \geqq 0). \tag{3.15}$$

This means that the function V, defined by

$$V(x)=1-F_{\log x}(0) \qquad (x \geqq 1), \tag{3.16}$$

varies regularly at ∞ with exponent $\log m$, i.e., $V(x)=x^{\log m} L(x)$ for some slowly varying L. If $\sum h_n n \log n<\infty$, then even

$$V(x) \sim x^{\log m} \qquad (x \to \infty). \tag{3.17}$$

Finally we prove a property of the function A (and hence of the *pgf* B) that we need in Sect. 5.

Lemma 3.2. *Let* $m<1$.

(i) *For all* $\tau>0$: $A(1-\tau x)\sim\tau^{1/(-\log m)}A(1-x)$ $(x\downarrow 0)$.

(ii) *If* $p(x)\downarrow 0$, $q(x)\downarrow 0$, $p(x)\sim q(x)$ $(x\downarrow 0)$, *then* $A(1-p(x))\sim A(1-q(x))$ $(x\downarrow 0)$.

Proof. From (3.11) with $z=0$ it is seen that the function V, defined by (3.16), satisfies

$$V(x)=1-A^{\sim}(1/x)\qquad(x\geqq 1),\tag{3.18}$$

where $A^{\sim}$ denotes the inverse function of A. From De Haan (1970), p. 22 or Seneta (1976), p. 24 it follows that the inverse $V^{\sim}$ of V, which satisfies

$$V^{\sim}(y)=\{A(1-y)\}^{-1}\qquad(0<y\leqq 1),$$

varies regularly at $0=V(\infty)$ with exponent $(\log m)^{-1}$. This is equivalent to (i), and has part (ii) as a consequence (cf. De Haan (1970), p. 21). □

We conclude this section with three examples of continuous semigroups $(F_t)_{t\geqq 0}$ with $m\leqq 1$. In each case we start from (a,H) and calculate U,m,F_t,A and (if $m<1$) B by means of the relations (3.4), (3.5), (3.9), (3.10) and (3.13), respectively.

Example 3.3. Take $a>0$ and $H\equiv 1$. Then

$$U(z)=a(1-z);\qquad m=e^{-a};\qquad F_t(z)=1-m^t(1-z);$$
$$A(z)=(1-z)^{1/a};\qquad B(z)=z.$$

Example 3.4. Take $a>0$ and $H(z)=1-p+pz^2$ with $0<p\leqq\frac{1}{2}$. Then

(i) $p=\frac{1}{2}$: $H(z)=\frac{1}{2}(1+z^2)$; $U(z)=\frac{1}{2}a(1-z)^2$; $m=1$;

$$F_t(z)=1-\frac{1-z}{1+\frac{1}{2}at(1-z)};\qquad A(z)=\exp\left[-\frac{2}{a}\frac{z}{1-z}\right].$$

(ii) $p<\frac{1}{2}$: $U(z)=a(1-z)(1-p-pz)$; $m=e^{-a(1-2p)}$;

$$F_t(z)=1-\frac{m^t(1-z)}{1+p(1-2p)^{-1}(1-m^t)(1-z)};$$
$$A(z)=\left\{\frac{(1-p)(1-z)}{1-p-pz}\right\}^{1/(-\log m)};\qquad B(z)=\frac{(1-2p)z}{1-p-pz}.$$

Example 3.5. Take $a>0$ and $H(z)=z+(1+\rho)^{-1}(1-z)^{1+\rho}$ with $0<\rho<1$. Then

$$U(z)=a(1+\rho)^{-1}(1-z)^{1+\rho};\qquad m=1;$$
$$F_t(z)=1-\{\rho(1+\rho)^{-1}at+(1-z)^{-\rho}\}^{-1/\rho};$$
$$A(z)=\exp[-(a\rho)^{-1}(1+\rho)\{(1-z)^{-\rho}-1\}].$$

4. Self-Decomposability and Stability with Respect to $(F_t)_{t\geqq 0}$

For probability measures on $[0,\infty)$ the classical concepts of self-decomposability and stability can be introduced as follows (cf. Loève (1977) and Feller

(1971)). With the notation of (2.1), $\mu\in\mathscr{P}$ is said to be *self-decomposable* if

$$\mu=(T_\alpha\mu)*\mu_\alpha \qquad (0<\alpha<1), \tag{4.1}$$

where $\mu_\alpha\in\mathscr{P}$. In terms of *LS* transforms and *rv*'s:

$$\begin{gathered}\hat{\mu}(\tau)=\hat{\mu}(\alpha\tau)\,\hat{\mu}_\alpha(\tau) \qquad (\tau\geqq 0),\\ X\overset{d}{=}\alpha X+X_\alpha \qquad (X, X_\alpha \text{ independent}).\end{gathered} \tag{4.1'}$$

More specially, $\mu\in\mathscr{P}$ is said to be *stable* if

$$\mu=(T_{1/c_n}\mu)^{*n} \qquad (n\in\mathbb{N}), \tag{4.2}$$

with $c_n>0$, i.e., in terms of *LS* transforms and *rv*'s:

$$\hat{\mu}(\tau)=\{\hat{\mu}(\tau/c_n)\}^n \quad (\tau\geqq 0), \qquad X\overset{d}{=}c_n^{-1}(X_1+\ldots+X_n), \tag{4.2'}$$

where $X_1, X_2, \ldots$ are independent and $X_k\overset{d}{=}X$ $(k\in\mathbb{N})$. There exists $\gamma\in(0,1]$ such that $c_n=n^{1/\gamma}$ $(n\in\mathbb{N})$, in which case μ is called *stable with exponent* γ and (4.2) is equivalent to (cf. Feller (1971), p. 171)

$$\mu=(T_\alpha\mu)*(T_{(1-\alpha^\gamma)^{1/\gamma}}\mu) \qquad (0<\alpha<1). \tag{4.3}$$

Contrary to general conventions, we do not exclude degenerate distributions in $\mathscr{P}$ from our definition of stability, so these are stable with exponent $\gamma=1$.

We now want to generalize these concepts to the situation where multiplying nonnegative *rv*'s by positive constants is replaced by applying semigroups $(T_\alpha)_{0<\alpha\leqq 1}$ satisfying (2.3) to *pgf*'s. By Theorem 2.1 such a semigroup is characterized by a composition semigroup $(F_t)_{t\geqq 0}\subset PGF$, and applying T_α to $P\in PGF$ corresponds to composing P with $F_{-\log\alpha}$. Thus, we are led to the following definition, in which, as in Sect. 3, $F=(F_t)_{t\geqq 0}\subset PGF$ is required to be a continuous semigroup with $F_t\neq I$ $(t>0)$.

Definition 4.1. $P\in PGF$ is said to be *F-self-decomposable* if

$$P=(P\circ F_t)\,P_t \qquad (t>0), \tag{4.4}$$

where $P_t\in PGF$. $P\in PGF$ is said to be *F-stable* if

$$P=P^n\circ F_{\log c_n} \qquad (n\in\mathbb{N}), \tag{4.5}$$

with $c_n\geqq 1$.

Using (2.3), the continuity of the semigroup F and the following obvious implications:

$$\begin{aligned} T_\alpha P&=T_\beta P \quad \text{for some } P\in PGF \Rightarrow \alpha=\beta,\\ T_\alpha P&=T_\alpha Q \quad \text{for some } \alpha\in(0,1] \Rightarrow \mathrm{P}=Q,\end{aligned}$$

we can adapt p. 77, 78 of Lamperti (1966) to conclude that if P is F-stable then for some $\gamma>0$

$$P=P^x\circ F_{(\log x)/\gamma} \qquad (x\geqq 1). \tag{4.6}$$

In this case, we again call *PF-stable with exponent* γ, and it can easily be shown that this is equivalent to (cf. (4.3))

$$P=(P\circ F_{-\log\alpha})(P\circ F_{-\log(1-\alpha^{\gamma})^{1/\gamma}}) \qquad (0<\alpha<1), \tag{4.7}$$

or also

$$P=(P\circ F_t)(P\circ F_s) \qquad (s,t>0;\ e^{-\gamma t}+e^{-\gamma s}=1). \tag{4.7'}$$

Hence we can state the following result.

Theorem 4.2. *An F-stable* $P\in PGF$ *is F-self-dec.*

The concepts of F-self-dec and F-stability can be interpreted in the Markov branching process $(Z_t)_{t\geqq 0}$ corresponding to F (cf. Sect. 3). If X has *pgf* P and $Z_t(X)$ denotes the number of individuals at time t, given X individuals at time 0, then $Z_t(X)$ has *pgf* $P\circ F_t$, so that, for instance, (4.4) can be written in the form

$$Z_0(X)\overset{d}{=}Z_t(X)+X_t \qquad (t>0), \tag{4.8}$$

with $X_t\in\mathbb{N}_0$ independent of $Z_t(X)$. In view of (4.8) one may expect that only semigroups corresponding to branching processes with extinction probability $q=1$ can have self-dec *pgf*'s. Indeed, we have the following properties (recall that $m=F_1'(1)$).

Lemma 4.3. (i) *If there exists an F-self-dec* $P\in PGF$*, then necessarily* $m\leqq 1$*. If in addition* $P'(1)<\infty$*, then* $m<1$.

(ii) *If there exists an F-stable* $P\in PGF$ *(exponent* γ*), then necessarily* $m<1$ *and* $\gamma\leqq -\log m$*. If in addition* $P'(1)<\infty$*, then* $\gamma=-\log m$.

(iii) *If* $P\in PGF$ *is F-self-dec, then necessarily* $P\in PGF_+$ *(so* $P(0)>0$*).*

Proof. (i) Let P be F-self-dec, and suppose $m>1$ (possibly $m=\infty$). Then necessarily $q<1$. As $F_t(q)=q\,(t>0)$, we see from (4.4) that $P_t(q)=1$ $(t>0)$ and hence $P_t\equiv 1$ $(t>0)$. But then it follows from (4.4) that $F_t=I\,(t>0)$, which contradicts $m>1$. (Moreover, $F_t=I\,(t>0)$ has been excluded.) Hence $m\leqq 1$. If $P'(1)<\infty$, then differentiation of (4.4) with respect to z and letting $z\uparrow 1$ yield

$$P'(1)\{1-m^t\}=\lim_{z\uparrow 1}P_t'(z)>0,$$

which implies $m<1$.

(ii) Let P be F-stable with exponent γ. Differentiation of (4.6) with respect to z gives

$$\lim_{z\uparrow 1}P'(z)/P'(F_{(\log x)/\gamma}(z))=x\,m^{(\log x)/\gamma}=x^{1+(\log m)/\gamma} \tag{4.9}$$

Because of Theorem 4.2 and part (i) of the present lemma, we know that $m\leqq 1$. Hence

$$F_t(z)>z \qquad (t>0;\ 0\leqq z<1), \tag{4.10}$$

and as P' is increasing, it follows that the limit in (4.9) does not exceed 1, i.e., $\gamma\leqq -\log m$. As $\gamma>0$, we also have $m<1$. Finally, if $P'(1)<\infty$, then the limit in (4.9) is equal to 1, and hence $\gamma=-\log m$.

(iii) Let $P \in PGF$ satisfy (4.4). Then $\lim_{t \downarrow 0} P_t(z) = 1$ $(0 < z \leqq 1)$, so $P_t(0) > 0$ for all t sufficiently small. Again we have (4.10), hence also $P(F_t(0)) > 0$ for all $t > 0$. By (4.4) it now follows that $P(0) > 0$. ☐

5. A Relation Between the Self-Dec Distributions on $[0, \infty)$ and Those on $\mathbb{N}_0$

It will turn out that the necessary condition $m \leqq 1$ in Lemma 4.3 is not sufficient. However, if $m < 1$ then there always exist F-self-dec *pgf*'s. This can be shown by means of a relation with the self-dec distributions on $[0, \infty)$, which also gives a characterization of these distributions. So, consider throughout this section a fixed continuous semigroup $(F_t)_{t \geqq 0}$ with $m = e^{-1}$ (cf. Remark 3.1) and with A as in (3.10). Note that now $A = 1 - B$ with B the *pgf* in (3.12). For $\theta > 0$ define the map $\pi_\theta = \pi_\theta^F \colon \mathscr{P} \to PGF_+$ as follows:

$$\pi_\theta \mu(z) = \hat{\mu}(\theta A(z)) \qquad (\mu \in \mathscr{P};\ 0 \leqq z \leqq 1).$$

Indeed, $\pi_\theta \mu \in PGF_+$, as it is a mixture of compound Poisson *pgf*'s and $\pi_\theta \mu(0) = \hat{\mu}(\theta) > 0$. In the following lemma we summarize some simple properties of $(\pi_\theta)_{\theta > 0}$ for later use. The operators T_α $(0 < \alpha \leqq 1)$ are defined on $\mathscr{P}$ by (2.1) and on PGF by (2.5).

Lemma 5.1. (i) π_θ *is continuous with respect to weak convergence* $(\theta > 0)$.

(ii) $\pi_\theta \mu = \pi_\theta \nu$ *for some* $\theta > 0 \Leftrightarrow \mu = \nu \Leftrightarrow \pi_\theta \mu = \pi_\theta \nu$ *for all* $\theta > 0$ $(\mu, \nu \in \mathscr{P})$.

(iii) $\pi_\theta(\mu * \nu) = (\pi_\theta \mu)(\pi_\theta \nu)$ $(\theta > 0;\ \mu, \nu \in \mathscr{P})$.

(iv) $\pi_\theta(T_\alpha \mu) = T_\alpha(\pi_\theta \mu)$ $(\theta > 0;\ 0 < \alpha \leqq 1;\ \mu \in \mathscr{P})$.

(v) *For all* $\mu \in \mathscr{P}$

$$\hat{\mu}(\tau) = \lim_{\theta \to \infty} \pi_\theta \mu(\exp[-\tau V(\theta)]) \qquad (\tau \geqq 0), \tag{5.1}$$

where V *is defined by* (3.16) (*or* (3.17)).

Proof. (i) and (iii) are trivial. (ii) expresses the well-known fact that an *LS* transform is determined by its values on a finite interval (cf. Widder (1946), Chap. IV).

(iv) By (3.11) we can write

$$\begin{aligned}\pi_\theta(T_\alpha \mu)(z) &= \hat{\mu}(\alpha \theta A(z)) = \hat{\mu}(\theta A(F_{-\log \alpha}(z))) \\ &= \pi_\theta \mu(F_{-\log \alpha}(z)) = T_\alpha(\pi_\theta \mu)(z).\end{aligned}$$

(v) Let $\mu \in \mathscr{P}$ and $\tau > 0$. As $\hat{\mu}$ is continuous, for (5.1) it is sufficient to prove that

$$A(\exp[-\tau V(y^{-1})]) \sim \tau y \qquad (y \downarrow 0). \tag{5.2}$$

By the second part of (3.6) we have $V(y^{-1})\to 0$ as $y\downarrow 0$. Now, taking $x=V(y^{-1})$ in Lemma 3.2(i) and using (3.18), we see that

$$A(1-\tau V(y^{-1}))\sim\tau y \quad (y\downarrow 0),$$

which by Lemma 3.2(ii) is equivalent to (5.2). ☐

Remark 5.2. Relation (5.1) expresses the fact that μ is the weak limit (as $\theta\to\infty$) of the measures $\mu_\theta\in\mathscr{P}$, where μ_θ assigns masses $p_n(\theta)$ to the points $nV(\theta)$ $(n\in\mathbb{N}_0)$ and $(p_n(\theta))_{n\in\mathbb{N}_0}$ has *pgf* $\pi_\theta\mu$.

Now, using Lemma 5.1, we can easily show that $(\pi_\theta)_{\theta>0}$ maps the class of self-dec (stable) elements of $\mathscr{P}$ into that of *PGF*.

Theorem 5.3. *Let $\mu\in\mathscr{P}$. Then μ is self-dec iff $\pi_\theta\mu$ is F-self-dec for all $\theta>0$. Similarly, if $\gamma\in(0,1]$, then μ is stable with exponent γ iff $\pi_\theta\mu$ is F-stable with exponent γ for some, and then for all, $\theta>0$.*

Proof. Let $\mu\in\mathscr{P}$ be self-dec and $\theta>0$. Then, using (4.1) and Lemma 5.1, we can write for all $\alpha\in(0,1)$

$$\pi_\theta\mu=\pi_\theta((T_\alpha\mu)*\mu_\alpha)=\pi_\theta(T_\alpha\mu)(\pi_\theta\mu_\alpha)=T_\alpha(\pi_\theta\mu)(\pi_\theta\mu_\alpha).$$

As $\mu_\alpha\in\mathscr{P}$, we have $\pi_\theta\mu_\alpha\in PGF$, and hence $\pi_\theta\mu$ is F-self-dec. Conversely, suppose $\pi_\theta\mu$ to be F-self-dec for all $\theta>0$. Then there exist *pgf*'s $P_{\theta,\alpha}$ $(\theta>0;\ 0<\alpha<1)$ such that

$$\pi_\theta\mu=T_\alpha(\pi_\theta\mu)P_{\theta,\alpha}=\pi_\theta(T_\alpha\mu)P_{\theta,\alpha}.$$

Now by Lemma 5.1(v) it follows that

$$\lim_{\theta\to\infty}P_{\theta,\alpha}(\exp[-\tau V(\theta)])=\hat\mu(\tau)/\hat\mu(\alpha\tau) \quad (\tau>0),$$

i.e. (cf. Remark 5.2), $\hat\mu(\tau)/\hat\mu(\alpha\tau)$ is the limit of a sequence of *LS* transforms, and hence, as $\hat\mu(\tau)/\hat\mu(\alpha\tau)\to 1$ for $\tau\downarrow 0$, is itself the *LS* transform of some $\mu_\alpha\in\mathscr{P}$. In view of (4.1′) we conclude that μ is self-dec.

The final statement of the theorem is an immediate consequence (use Lemma 5.1(ii) for the "if"-part) of the following relation:

$$\pi_\theta((T_{n^{-1/\gamma}}\mu)^{*n})=\{T_{n^{-1/\gamma}}(\pi_\theta\mu)\}^n \quad (n\in\mathbb{N}), \tag{5.3}$$

which easily follows from Lemma 5.1(iii) and (iv). ☐

In case $(F_t)_{t\geq 0}$ is given by Example 3.3, the preceding theorem has been proved by Forst (1979). Although he uses a relation like (5.1), his proof is rather indirect, via the canonical measures of the infinitely divisible (infdiv) distributions involved. In the case of Example 3.3 several other subclasses of $\mathscr{P}$ and of PGF_+ have been connected by means of $(\pi_\theta)_{\theta>0}$ (cf. Goldie (1967), Hirsch (1975) and Forst (1978)). We now generalize the relation, implicitly given by Goldie.

Theorem 5.4. *Let $\mu\in\mathscr{P}$. Then μ is infdiv iff $\pi_\theta\mu$ is infdiv for all $\theta>0$.*

Proof. The "only if"-part is a consequence of the subordination theorem of Feller (1971), Chap. XVII (see also Steutel (1970)), but is also immediate from

$$\mu=(\mu_n)^{*n} \Rightarrow \pi_\theta \mu=(\pi_\theta \mu_n)^n \qquad (n \in \mathbb{N}).$$

Conversely, if $\pi_\theta \mu$ is infdiv for all $\theta>0$, then from (5.1) and Remark 5.2 it is seen that μ is the weak limit of infdiv measures, and hence is itself infdiv. □

On the one hand Theorem 5.3 gives necessary and sufficient conditions for $\mu \in \mathscr{P}$ to be self-dec or stable. On the other hand, starting from well-known self-dec or stable measures $\mu \in \mathscr{P}$, by means of this theorem we can construct *pgf*'s that are *F*-self-dec or *F*-stable. It is well known that the *LS* transforms of the stable measures $\mu \in \mathscr{P}$ with exponent γ $(0<\gamma \leqq 1)$ are given by the functions of the following form (cf. Feller (1971), Chap. XIII):

$$\hat{\mu}(\tau)=\exp[-\lambda \tau^\gamma] \qquad (\tau \geqq 0), \tag{5.4}$$

where $\lambda>0$. Hence, by Theorem 5.3, all functions P of the form

$$P(z)=\exp[-\lambda A(z)^\gamma] \qquad (0 \leqq z \leqq 1), \tag{5.5}$$

with $\lambda>0$, are *F*-stable *pgf*'s with exponent γ. In Sect. 7 (and again in Sect. 8) we shall show that there are no other *F*-stable *pgf*'s.

For self-dec measures no representation like (5.4) in the stable case seems to be known. However, using the method of proof to be applied for Theorem 6.1, we obtain the following representation theorem (already mentioned briefly in Steutel and Van Harn (1979)).

Theorem 5.5. *A function ϕ on $[0,\infty)$ is the LS transform of a self-dec $\mu \in \mathscr{P}$ iff ϕ has the form*

$$\phi(\tau)=\exp\Big[\int_{(0,\tau)} \sigma^{-1} \log \hat{\nu}(\sigma)\, d\sigma\Big] \qquad (\tau \geqq 0), \tag{5.6}$$

where $\nu \in \mathscr{P}$ is infdiv and such that the integral n (5.6) is finite.

Let $\mu \in \mathscr{P}$ have an *LS* transform of the form (5.6) with ν inf div. Then, by Theorem 5.3, for all $\theta>0$ the following function is an *F*-self-dec *pgf* (see also (3.10)):

$$\begin{aligned}\pi_\theta \mu(z)&=\exp\Big[\int_{(0,\theta A(z))} \sigma^{-1} \log \hat{\nu}(\sigma)\, d\sigma\Big]\\ &=\exp\Big[\int_{(z,1)} U(x)^{-1} \log \hat{\nu}(\theta A(x))\, dx\Big].\end{aligned} \tag{5.7}$$

By Theorem 5.4 it now follows that $\pi_\theta \mu$ has the form

$$\pi_\theta \mu(z)=\exp\Big[\int_{(z,1)} U(x)^{-1} \log S(x)\, dx\Big] \qquad (0 \leqq z \leqq 1), \tag{5.8}$$

where S is an infdiv *pgf* such that the integral in (5.8) is finite. In the next section we shall show that the class of *F*-self-dec *pgf*'s coincides with the class of functions of the form (5.8), also if $m=1$. However, (5.8) is more general than (5.7) (cf. Example 6.6).

6. *F*-Self-Dec *PGF*'s and Branching Processes with Immigration

Let $(F_t)_{t \geqq 0} \subset PGF$ be a continuous semigroup with $m \leqq 1$ (cf. Lemma 4.3(i)). Here we derive a representation theorem for the F-self-dec *pgf*'s. The method of proof is similar to that used for the discrete self-dec *pgf*'s (see Steutel and Van Harn (1979)), i.e., the *pgf*'s that are self-dec with respect to the semigroup of Example 3.3.

Theorem 6.1. *A function P on $[0,1]$ is an F-self-dec element of PGF iff P has the form*

$$P(z) = \exp\left[-\lambda \int_{(z,1)} \frac{1-Q(x)}{U(x)}\,dx\right] \quad (0 \leqq z \leqq 1), \tag{6.1a}$$

or, equivalently,

$$P(z) = \exp\left[-\lambda \int_{(0,\infty)} \{1 - Q(F_t(z))\}\,dt\right] \quad (0 \leqq z \leqq 1), \tag{6.1b}$$

where $\lambda > 0$ and Q is a pgf with $Q(0) = 0$ such that

$$\int_{(0,1)} \frac{1-Q(x)}{U(x)}\,dx < \infty. \tag{6.2}$$

The representation (λ, Q) in (6.1) is unique.

Proof. Let $P \in PGF$ be F-self-dec, i.e., for all $t > 0$ let there be $P_t \in PGF$ such that (4.4) holds. As noted in (3.8), we have $U(z) > 0$ for all $z \in [0,1)$. Since by (3.2)

$$F_t(z) - z = t\,U(z) + o(t) \quad (t \downarrow 0;\ 0 \leqq z < 1),$$

we have

$$P(F_t(z)) - P(z) = t\,U(z)\,P'(z) + o(t) \quad (t \downarrow 0;\ 0 \leqq z < 1),$$

and hence for $0 \leqq z < 1$ and $t \downarrow 0$

$$P_t(z) = \left\{1 + \frac{P(F_t(z)) - P(z)}{P(z)}\right\}^{-1} = \{1 + t\,U(z)\,P'(z)/P(z) + o(t)\}^{-1}.$$

Now, take $\gamma > 0$ and $t_n = \gamma/n$ $(n \in \mathbb{N})$. Then it follows that

$$\begin{aligned}\lim_{n\to\infty} \{P_{t_n}(z)\}^n &= \lim_{n\to\infty} \left\{1 + \frac{\gamma}{n} U(z)\,P'(z)/P(z) + o\left(\frac{1}{n}\right)\right\}^{-n} \\ &= \exp[-\gamma\,U(z)\,P'(z)/P(z)].\end{aligned}$$

Since by (3.5) and Lemma 1.1 in Steutel and Van Harn (1979)

$$\lim_{z\uparrow 1} U(z)\,P'(z)/P(z) = (-\log m) \lim_{z\uparrow 1} (1-z)\,P'(z) = 0,$$

we conclude from the continuity theorem for *pgf*'s (cf. Feller (1968)) that S^γ, with $S(z) = \exp[-U(z)\,P'(z)/P(z)]$, is a *pgf* for all $\gamma > 0$, i.e., S is an inf div *pgf*, or, equivalently (cf. Feller (1968)), S is compound Poisson. It follows that there exist $\lambda>0$ and a *pgf* Q with $Q(0)=0$ such that

$$R(z)=\frac{d}{dz}\log P(z)=\frac{-\log S(z)}{U(z)}=\lambda\frac{1-Q(z)}{U(z)}, \tag{6.3}$$

which yields (6.1a). Clearly, (6.2) must hold, otherwise we would have $P\equiv 0$. For the equivalent form (6.1b) we note that by (2.8′), the second part of (3.6) and (3.1), together with Fubini's theorem

$$\begin{aligned}\int_{(z,1)}\frac{1-Q(x)}{U(x)}dx&=\int_{(z,1)}U(x)^{-1}\int_{(0,\infty)}Q'(F_t(x))\frac{\partial}{\partial t}F_t(x)\,dt\,dx\\&=\int_{(0,\infty)}\int_{(z,1)}Q'(F_t(x))F_t'(x)\,dx\,dt\\&=\int_{(0,\infty)}\{1-Q(F_t(z))\}\,dt,\end{aligned}$$

also when the integrals are infinite.

Conversely, let P be a function of the form (6.1b) with λ and Q as indicated. Then the function R in (6.3) satisfies

$$R(z)=\lambda\int_{(0,\infty)}\frac{\partial}{\partial z}Q(F_t(z))\,dt,$$

which is an absolutely monotone functon. As $\lim_{z\uparrow 1}P(z)=1$, it follows that P is a *pgf*, even an inf div *pgf* (cf. Feller (1968)). Similarly, it can be shown that for all $t>0$ the function $P_t(z)=P(z)/P(F_t(z))$, which by the semigroup property satisfies

$$P_t(z)=\exp\left[-\lambda\int_{(0,t)}\{1-Q(F_s(z))\}\,ds\right]\qquad(0\leqq z\leqq 1), \tag{6.4}$$

is an inf div *pgf*. Thus we have verified Definition 4.1, and P is an F-self-dec *pgf*. □

Corollary 6.2. *If $P\in PGF$ is F-self-dec, then P, and its factors P_t ($t>0$; cf.* (4.4)), *are inf div. Furthermore, the distribution $(p_n)_{n\in\mathbb{N}_0}$ of which P is the pgf, has $p_n>0$ for all $n\in\mathbb{N}_0$.*

Proof. From (6.3) it follows that $p_1/p_0=R(0)=\lambda/U(0)>0$, so $p_1>0$. But then all $p_n>0$, as P is inf div (cf. Steutel (1970)). □

We now examine Condition (6.2).

Theorem 6.3. (i) *If $m<1$, then* (6.2) *is equivalent to*

$$\sum_{n=1}^{\infty}q_n\log n<\infty. \tag{6.5}$$

(ii) *If $m=1$, then for* (6.2) *it is necessary that* (6.5) *holds and $U''(1)=H''(1)=\infty$, and sufficient that for some $c>0$ and $\rho\in(0,1)$*

$$Q'(1)<\infty\quad\textit{and}\quad U(z)\sim c(1-z)^{1+\rho}\qquad(z\uparrow 1), \tag{6.6}$$

or, more generally, that $1-Q(1-.)$ *varies regularly at* 0 *with exponent* $\alpha\in[0,1]$ *and* $U(1-.)$ *varies regularly at* 0 *with exponent* $\beta\in[1,2]$ *and* $\beta-\alpha<1$.

Proof. (i) By (3.7) we have $U(x)\sim(-\log m)(1-x)$ as $x\uparrow 1$. Hence (6.2) is equivalent to

$$\int_{(0,1)}\frac{1-Q(x)}{1-x}dx<\infty,\ \text{i.e.,}\quad \sum_{k=1}^{\infty}q_k\sum_{n=1}^{k}n<\infty, \tag{6.7}$$

which is equivalent to (6.5).

(ii) If $m=1$, then $U'(1)=\log m=0$, so for x close to 1 we have

$$\frac{1-Q(x)}{U(x)}\geqq\frac{1-Q(x)}{1-x}.$$

By (6.7) it follows that (6.2) implies (6.5). If $U''(1)<\infty$, then the middle factor in the right-hand side of

$$\frac{1-Q(x)}{U(x)}=\frac{1-Q(x)}{1-x}\frac{(1-x)^2}{U(x)}\frac{1}{1-x}$$

tends to $2U''(1)^{-1}$ as $x\uparrow 1$, while the first factor tends to $Q'(1)\in(0,\infty]$, implying the divergence of the integral in (6.2). Hence $U''(1)=\infty$ if (6.2) holds. If (6.6) holds, or more generally, the condition following (6.6), then the integrand in (6.2) varies regularly at 1 with exponent $\alpha-\beta\in(-1,0]$, so (6.2) holds by De Haan (1970) or Seneta (1976). □

Remark 6.4. By similar considerations we see that if P is F-self-dec with representation (6.1) then $P'(1)<\infty$ iff $m<1$ and $Q'(1)<\infty$.

In Steutel, Vervaat and Wolfe (1980) the set of F-self-dec *pgf*'s is shown to coincide with the set of invariant distributions of F-branching processes with immigration. More specifically, consider a branching process with immigration, where branching is governed by $(F_t)_{t\geqq 0}$ and immigration occurs according to an increasing $\mathbb{N}_0$-valued process with stationary independent increments, i.e., according to a compound Poisson process with, say, intensity $\lambda>0$ and batch size *pgf* Q. Then (6.4) gives the *pgf* of the process at time t, starting with 0 individuals at time 0 (cf. Harris (1963), p. 118 and Sevast'janov (1957, 1971)), from which by letting $t\to\infty$ it follows that the invariant distribution exists and is given by (6.1b) iff (6.1b) makes sense, i.e., iff (6.2) holds.

Thus, Condition (6.2) is necessary and sufficient for ergodicity of an F-branching process with immigration according to a compound-Poisson-(λ,Q) process. This generalizes results of Sevast'janov (1957) and Foster and Williamson (1971). In case $m<1$ Sevast'janov proves ergodicity assuming $Q'(1)<\infty$, which condition is stronger than (6.2) cf. Theorem 6.3(i)). Foster and Williamson consider discrete-time processes, and obtain instead of (6.2)

$$\int_{(0,1)}\frac{1-Q(x)}{F_1(x)-x}dx<\infty \tag{6.8}$$

as a criterion, which can be shown to be equivalent to (6.2) in the continuous-time case. They also obtain Theorem 6.3(i) with (6.2) replaced by (6.8).

We conclude this section with two examples.

Example 6.5. Consider the discrete stable *pgf*'s (i.e., F-stable with F as in Example 3.3) with exponent $\gamma \in (0,1)$, i.e., the functions P of the form

$$P(z) = \exp[-\lambda(1-z)^{\gamma}] \quad (0 \leqq z \leqq 1), \tag{6.9}$$

with $\lambda > 0$. From Theorems 6.1 and 6.3(ii) with $Q(z) = z$ we see that these *pgf*'s are self-dec with respect to the semigroup $(F_t)_{t \geqq 0}$ of Example 3.5 with $\rho = 1 - \gamma$, that is with respect to a semigroup with $m = 1$.

Example 6.6. There exist semigroups $(F_t)_{t \geqq 0}$ with $m = e^{-1}$, for which not all F-self-dec *pgf*'s are of the form $\pi_\theta \mu$ with $\mu \in \mathscr{P}$ self-dec (cf. Section 5). In fact, suppose that $P = \pi_\theta \mu$, with μ self-dec, has the form (6.1a). Then by (5.7) there exists an infdiv $\nu \in \mathscr{P}$ such that

$$\lambda\{Q(A^{\sim}(\tau)) - 1\} = \log \hat{\nu}(\tau) \quad (0 \leqq \tau \leqq 1).$$

As ν is infdiv, $-\dfrac{d}{d\tau} \log \hat{\nu}(\tau)$ is completely monotone (comp mon) on $(0, \infty)$, and so necessarily

$$-\frac{d}{d\tau} Q(A^{\sim}(\tau)) \quad \text{is comp mon on } (0,1]. \tag{6.10}$$

Now, consider $(F_t)_{t \geqq 0}$ from Example 3.4(ii) with $a = (1-2p)^{-1}$, take $Q(z) = z$ and $\lambda = p/(1-2p)$. Then it follows that

$$P(z) = (1-2p)/(1-p-pz)$$

is F-self-dec. But, as $A^{\sim} = A$ in this case, we see that the function in (6.10) is equal to $B'(\tau)$, which is not comp mon on $(0,1]$. It can even be shown that $P \notin \pi_\theta(\mathscr{P})$.

In Sect. 8 it will be shown that the class of F-self-dec *pgf*'s of the form $\pi_\theta \mu$ with $\mu \in \mathscr{P}$ self-dec coincides with the set of limit distributions in a "central limit problem".

7. Canonical Representation of *F*-Stable *PGF*'s

Let $(F_t)_{t \geqq 0} \subset PGF$ be a continuous semigroup with $m < 1$. By Lemma 4.3 (ii) only such semigroups can have F-stable *pgf*'s and the exponent γ of an F-stable *pgf* satisfies $0 < \gamma \leqq -\log m$. We have the following representation theorem.

Theorem 7.1. *Let $0 < \gamma \leqq -\log m$. Then a function P on $[0,1]$ is an F-stable element of PGF with exponent γ iff P has the form*

$$P(z) = \exp[-\lambda A(z)^{\gamma}] \quad (0 \leqq z \leqq 1), \tag{7.1}$$

with $\lambda > 0$.

Proof. Let P be an F-stable *pgf* with exponent γ. Then we have (4.6), from which by taking $x=\alpha^{-\gamma}$ $(0<\alpha\leqq 1)$ and $z=0$ we see that

$$\log P(F_{-\log\alpha}(0))=-\lambda\alpha^{\gamma} \qquad (0<\alpha\leqq 1), \tag{7.2}$$

with $\lambda=-\log P(0)$. From (3.11) with $z=0$ and $t=-\log\alpha$ we obtain $A(F_{-\log\alpha}(0))=\alpha$, which together with (7.2) and the continuity of the semigroup implies (7.1).

The converse statement follows by reduction to the case $m=e^{-1}$ (cf. Remark 3.1) and applying the results of Sect. 5. We can also argue, however, in a direct way as follows. If P has the form (7.1), then $P(z)=\exp[\lambda(Q(z)-1)]$, where $Q(z)=1-A(z)^{\gamma}$ is a *pgf* because of (3.13). Hence P is a (compound Poisson) *pgf*, which by (3.11) satisfies (4.6), i.e., P is an F-stable *pgf* with exponent γ. □

Contrary to the self-dec case (cf. Example 6.6), by (5.4) we can state the following correspondence.

Corollary 7.2. *Let $m=e^{-1}$ and $0<\gamma\leqq 1$. Then $P\in PGF$ is F-stable with exponent γ iff there exists a stable $\mu\in\mathscr{P}$ with exponent γ such that $P=\hat{\mu}\circ A$.*

Remark 7.3. Representation (6.1a) reduces to (7.1) if we take $Q(z)=1-A(z)^{\gamma}$. Hence from Remark 6.4 we obtain the following improvement of the last part of Lemma 4.3(ii): if P is F-stable with exponent γ then $P'(1)<\infty$ iff $\gamma=-\log m$ and $B'(1)<\infty$, i.e. (cf. Athreya and Ney (1972)), iff $\gamma=-\log m$ and $\sum h_n n\log n<\infty$.

8. Central Limit Problem

The self-dec and stable elements of $\mathscr{P}$ can be identified as the limits of certain central limit problems. Recall that $\mathscr{P}$ consists of all probability measures on $[0,\infty)$ that are not concentrated at zero.

Definition 8.1. Let $\mu\in\mathscr{P}$, and suppose there exist $a_n>0$ $(n\in\mathbb{N})$ and independent, nonnegative *rv*'s $X_1, X_2, \ldots$ such that the distribution of $a_n^{-1}S_n$, with $S_n=X_1+\ldots+X_n$ $(n\in\mathbb{N})$, converges to μ $(n\to\infty)$.

(i) If in addition

$$\lim_{n\to\infty}\max_{k\leqq n}\Pr[a_n^{-1}X_k\geqq\varepsilon]=0 \qquad (\varepsilon>0), \tag{8.1}$$

then μ is said to be in *the class* $\mathscr{L}$.

(ii) If in addition all X_k have the same distribution, then μ is said to be in *the class* $\mathscr{S}$.

Let ν_k denote the distribution of X_k in Definition 8.1 $(k\in\mathbb{N})$. Then the distribution of $a_n^{-1}S_n$ converges to μ iff

$$\lim_{n\to\infty}\prod_{k=1}^{n}\hat{\nu}_k(\tau/a_n)=\hat{\mu}(\tau) \qquad (\tau\geqq 0), \tag{8.2}$$

and, adapting Sect. 23.2A of Loève (1977), one can show that (8.1) is equivalent to

$$\lim_{n\to\infty} \min_{k\leqq n} \hat{v}_k(\tau/a_n)=1 \qquad (\tau\geqq 0). \tag{8.3}$$

By Schwarz' inequality $\phi(\tau)=\hat{v}_k(\tau/a_n)$ $(\tau\geqq 0)$ is log-convex, so $\phi(\tau)\geqq\phi(\frac{1}{2}\tau)^2$ $(\tau\geqq 0)$; hence (8.3) is equivalent to

$$\lim_{n\to\infty} \min_{k\leqq n} \hat{v}_k(1/a_n)=1. \tag{8.3'}$$

Finally, we note (cf. Loève (1977)) that for $\mu\in\mathscr{L}$ every sequence $(a_n)_{n\in\mathbb{N}}$ occurring in Definition 8.1 necessarily satisfies

$$\lim_{n\to\infty} a_n=\infty. \tag{8.4}$$

Theorem 8.2. *Let $\mu\in\mathscr{P}$. Then*

(i) *$\mu\in\mathscr{L}$ iff μ is self-dec;*

(ii) *$\mu\in\mathscr{S}$ iff μ is stable (with some exponent $\gamma\in(0,1]$).*

Proof. The major part of the theorem is classical (cf. Loève (1977), Sects. 24.3, 4, 5). Adapting Sect. 24.3 of Loève to our restrictions, we obtain (i); note in particular that no subtraction by constants occurs in the first part of the proof of his criterion *A*. Similar considerations prove (ii) for nondegenerate μ. On the other hand, we have termed degenerate $\mu\in\mathscr{P}$ stable with exponent $\gamma=1$, but for these μ (ii) is trivial. □

Now, let $F=(F_t)_{t\geqq 0}\subset PGF$ be a fixed continuous semigroup. We want to solve the analogous central limit problem for $\mathbb{N}_0$-valued *rv*'s, replacing ordinary scalar multiplication αX by the multiplication $\alpha\odot X$, as defined by (1.2) and (1.3). Thus we obtain the following analogue of Definition 8.1.

Definition 8.3. Let $P\in PGF$, and suppose there exist $c_n\geqq 1$ $(n\in\mathbb{N})$ and independent, $\mathbb{N}_0$-valued *rv*'s $X_1, X_2, \ldots$ such that the *pgf* of $c_n^{-1}\odot S_n$, with $S_n=X_1+\ldots+X_n$ $(n\in\mathbb{N})$, converges to P $(n\to\infty)$.

(i) If in addition

$$\lim_{n\to\infty} \max_{k\leqq n} \Pr[c_n^{-1}\odot X_k\geqq\varepsilon]=0 \qquad (\varepsilon>0), \tag{8.5}$$

then P is said to be in *the class* $\mathscr{L}_F$.

(ii) If in addition all X_k have the same distribution, then P is said to be in *the class* $\mathscr{S}_F$.

Let Q_k denote the *pgf* of X_k in Definition 8.3 $(k\in\mathbb{N})$. Then the *pgf* of $c_n^{-1}\odot S_n$ converges to P iff

$$\lim_{n\to\infty} \prod_{k=1}^{n} Q_k\circ F_{\log c_n}=P, \tag{8.6}$$

and it is easily shown that Condition (8.5) is equivalent to

$$\lim_{n\to\infty} \min_{k\leqq n} Q_k(F_{\log c_n}(0))=1. \tag{8.7}$$

Furthermore, if $P \in \mathscr{L}_F$ then the c_n in Definition 8.3 necessarily satisfy (cf. (8.4))

$$\lim_{n\to\infty} c_n = \infty. \tag{8.8}$$

In fact, if not so, then there is a subsequence $(c_{n_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} c_{n_j} = c < \infty$. As $P \not\equiv 1$, there exists $k_0 \in \mathbb{N}$ such that $Q_{k_0} \not\equiv 1$. Since from (8.7) we see that $Q_{k_0}(F_{\log c_{n_j}}(0)) \to 1$ as $j \to \infty$, it follows that $F_{\log c}(0) = 1$, i.e., $F_{\log c} \equiv 1$, which has been excluded.

Basic to our further considerations is the following theorem. Here again we restrict ourselves to the case $m = e^{-1}$ if $m < 1$ (cf. Remark 3.1), and we use the function V defined in (3.16) and the map π_θ^F defined in Sect. 5.

Theorem 8.4. *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{N}_0$-valued rv's with pgf's P_n $(n \in \mathbb{N})$.*

(i) Let $m = e^{-1}$. Then there exist $c_n \to \infty$ and $P \in PGF$ such that $\lim_{n\to\infty} P_n \circ F_{\log c_n} = P$ iff there exist $a_n \to \infty$ and $\mu \in \mathscr{P}$ such that the distribution of $a_n^{-1} S_n$ converges to μ $(n \to \infty)$. In this case

$$\lim_{n\to\infty} a_n V(c_n) = \theta \quad \text{for some } \theta > 0. \tag{8.9}$$

and

$$P = \pi_\theta^F \mu. \tag{8.10}$$

(ii) Let $m \geqq 1$. Then $P_n \circ F_{\log c_n}$ does not converge to a pgf $\not\equiv 1$ for any sequence $c_n \to \infty$.

Proof. Convergence of the distribution of $a_n^{-1} S_n$ to $\mu \in \mathscr{P}$ is equivalent to

$$\lim_{n\to\infty} P_n(e^{-\tau/a_n}) = \phi(\tau), \tag{8.11}$$

with $\phi = \hat{\mu}$, pointwise for $\tau \in [0, \infty)$, or pointwise for $\tau \in [0, \varepsilon]$ (some $\varepsilon > 0$), or, as we have continuous monotone functions, uniformly in $\tau \in [0, \varepsilon]$ (some $\varepsilon > 0$). Consequently, the same convergence is equivalent to

$$\lim_{n\to\infty} P_n(1 - \tau/a_n) = \phi(\tau) \tag{8.12}$$

in any mode of convergence stated for (8.11). Moreover, if the limit ϕ in (8.11) or (8.12) exists for $\tau \in [0, \varepsilon]$ (some $\varepsilon > 0$) and is continuous at zero, then there exists a unique $\mu \in \mathscr{P}$ such that $\phi(\tau) = \hat{\mu}(\tau)$ for $\tau \in [0, \varepsilon]$.

(i) Let $m = e^{-1}$. Then we can write

$$P_n(F_{\log c_n}(z)) = P_n(1 - a_n V(c_n)\, C(c_n, z)/a_n) \quad (n \in \mathbb{N};\ 0 \leqq z \leqq 1), \tag{8.13}$$

where (cf. (3.13) and (3.14))

$$C(t, z) = \frac{1 - F_{\log t}(z)}{1 - F_{\log t}(0)} \to A(z) \quad (t \to \infty;\ \text{uniformly in } z \in [0, 1]). \tag{8.14}$$

Now, suppose that the distribution of $a_n^{-1} S_n$ converges to $\mu \in \mathscr{P}$ for $a_n \to \infty$. Then we have (8.12) with $\phi = \hat{\mu}$, and, as by (3.6) and (3.7) $\lim_{x\to\infty} V(x) = 0$, we can choose $\theta>0$ and $c_n\to\infty$ such that (8.9) holds. By (8.13) and (8.14) it follows that

$$\lim_{n\to\infty} P_n(F_{\log c_n}(z))=\hat{\mu}(\theta A(z))=\pi_\theta^F\,\mu(z),$$

which indeed belongs to PGF.

Conversely, suppose that $\lim_{n\to\infty} P_n\circ F_{\log c_n}=P\in PGF$ for $c_n\to\infty$. Choose $\theta>0$ and $a_n\to\infty$ such that (8.9) holds. Then by (8.13) and (8.14) it follows that

$$\lim_{n\to\infty} P_n(1-\theta A(z)/a_n)=P(z) \qquad (\text{uniformly in } z\in[0,1]),$$

so

$$\lim_{n\to\infty} P_n(1-\tau/a_n)=P(A^{\sim}(\tau/\theta)) \qquad (\tau\in[0,\theta]),$$

which function is continuous at $\tau=0$. From the observations in the beginning of the present proof we now conclude that the distribution of $a_n^{-1}S_n$ converges to some $\mu\in\mathscr{P}$ as $n\to\infty$.

Finally, let $a_n\to\infty$ and $c_n\to\infty$ be such that both $a_n^{-1}S_n$ and $c_n^{-1}\odot S_n$ converge in distribution (in $\mathscr{P}$ and PGF). Choose $\alpha_n\to\infty$ such that $\alpha_n V(c_n)\to 1$. By the preceding paragraph also $\alpha_n^{-1}S_n$ converges in distribution (in $\mathscr{P}$), so by the convergence of types theorem $a_n/\alpha_n\to\theta$ for some $\theta>0$, and (8.9) follows.

(ii) Let $m\geqq 1$, and suppose that $P_n\circ F_{\log c_n}$ converges to a $pgf\,P$ as $n\to\infty$ with $c_n\to\infty$. For all $\varepsilon\in(0,1)$ we have

$$F_t(z)=1-\{1-F_t(0)\}(1+o(1)) \qquad (t\to\infty;\ \text{uniformly in } z\in[0,\varepsilon]).$$

For $m>1$ this follows from (3.6) and (3.7), and for $m=1$ from Athreya and Ney (1972), p. 16–18. Consequently, for $z\in[0,\varepsilon]$

$$P(z)=\lim_{n\to\infty} P_n(1-\{1-F_{\log c_n}(0)\}(1+o(1)))=P(0),$$

and hence $P\equiv 1$. □

From Theorem 8.4 we see that we may expect solutions to our central limit problem only if $m<1$. The next theorem characterizes the sets of solutions.

Theorem 8.5. *If* $m=e^{-1}$, *then for all* $\theta>0$

$$\mathscr{L}_F=\pi_\theta^F(\mathscr{L}) \quad \textit{and} \quad \mathscr{S}_F=\pi_\theta^F(\mathscr{S}).$$

If $m\geqq 1$, *then* $\mathscr{L}_F=\mathscr{S}_F=\emptyset$.

Proof. By (8.6) and (8.8) the final statement is an immediate cnsequence of Theorem 8.4(ii). So let $m=e^{-1}$. We set again $\pi_\theta=\pi_\theta^F$, and because $T_\theta\mu\in\mathscr{L}$ if $\mu\in\mathscr{L}$ $(\theta>0)$, it is sufficient to consider the case $\theta=1$. First, let $P\in\mathscr{L}_F$. Then there exist $c_n\to\infty$ and pgf's Q_k $(k\in\mathbb{N})$ such that (8.6) and (8.7) hold. By Theorem 8.4(i) it follows that $P=\pi_1\mu$ with $\mu\in\mathscr{L}$, as soon as we have proved (8.3′), with $\hat{v}_k(\tau)=Q_k(e^{-\tau})$. Indeed, by (8.9) with $\theta=1$ we have

$$\begin{aligned}\lim_{n\to\infty}\min_{k\leqq n}\hat{\nu}_k(1/a_n) &= \lim_{n\to\infty}\min_{k\leqq n} Q_k(1-1/a_n)\\ &= \lim_{n\to\infty}\min_{k\leqq n} Q_k(1-V(c_n))\\ &= \lim_{n\to\infty}\min_{k\leqq n} Q_k(F_{\log c_n}(0))=1. \end{aligned}\tag{8.15}$$

Conversely, let $P=\pi_1\mu$ with $\mu\in\mathscr{L}$. Then there exist $a_n\to\infty$ and $\nu_k\in\mathscr{P}$ $(k\in\mathbb{N})$ such that (8.2) and (8.3′) hold. By Lemma 5.1 it follows that

$$\begin{aligned}P=\pi_1\mu=\lim_{n\to\infty}\pi_1\left(\mathop{*}_{k=1}^{n} T_{1/a_n}\nu_k\right)=\lim_{n\to\infty}\prod_{k=1}^{n} T_{1/a_n}(\pi_1\nu_k)\\ =\lim_{n\to\infty}\prod_{k=1}^{n}(\pi_1\nu_k)\circ F_{\log a_n}.\end{aligned}$$

Hence P satisfies (8.6) with $Q_k=\pi_1\nu_k$ and $c_n=a_n$. As

$$Q_k(F_{\log c_n}(0))=\pi_1(T_{1/a_n}\nu_k)(0)=\hat{\nu}_k(1/a_n),$$

we also have (8.7), so that $P\in\mathscr{L}_F$.

The identity $\mathscr{S}_F=\pi_\theta^F(\mathscr{S})$ is proved similarly. □

Remark 8.6. By (3.17) we conclude from (8.15) that if $\sum h_n n\log n<\infty$, then Condition (8.5) is equivalent to (8.1) with a_n replaced by c_n.

We make some concluding remarks. If $m<1$, then by Theorems 5.3 and 8.5 $\mathscr{L}_F$ is a subset of the set of all F-self-dec *pgf*'s. However, contrary to the classical case, this subset can be *proper* (cf. Example 6.6). Also, if $m=1$, then $\mathscr{L}_F=\emptyset$, but there may exist F-self-dec *pgf*'s (cf. Example 6.5). By Corollary 7.2 $\mathscr{S}_F$ does coincide with the set of all F-stable *pgf*'s.

From Theorems 8.4(ii) and 8.5 one can deduce once more that F-stable *pgf*'s exist only when $m<1$ and belong to $\pi_\theta^F(\mathscr{S})$ (so, have the form (7.1)).

From Theorems 8.2, 8.4(i) and 8.5 it follows that the domains of attraction of all F-stable distributions with exponent γ $(m=e^{-1})$ are the same as the intersections with *PGF* of the domains of attraction of the stable distributions in $\mathscr{P}$ with exponent γ.

Finally, we note that Lamperti (1967b, c) studies the possible limit distributions of $(Z_n(c_n)-b_n)/a_n$. The particular case $b_n=0$, $a_n=1$ (Lamperti (1967b), Theorem 2.2) coincides with a special case of the central limit problem of this section, with $S_n=c_n$ degenerate.

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Received January 29, 1981