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# Perturbations of Random Matrix Products

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**Summary.** If $X_1, X_2, \ldots$ are identically distributed independent random matrices with a common distribution $\mu$ then with the probability 1 the limit

$$\Lambda_\mu = \lim_{n\to\infty} \frac{1}{n} \ln \|X_n \ldots X_1\|$$

exists. The paper treats the problem: is it true that $\Lambda_{\mu_k} \to \Lambda_\mu$ if $\mu_k \to \mu$ in the weak sense?

## § 0. Introduction

Let $X_1, X_2, \ldots$ be identically distributed independent random $m\times m$ matrices with a common distribution $\mu$ on the real unimodular group $\mathrm{SL}(m, R)$. The theorem of Furstenberg and Kesten (see [3] and [4]) establishes that with the probability 1 there exists

$$\Lambda_\mu = \lim_{n\to\infty} \frac{1}{n} \ln \|X_n X_{n-1} \ldots X_1\| \tag{0.1}$$

provided

$$\int \ln \|g\| \, \mu(dg) < \infty, \tag{0.2}$$

where $\Lambda_\mu$ is a constant depending only on $\mu$. Now let $\{\mu_k\}$ be a sequence of probability measures on $\mathrm{SL}(m, R)$ satisfying (0.2) such that

$$\mu_k \to \mu \quad \text{in the weak sense as} \quad k\to\infty. \tag{0.3}$$

This paper studies conditions under which (0.3) yields

$$\Lambda_{\mu_k} \to \Lambda_\mu \quad \text{as} \quad k\to\infty. \tag{0.4}$$

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The products of random matrices naturally arise in certain models of physics (see, for instance, [5] and [8]). The assertion $\Lambda_\mu>0$ plays there a decisive part. But physical results have to be stable under perturbations of parameters of a model since they cannot be determined exactly. In particular, it is relevant to the study of a probability distribution of some physical process that leads directly to the problem of the present paper.

It is easy to see that under the "equiintegrability" condition (1.9) of § 1 below, (0.3) always implies

$$\limsup_{k\to\infty} \Lambda_{\mu_k} \leqq \Lambda_\mu. \tag{0.5}$$

Indeed, let the norm satisfy the property

$$\|A\cdot B\| \leqq \|A\|\cdot\|B\| \tag{0.6}$$

for any two matrix $A$ and $B$. Then

$$a_\mu^{(n)} = \int \ln \|g_n \dots g_1\| \, \mu(dg_1)\dots\mu(dg_n) \tag{0.7}$$

is a subadditive sequence i.e. $a_\mu^{(n+m)} \leqq a_\mu^{(n)} + a_\mu^{(m)}$ and that is known to imply

$$\Lambda_\mu = \inf_{n>1} \frac{1}{n} a_\mu^{(n)} \quad \text{and} \quad \Lambda_{\mu_k} = \inf_{n>1} \frac{1}{n} a_{\mu_k}^{(n)}. \tag{0.8}$$

By (0.3) it follows that $a_{\mu_k}^{(n)} \to a_\mu^{(n)}$ as $k\to\infty$ and so

$$\limsup_{k\to\infty} \Lambda_{\mu_k} = \limsup_{k\to\infty} \inf_{n\geqq 1} \frac{1}{n} a_{\mu_k}^{(n)} \leqq \inf_{n\geqq 1} \frac{1}{n} \limsup_{k\to\infty} a_{\mu_k}^{(n)} = \inf_{n\geqq 1} \frac{1}{n} a_\mu^{(n)} = \Lambda_\mu$$

that gives (0.5).

The inequality (0.5) does not help even in the physical problem referred to above since the case $\Lambda_\mu>0$ and $\Lambda_{\mu_k}=0$ for all $k$, does not contradict (0.5) (and, indeed, this case occurs (see § 2)). But in the special case when $\Lambda_\mu=0$ the relation (0.5) implies (0.4) since, clearly, $\Lambda_{\mu_k}\geqq 0$ for any distribution $\mu_k$ on $\mathrm{SL}(m,R)$. In particular, it follows from [10] that $\Lambda_\mu=0$ when the support of $\mu$ contains in the subgroup of unipotent matrices, solving the problem for this case. In this paper we shall improve (0.5) to (0.4) for some other classes of measures $\mu$.

We shall consider products of random matrices only with determinants equal to one, but actually this is no restriction. Indeed, if $X_1, X_2, \dots$ are independent random matrices with a common distribution $\eta$ on the group $\mathrm{GL}(m,R)$ of nondegenerate matrices, then one can write

$$\frac{1}{n}\ln\|X_n\dots X_1\| = \frac{1}{n}\ln\left\|\frac{X_n}{(\det X_n)^{1/m}}\cdots\frac{X_1}{(\det X_1)^{1/m}}\right\| + \frac{1}{n}\sum_{k=1}^{n}\ln|\det X_k|^{1/m}.$$

If $\int \ln|\det g|\,\eta(dg)<\infty$ then by the strong law of large numbers (see [2]) with the probability 1 as $n\to\infty$,

$$\frac{1}{n}\sum_{k=1}^{n}\ln|\det X_k| \to \int \ln|\det g|\,\eta(dg). \tag{0.9}$$

The right hand side of (0.9) is continuously dependent on $\eta$ with respect to the weak convergence of measures. Therefore the remaining problem concerns just the product of the matrices

$$\frac{X_1}{(\det X_1)^{1/m}}, \frac{X_2}{(\det X_2)^{1/m}}, \ldots$$

having determinants equal to one.

Actually, this paper studies the dependence on the distribution $\mu$ of the biggest Lyapunov characteristic exponent in the sense of Oseledec (see [11]) for products of independent matrices.

To treat the other characteristic exponents one notices that by [4] with the probability one there exists the limit

$$\Lambda_{\mu_k}^{\wedge p} = \lim_{n\to\infty} \frac{1}{n} \ln \|X_n^{\wedge p} \ldots X_1^{\wedge p}\|, \tag{0.10}$$

where $X_1, \ldots, X_n, \ldots$ are the same as in (0.1) and $g^{\wedge p}$ denotes the $p$th exterior power of $g \in \mathrm{SL}(m, R)$. According to [10] the number $\Lambda_{\mu_k}^{\wedge p}$ is equal to the sum of the biggest $p$ characteristic exponents. Hence if $\Lambda_{\mu_k}^{\wedge p} \to \Lambda_{\mu}^{\wedge p}$ as $k \to \infty$ for all $p = 1, 2, \ldots, m-1$ then one obtains the convergence of all characteristic exponents, as well. The minimal characteristic exponent $\lambda_\mu^{\min}$ can also be represented with the probability 1 as

$$\lambda_\mu^{\min} = -\lim_{n\to\infty} \frac{1}{n} \ln \|X_1^{-1} \ldots X_n^{-1}\| = -\Lambda_\eta$$

where $\eta(\Gamma) = \mu(\{g\colon g^{-1} \in \Gamma\})$ for any Borel set $\Gamma \subset \mathrm{SL}(m, R)$.

This paper consists of two sections. In § 1 we shall show that (0.4) is true provided the support of the measure $\mu$ is not contained in a reducible subgroup of $\mathrm{SL}(m, R)$. We shall see that such measures form an open everywhere dense set i.e. (0.4) holds in a "generic" case. Nevertheless, we shall see in § 2 that (0.4) is not always true. We exhibit an example suggested by Furstenberg with $\mu$ concentrated in one point $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and show that (0.4) fails for it. But still, we shall prove in § 2 that if $\mu$ concentrated in one point $A \in \mathrm{SL}(m, R)$ and the measure $\mu_k$ have supports contained in some neighbourhood of $A$ then (0.4) is true. Moreover in this case one obtains some assertion on the stability of eigenvalues of the matrix $A$ with respect to its random perturbations.

In view of the papers [7] and [12] the results can be extended to the case when $X_1, X_2, \ldots$ form a matrix-valued Markov stationary process. Using the theory of representations similar results can be proved for general semi-simple groups in the spirit of [4].

*Acknowledgement.* The problem studied here is the generalization of a question suggested by A. Katok who also pointed out to me the preprint [11] where (0.4) is proved for some examples. I would also like to thank H. Furstenberg for a number of very useful discussions and remarks. I appreciate very much the referee for certain suggestions improving the proofs of Theorems 1.1 and 1.2 and for pointing out to me the more simple proof (referred to Y. Guivarch) than the original one for the counterexample of § 2.

## § 1. The "Generic" Case

Let us first specify some notions. All measures we consider in this paper are Borel probability measures on the corresponding spaces. As above, we do not indicate the domain of integration if an integral is taken over the whole space under consideration. Suppose we are given a sequence of measures $\eta_k$ together with a measure $\eta$ on a topological space $X$. Then we say that $\eta_k$ converges to $\eta$ (i.e. $\eta_k \to \eta$) in the weak sense if $\int f(x)\,\eta_k(dx) \to \int f(x)\,\eta(dx)$ for any bounded continuous function $f$ on $X$.

The constant $\Lambda_\mu$ in (0.1) does not depend on a matrix norm and we shall take some vector norm and a corresponding matrix norm satisfying (0.6).

A group $G$ of $m \times m$ real matrices acts in a natural way on the $m$-dimensional vector space $R^m$ by left multiplication. The group $G$ is called irreducible if the only subspaces left fixed by all matrices of $G$ are $R^m$ and $\{0\}$. Otherwise it is called reducible.

The real $(m-1)$-dimensional projective space $P^{m-1}$ is obtained from $R^m \setminus \{0\}$ by identifying two vectors if each is a scalar multiple of the other. Clearly, if $G$ is a group of matrices, then the action of $G$ on $R^m$ induces the natural action of $G$ on $P^{m-1}$.

Let $\mu$ be a probability measure on the real unimodular group $\mathrm{SL}(m,R)$ and let $G_\mu$ be the smallest closed subgroup of $\mathrm{SL}(m,R)$ containing the support of $\mu$ $(\operatorname{supp}\mu)$ which is the minimal closed set having $\mu$-measure equal one.

A space of all Borel probability measures on $\mathrm{SL}(m,R)$ we denote by $\mathcal{M}$. The space $\mathcal{M}$ is a topological one with respect to the weak convergence of measures.

Set

$$\mathcal{M}_{\mathrm{ir}} = \{\mu \in \mathcal{M} \colon G_\mu \text{ is irreducible}\}.$$

The following result shows that the property $\mu \in \mathcal{M}_{\mathrm{ir}}$ is a "generic" one.

**Theorem 1.1.** *The set $\mathcal{M}_{\mathrm{ir}}$ is open and everywhere dense in $\mathcal{M}$.*

*Proof.* First let us show that $\mathcal{M}_{\mathrm{ir}}$ is open in $\mathcal{M}$. To do this we shall prove that $\mathcal{M}_{\mathrm{r}} = \mathcal{M} \setminus \mathcal{M}_{\mathrm{ir}}$ is a closed set.

Consider a sequence of measures $\mu_n$ such that

$$\mu_n \in \mathcal{M}_r \quad \text{and} \quad \mu_n \to \mu \quad \text{in the weak sense as} \quad n \to \infty. \tag{1.1}$$

By the definition, the subgroup $G_{\mu_n}$ leaves invariant some nontrivial $m_n$-dimensional subspace $\Gamma_n$ of $R^m$ i.e. $0 < m_n < m$. Let $e_1^{(n)}, e_2^{(n)}, \ldots, e_{m_n}^{(n)}$ be an orthonormal basis of $\Gamma_n$. Clearly, one can choose a subsequence $\{n_i\}$ such that for some $m^* > 0$ we have $m_{n_i} = m^*$ for all $i = 1, 2, \ldots$ and there exists

$$\lim_{i \to \infty} e_k^{(n_i)} = e_k^* \qquad \text{for any } k = 1, 2, \ldots, m^*. \tag{1.2}$$

Set

$$W = \{g \in \operatorname{supp}\mu \colon \text{there are a subsequence } \{n_{i_j}\} \text{ of } \{n_i\} \text{ and matrices } g_j \in \operatorname{supp}\mu_{n_{i_j}} \text{ such that } \lim_{j \to \infty} g_j = g\}$$

where the convergence is taken with respect to a matrix norm.

Let $\Gamma^*$ be the subspace generated by the basis $e_1^*, e_2^*, \ldots, e_{m^*}^*$. Take arbitrary $g \in W$ and $\xi \in \Gamma^*$. Then $g = \lim_{j\to\infty} g_j$ for some sequence $g_j \in \operatorname{supp} \mu_{n_{i_j}}$. Therefore there exists $M_1 > 0$ such that

$$\|g_j\| \leqq M_1 \qquad \text{for all } j = 1, 2, \ldots, \tag{1.3}$$

and so by (1.2),

$$g e_k^* = (\lim_{j\to\infty} g_j) \lim_{j\to\infty} e_k^{(n_{i_j})} = \lim_{j\to\infty} g_j e_k^{(n_{i_j})}. \tag{1.4}$$

But $g_j e_k^{(n_{i_j})} \in \Gamma_{n_{i_j}}$ and $\Gamma_{n_{i_j}} \to \Gamma^*$ as $j \to \infty$ in the natural topology of $m^*$-dimensional subspaces of $R^m$. This together with (1.4) implies that $g e_k^* \in \Gamma^*$ for all $k = 1, 2, \ldots, m^*$ and so

$$g\Gamma^* = \Gamma^* \qquad \text{for any } g \in W. \tag{1.5}$$

We shall now show that

$$\mu(W) = 1. \tag{1.6}$$

Indeed, for any closed subset $Q \subset \mathrm{SL}(m, R)$,

$$\limsup_{i\to\infty} \mu_{n_i}(Q) \leqq \mu(Q) \tag{1.7}$$

(see [1], Theorem 2.1). Taking

$$Q_k = \overline{\bigcup_{i \geqq k} \operatorname{supp} \mu_{n_i}}$$

one obtains from (1.7) that

$$\mu(Q_k) = 1 \qquad \text{for all } k = 1, 2, \ldots,$$

where $\bar{A}$ is the closure of the set $A$.

Therefore

$$\operatorname{supp} \mu \subset \bigcap_{k=1}^{\infty} Q_k = W$$

that proves (1.6).

It remains to show that $\mathscr{M}_{\mathrm{ir}}$ is everywhere dense. First, notice that $\mathscr{M}_{\mathrm{ir}}$ is not empty. Indeed, it is easy to see that, for example, the normalized Haar measure on the group $\mathrm{SO}(m, R)$ of orthogonal matrices with determinants equal one, belongs to $\mathscr{M}_{\mathrm{ir}}$.

Now take arbitrary $\mu \in \mathscr{M}$ and $\eta \in \mathscr{M}_{\mathrm{ir}}$. Define

$$\mu_n = \left(1 - \frac{1}{n}\right) \mu + \frac{1}{n} \eta$$

then $\mu_n \in \mathscr{M}_{\mathrm{ir}}$ since $\operatorname{supp} \mu_n = \operatorname{supp} \mu \cup \operatorname{supp} \eta$. Besides, $\mu_n \to \mu$ in the weak sense as $n \to \infty$. This completes the proof of Theorem 1.1.

The main result of the present section is the following

**Theorem 1.2.** *Let $\mu \in \mathscr{M}_{\mathrm{ir}}$. For any sequence of measures $\mu_k \in \mathscr{M}$ such that*

$$\mu_k \to \mu \quad \text{in the weak sense as} \quad k \to \infty \tag{1.8}$$

*and*

$$\lim_{N\to\infty} \sup_k \int_{\{g:\, \|g\| \geqq N\}} \ln(\|g\|)\, \mu_k(dg) = 0 \tag{1.9}$$

*one has*

$$\Lambda_{\mu_k} \to \Lambda_\mu \quad \text{as} \quad k \to \infty, \tag{1.10}$$

*where* $\Lambda_\mu$ *is defined in* (0.1).

*Remark 1.1.* A partial case of Theorem 1.2 is proved in [6].

*Proof of Theorem 1.2.* If $g \in \mathrm{SL}(m, R)$ and $\|g\| = m \max_{i,j} |g_{ij}|$ then one can easily check that $\|g^{-1}\| \leqq \|g\|^{m-1}$. Since all matrix norms are equivalent then

$$\|g^{-1}\| \leqq K \|g\|^{m-1}$$

where $K$ depends just on the norm $\|.\|$ but not on $g$. This together with (1.9) gives that also

$$\lim_{N\to\infty} \sup_k \int_{\{g:\, \|g\| \geqq N\}} \ln(\|g^{-1}\|) \mu_k(dg) = 0. \tag{1.11}$$

Theorem 1.1 asserts that $\mathcal{M}_{\mathrm{ir}}$ is an open set. Since $\mu \in \mathcal{M}_{\mathrm{ir}}$ then (1.8) implies that $\mu_k \in \mathcal{M}_{\mathrm{ir}}$ for all $k$ greater than some $k_0$. Without loss of generality we can assume that $\mu_k \in \mathcal{M}_{\mathrm{ir}}$ for all $k = 1, 2, \ldots$.

Define the function $\rho(g, \xi)$ on the product of $\mathrm{SL}(m, R)$ and the projective space $P^{m-1}$ by the formula

$$\rho(g, \xi) = \ln \frac{\|gu\|}{\|u\|}, \tag{1.12}$$

where the vector $u \in R^m \setminus \{0\}$ and its scalar multiples represent after idenfication, the point $\xi \in P^{m-1}$.

Employing Theorem 8.5 of [4] we obtain that

$$\Lambda_{\mu_k} = \iint \rho(g, \xi)\, \mu_k(dg)\, \nu_k(d\xi), \tag{1.13}$$

for any probability measure $\nu_k$ on $P^{m-1}$ satisfying the equation

$$\mu_k * \nu_k = \nu_k, \tag{1.14}$$

where $\mu_k * \nu_k$ is the measure on $P^{m-1}$ defined by the equality

$$\int f(\xi)\, \mu_k * \nu_k(d\xi) = \iint f(g\xi)\, \mu_k(dg)\, \nu_k(d\xi) \tag{1.15}$$

which holds for all Borel functions $f$ on $P^{m-1}$. Here $g\xi \in P^{m-1}$ denotes the result of applying $g \in \mathrm{SL}(m, R)$ to $\xi \in P^{m-1}$ according to the natural action of $\mathrm{SL}(m, R)$ on $P^{m-1}$.

The space $P^{m-1}$ is a compact one, hence there is a subsequence $k_i \to \infty$ and a probability measure $\nu$ on $P^{m-1}$ such that

$$\nu_{k_i} \to \nu \quad \text{in the weak sense as} \quad i \to \infty. \tag{1.16}$$

This together with (1.8), (1.15) and the theorem on the weak convergence of product measures (see, for instance, Theorem 3.2 in Chapter 1 of [1]) imply that

$$\mu_{k_i} * \nu_{k_i} \to \mu * \nu \quad \text{in the weak sense as} \quad i \to \infty. \tag{1.17}$$

Therefore by (1.14) and (1.16),

$$\mu * \nu = \nu. \tag{1.18}$$

Hence by the Theorem 8.5 of [4] it follows that

$$\Lambda_\mu = \iint \rho(g, \xi)\, \mu(dg)\, \nu(d\xi) \tag{1.19}$$

and this expression does not depend on the measure $\nu$ provided it satisfies (1.18).

Notice that if $g \in \mathrm{SL}(m, R)$ then

$$\ln \|g\| \geqq 0 \quad \text{and} \quad \ln \|g^{-1}\| \geqq 0.$$

Indeed, $\|g\| \geqq \operatorname{specrad} g \geqq |\det g|^{1/m} = 1$. Since $g^{-1} \in \mathrm{SL}(m, R)$ then the second inequality in (1.26) also follows.

It is easy to see that

$$-\ln \|g^{-1}\| \leqq \rho(g \cdot \xi) \leqq \ln \|g\| \tag{1.21}$$

for any $g \in \mathrm{SL}(m, R)$ and $\xi \in P^{m-1}$.

Define the sequence of functions $\rho_N(g, \xi) = \max(-N, \min(N, (g, \xi)))$. For $N = 1, 2, \ldots$. Set

$$\Lambda_\mu^N = \iint \rho_N(g, \xi)\, \mu(dg)\, \nu(d\xi). \tag{1.22}$$

The numbers $\Lambda_{\mu_k}^N$ are defined in the same way by substituting $\mu_k$ for $\mu$ in (1.22).

By the theorem on the weak convergence of product measures ([1], Chapter 1) and (1.17) it follows that

$$\lim_{i \to \infty} \Lambda_{\mu_{k_i}}^N = \Lambda_\mu^N. \tag{1.23}$$

Now by (1.13), (1.19), (1.21) and (1.22) it follows that

$$\begin{aligned} |\Lambda_{\mu_{k_i}} - \Lambda_\mu| \leqq |\Lambda_{\mu_{k_i}}^N - \Lambda_\mu^N| &+ \sup_n \int_{\{g:\, \|g\| \geqq e^N\}} \ln(\|g\|)\, \mu_n(dg) \\ &+ \sup_n \int_{\{g:\, \|g^{-1}\| \geqq e^N\}} \ln(\|g^{-1}\|)\, \mu_n(dg). \end{aligned} \tag{1.24}$$

Setting $k_i \to \infty$ and using (1.23) and then letting $N \to \infty$ one obtains by (1.9) and (1.11) that

$$\lim_{i \to \infty} |\Lambda_\mu - \Lambda_{\mu_{k_i}}| = 0. \tag{1.25}$$

Actually, we have proved that from any sequence of integers $\ell_i$ such that $\ell_i \to \infty$ as $i \to \infty$ one can choose a subsequence $\{\ell_{i_j}\}$ with the property

$$\Lambda_{\mu_{\ell_{i_j}}} \to \Lambda_\mu \quad \text{as} \quad j \to \infty.$$

This enables us to assert that (1.25) holds for any subsequence $\{k_i\}$, completing the proof of (1.10).

## § 2. Perturbations of a Single Matrix

First consider the following counterexample which shows that (0.4) is not always true. Set

$$A=\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a>0 \quad \text{and} \quad J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mu^\varepsilon$ be the family of the probability measures on $\mathrm{SL}(2,R)$ defined by the formula

$$\mu^\varepsilon(\{A\})=1-\varepsilon \quad \text{and} \quad \mu^\varepsilon(\{J\})=\varepsilon,$$

where $\{A\}$ and $\{J\}$ are the sets containing one element $A$ and $J$, respectively.

Clearly, $\mu^\varepsilon \to \mu^0$ in the weak sense as $\varepsilon \to 0$, where $\mu^0$ is concentrated in the one point $A$. But we shall see that $\Lambda_{\mu^\varepsilon}$ defined by (0.1) does not converge to $\Lambda_{\mu^0}$. Namely

**Proposition 2.1.** *Let $X_1^\varepsilon, X_2^\varepsilon, \ldots$ be identically distributed independent random matrices with the common distribution $\mu^\varepsilon$ then*

$$\Lambda_{\mu^\varepsilon}=0 \quad \textit{if } 1>\varepsilon>0 \quad \textit{and} \quad \Lambda_{\mu^0}=|\ln a|.$$

*Remark 2.1.* The counterexample is based on the property of the matrix $J$ to exchange the expanding and contracting directions of the matrix $A$. Similar examples can also be built in the multidimensional case. This is the only effect I know which results in the discountinuity of $\Lambda_\mu$ in $\mu$.

*Proof of Proposition 2.1.* It is easy to see that $\mu^\varepsilon \in \mathcal{M}_{\mathrm{ir}}$ (but, of course, $\mu^0 \notin \mathcal{M}_{\mathrm{ir}}$ that distinguishes this case from the one of § 1). Let $e$ and $f$ be the points of $P^1$ represented by the vectors (0, 1) and (1, 0), correspondingly. Set $\nu=\frac{1}{2}(\delta_e+\delta_f)$, where $\delta_e$ and $\delta_f$ are probability measure on $P^1$ concentrated in $e$ and $f$, respectively. Clearly, for any $\varepsilon$

$$\mu^\varepsilon * \nu = \nu$$

and so by Furstenberg's formula (1.19) one computes readily that $\Lambda_{\mu^\varepsilon}=0$.

On the other hand, obviously, $\Lambda_{\mu^0}=\ln \operatorname{specrad} A=|\ln a|$ that comples the proof.

Now we shall prove that if the perturbation of a matrix $A$ is "local" in some sense then (0.4) still holds.

Let $\delta(A)$ be the probability measure concentrated in the one point $A \in \mathrm{SL}(m,R)$. By the spectral radius theorem (see [9]),

$$\Lambda(A) \equiv \Lambda_{\delta(A)}=\ln \operatorname{specrad} A=\ln \max_{1 \le i \le m} |\lambda_i|, \tag{2.1}$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $A$. Set

$$V_r(A)=\{g \in \mathrm{SL}(m,R) \colon \|g-A\| \le r\}.$$

The main result of this section is the following:

**Theorem 2.1.** *For any matrix $A \in \mathrm{SL}(m,R)$ there exists a number $r(A)>0$ such that if*

$$\mu_k \in \mathcal{M} \quad \text{and} \quad \operatorname{supp} \mu_k \subset V_{r(A)}(A) \quad \text{for all } k=1,2,\ldots \tag{2.2}$$

*and*

$$\mu_k \to \delta(A) \quad \text{in the weak sense as} \quad k \to \infty \tag{2.3}$$

*then*

$$\Lambda_{\mu_k} \to \Lambda(A) \quad \text{as} \quad k \to \infty. \tag{2.4}$$

*Remark 2.1.* Let $\lambda_1, \ldots, \lambda_m$ be eigenvalues of $A$ such that $|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_m|$ then $\Lambda(A^{\wedge \not{p}}) = \ln \operatorname{specrad} A^{\wedge \not{p}} = \sum_{1 \leqq i \leqq \not{p}} \ln |\lambda_i|$, where, recall, $A^{\wedge \not{p}}$ is the $\not{p}$-th exterior power of $A$. Let $\Lambda_{\mu_k^{\wedge \not{p}}}$ be defined by (0.10) with $\mu_k$ substituted for $\mu$ then one can easily generalize Theorem 2.1 to obtain that $\Lambda_{\mu_k^{\wedge \not{p}}} \to \Lambda(A^{\wedge \not{p}})$ as $k \to \infty$ for all $\not{p} = 1, 2, \ldots, m$, provided (2.2) and (2.3) are satisfied. Since by [10] the number $\Lambda_{\mu_k^{\wedge \not{p}}}$ is equal to the sum $\sum_{1 \leqq i \leqq \not{p}} \lambda_{\mu_k}^{(i)}$ of the biggest characteristic exponents of the product of independent matrices with the common distribution $\mu_k$ then $\lambda_{\mu_k}^{(i)} \to \ln |\lambda_i|$ as $k \to \infty$ for all $i = 1, \ldots, m$. This can be interpreted as the stability of the absolute values of eigenvalues of a matrix with respect to random perturbations that complements the classical result about the continuous dependence on the matrix of its eigenvalues (see [9]).

For the proof of Theorem 2.1 we shall need some auxiliary results from the matrix theory.

Let $\Gamma_{\max}$ and $\Gamma_{\min}$ be the eigenspaces of the matrix $A$ corresponding to the eigenvalues having absolute values equal to the spectral radius of $A$ and less than it, respectively. Clearly, $R^m = \Gamma_{\max} + \Gamma_{\min}$ and any vector $\zeta \in R^m$ has the unique representation

$$\zeta = \Pi_{\max} \zeta + \Pi_{\min} \zeta, \tag{2.5}$$

where $\Pi_{\max} \zeta \in \Gamma_{\max}$ and $\Pi_{\min} \zeta \in \Gamma_{\min}$. We shall need

**Lemma 2.1.** *There is a natural $N(A) > 0$ and a number $r(A) > 0$ depending on the matrix $A$ such that if matrices $B_i$ have the property: $B_i \in V_{r(A)}(A)$ for all $i = 1, \ldots, N(A)$ then*

$$B_{N(A)} \ldots B_1 U(A) \subset U(A), \tag{2.6}$$

*where*

$$U(A) = \{\zeta \in R^m \colon \|\Pi_{\max} \zeta\| \geqq \|\Pi_{\min} \zeta\|\}. \tag{2.7}$$

*Proof.* If $\|A\| \geqq r > 0$ and $B_i \in V_r(A)$ then, clearly, $\|B_i\| \leqq 2\|A\|$ and so

$$\|B_N \ldots B_1 - A^N\| \leqq 2^{N-1} \|A\|^{N-1} N r \tag{2.8}$$

for any natural $N$.

From the definition of $\Gamma_{\max}$ it is easy to see that for each $\kappa > 0$ there is $C_\kappa > 0$ so that for any $\xi \in \Gamma_{\max}$,

$$\|A^N \xi\| \geqq C_\kappa e^{(\Lambda(A) - \kappa)N} \|\xi\| \qquad \text{for all } N = 1, 2, \ldots. \tag{2.9}$$

On the other hand, by the definition of $\Gamma_{\min}$ one can find $\gamma > 0$ and $\tilde{C} > 0$ such that for any $\eta \in \Gamma_{\min}$,

$$\|A^N \eta\| \leqq \tilde{C} e^{(\Lambda(A) - \gamma)N} \|\eta\| \qquad \text{for all } N = 1, 2, \ldots. \tag{2.10}$$

By (2.8) and (2.9) we obtain for $B_1, \dots, B_N \in V_r(A)$ and $\zeta \in R^m$ that

$$\begin{aligned}\|\Pi_{\max} B_N \dots B_1 \zeta\| &\geqq \|\Pi_{\max} A^N \zeta\| - \|\Pi_{\max}(B_N \dots B_1 - A^N)\zeta\| \\ &\geqq C_\kappa e^{(\Lambda(A)-\kappa)N} \|\Pi_{\max} \zeta\| \\ &\quad - \|\Pi_{\max}\| (2\|A\|)^{N-1} N r(\|\Pi_{\max}\zeta\| + \|\Pi_{\min}\zeta\|) \end{aligned} \tag{2.11}$$

since $\Pi_{\max} A^N = A^N \Pi_{\max}$ and $\|\zeta\| \leqq \|\Pi_{\max}\zeta\| + \|\Pi_{\min}\zeta\|$, where $\Pi_{\max}$ is the projection operator acting according to (2.5).

Similarly, by (2.8) and (2.10)

$$\begin{aligned}\|\Pi_{\min} B_N \dots B_1 \zeta\| &\leqq \|\Pi_{\min} A^N \zeta\| + \|\Pi_{\min}(B_N \dots B_1 - A^N \zeta)\| \\ &\leqq \tilde{C} e^{(\Lambda(A)-\gamma)N} \|\Pi_{\min} \zeta\| \\ &\quad + \|\Pi_{\max}\|^{-1} (2\|A\|)^{N-1} N r(\|\Pi_{\max}\zeta\| + \|\Pi_{\min}\zeta\|). \end{aligned} \tag{2.12}$$

Now let $\zeta \in U(A)$ i.e. $\|\Pi_{\max}\zeta\| \geqq \|\Pi_{\min}\zeta\|$. It is easy to see from (2.11) and (2.12) that in order to obtain

$$\|\Pi_{\max} B_N \dots B_1 \zeta\| \geqq \|\Pi_{\min} B_N \dots B_1 \zeta\|$$

it suffices to find $r$, $\kappa$ and $N$ such that

$$C_\kappa e^{(\Lambda(A)-\kappa)N} \geqq \tilde{C} e^{(\Lambda(A)-\gamma)N} + 4\|\Pi_{\max}\|^{-1} (2\|A\|)^{N-1} N r. \tag{2.13}$$

To do this one can take $\kappa = \frac{\gamma}{2}$, then choose $N(A)$ so that $N(A) \geqq 2\gamma^{-1} \ln(2\tilde{C} C_{\gamma/2}^{-1})$ and, finally, set

$$r(A) = \min\left(\|A\|, \frac{1}{8N(A)} e^{\left(\Lambda(A) - \frac{\gamma}{2}\right)N(A)} (2\|A\|)^{1-N(A)}\right).$$

**Lemma 2.2** *For any $\varepsilon > 0$ there is a natural $K(\varepsilon) > 0$ and a number $r(\varepsilon, A) > 0$ such that if $B_i \in V_{r(\varepsilon,A)}(A)$ for all $i = 1, \dots, N(\varepsilon, A)$, with*

$$N(\varepsilon, A) = K(\varepsilon) N(A), \tag{2.14}$$

*then*

$$\|\Pi_{\max} B_{N(\varepsilon,A)} \dots B_1 \zeta\| \geqq e^{(\Lambda(A)-\varepsilon)N(\varepsilon,A)} \|\Pi_{\max}\zeta\| \tag{2.15}$$

*provided $\zeta \in U(A)$, where $N(A)$ and $U(A)$ are defined in Lemma 2.1.*

*Proof.* By (2.11) it follows for $\zeta \in U(A)$ that

$$\begin{aligned}\|\Pi_{\max} B_N \dots B_1 \zeta\| &\geqq (C_\kappa e^{(\Lambda(A)-\kappa)N} - 2\|\Pi_{\max}\| (2\|A\|)^{N-1} N r) \|\Pi_{\max}\zeta\| \\ &= e^{(\Lambda(A)-\varepsilon)N} \|\Pi_{\max}\zeta\| (C_\kappa e^{(\varepsilon-\kappa)N} \\ &\quad - 2\|\Pi_{\max}\| (2\|A\|)^{N-1} N r e^{(\varepsilon - \Lambda(A))N}). \end{aligned} \tag{2.16}$$

Now take $\kappa = \frac{\varepsilon}{2}$, then choose $K(\varepsilon)$ such that

$$K(\varepsilon) \geqq 2(\varepsilon N(A))^{-1} \ln(2C_{\varepsilon/2}^{-1})$$

and, finally, set $r(\varepsilon, A)=\min(r(A), r_\varepsilon)$, where

$$r_\varepsilon=\tfrac{1}{2}\|\Pi_{\max}\|^{-1}\cdot(2\|A\|)^{1-K(\varepsilon)N(A)}(K(\varepsilon)N(A))^{-1}e^{(A(A)-\varepsilon)K(\varepsilon)N(A)}$$

and $r(A)$ is defined in Lemma 2.1.

Then, clearly, (2.15) is satisfied.

*Proof of Theorem* 2.1. Let the number $r(A)$ of Theorem 2.1 be the same as in Lemma 2.1. Take an arbitrary $\varepsilon>0$ and by Lemma 2.2 find $K(\varepsilon)$ and $r(\varepsilon, A)\leqq r(A)$ satisfying (2.14) and (2.15).

From (2.2) and (2.4) it follows that for each $\alpha>0$ there is a natural $k(\alpha, \varepsilon)$ such that

$$\mu_k(V_{r(A)}(A)\setminus V_{r(\varepsilon,A)}(A))\leqq\alpha \quad \text{for any } k\geqq k(\alpha,\varepsilon). \tag{2.17}$$

It is easy to see that there is $q(N,r)>0$ so that

$$\|B_N\ldots B_1\zeta\|\geqq q(N,r)\|\zeta\| \tag{2.18}$$

for any $N=1,2,\ldots$ and $\zeta\in R^m$, provided $B_1,\ldots,B_N\in V_r(A)$.

Define $N(\varepsilon, A)$ by (2.14) then it follows from (2.6) that

$$B_{N(\varepsilon,A)}\ldots B_1U(A)\subset U(A) \tag{2.19}$$

provided $B_1,\ldots B_{N(\varepsilon,A)}\in V_{r(A)}(A)$. Hence by (2.18) for any $\zeta\in U(A)$.

$$\begin{aligned}2\|\Pi_{\max}B_{N(\varepsilon,A)}\ldots B_1\zeta\| &\geqq \|\Pi_{\max}B_{N(\varepsilon,A)}\ldots B_1\zeta\|+\|\Pi_{\min}B_{N(\varepsilon,A)}\ldots B_1\zeta\|\\ &\geqq\|B_{N(\varepsilon,A)}\ldots B_1\zeta\|\\ &\geqq q_\varepsilon\|\zeta\|\geqq q_\varepsilon\|\Pi_{\max}\|^{-1}\cdot\|\Pi_{\max}\zeta\|,\end{aligned} \tag{2.20}$$

where we put $q_\varepsilon=q(N(\varepsilon,A), r(A))$.

Now let $X_1^{(k)},\ldots,X_n^{(k)},\ldots$ be independent random matrices with the common distribution $\mu_k$. By (2.17),

$$P_{N,k}=P\{X_1^{(k)}\in V_{r(\varepsilon,A)}(A),\ldots,X_N^{(k)}\in V_{r(\varepsilon,A)}(A)\}\geqq(1-\alpha)^N, \tag{2.21}$$

for any $k\geqq k(\alpha,\varepsilon)$ and $N=1,2,\ldots$, where $P\{.\}$ denotes the probability of the event in brackets.

Introduce the events

$$Q_i(k)=\{X^{(k)}_{iN(\varepsilon,A)+j}\in V_{r(\varepsilon,A)} \text{ for all } j=1,\ldots,N(\varepsilon,A)\}$$

and define the random values $M(L,k)$ as follows

$$M(L,k)=\sum_{i=1}^{L}\chi_{Q_i(k)}, \tag{2.22}$$

where $\chi_Q$ denotes the indicator of the event $Q$. Then by (2.15) and (2.20) we conclude that

$$\frac{1}{L\cdot N(\varepsilon,A)}\ln\|\Pi_{\max}X^{(k)}_{L\cdot N(\varepsilon,A)}\cdots X^{(k)}_1\zeta\|\geqq L^{-1}M(L,k)(\Lambda(A)-\varepsilon)$$
$$+(LN(\varepsilon,A))^{-1}(L-M(L,k))\ln(\tfrac{1}{2}q_\varepsilon\|\Pi_{\max}\|^{-1}\|\Pi_{\max}\zeta\|) \qquad (2.23)$$

for any $\zeta\in U(A)$.

By the strong law of large numbers (see [2]) with the probability one,

$$\frac{M(L,k)}{L}\to P_{N,k} \quad \text{as} \quad L\to\infty, \qquad (2.24)$$

where $P_{N,k}$ is defined in (2.21).

Since

$$\|X^{(k)}_{L\cdot N(\varepsilon,A)}\cdots X^{(k)}_1\|\geqq\|\Pi_{\max}\|^{-1}\|\Pi_{\max}X^{(k)}_{L\cdot N(\varepsilon,A)}\cdots X^{(k)}_1\|$$

and

$$\Lambda_{\mu_k}=\lim_{L\to\infty}\frac{1}{L\cdot N(\varepsilon,A)}\ln\|X^{(k)}_{L\cdot N(\varepsilon,A)}\cdots X^{(k)}_1\|$$

then by (2.21), (2.23) and (2.24),

$$\liminf_{k\to\infty}\Lambda_{\mu_k}\geqq(1-\alpha)^{N(\varepsilon,A)}(\Lambda(A)-\varepsilon)+(1-(1-\alpha)^{N(\varepsilon,A)})\ln(\tfrac{1}{2}q_\varepsilon\|\Pi_{\max}\|^{-2}).$$

Here letting $\alpha\to 0$ one obtains

$$\liminf_{k\to\infty}\Lambda_{\mu_k}\geqq\Lambda(A)-\varepsilon$$

and since $\varepsilon$ is arbitrarily small then

$$\liminf_{k\to\infty}\Lambda_{\mu_k}\geqq\Lambda(A)=\Lambda_{\delta(A)}. \qquad (2.25)$$

By the general assertion (0.5)

$$\limsup_{k\to\infty}\Lambda_{\mu_k}\leqq\Lambda_{\delta(A)},$$

that together with (2.25) gives (2.4).

*Remark 2.2.* One can see from the proof that the assumption on $X^{(k)}_1,\ldots,X^{(k)}_n,\ldots$ to be independent is too strong. Some stationarity and ergodicity conditions on this matrix valued process would be enough.

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