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Limit Laws for the Maximum and Minimum of Stationary Sequences*

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Summary. The class of non-degenerate joint limiting distributions for the maximum and minimum of stationary mixing sequences is determined. These limit distributions are of the form, $H(x, \infty)-H(x, -y)$, where $H(x, y)$ is a bivariate extreme value distribution. Unlike the classical result for i.i.d. sequences, the maximum and minimum of stationary mixing sequences may be asymptotically dependent. We also give a sufficient condition for the asymptotic independence of the maximum and minimum. Finally, an example of a stationary sequence satisfying the mixing condition D in Leadbetter but which is not weakly mixing is constructed.

1. Introduction

The weak limit behavior of extreme values is well known for sequences of independent and identically distributed (iid) random variables. In an attempt to achieve similar results for stationary sequences, the processes are typically required to satisfy two types of dependence conditions. The first is a mixing condition requiring a certain class of events to become independent as their time separation increases. The second assumption is more of a local condition restricting the dependence between any two of the random variables when both are large. One of the weakest and most workable forms of these two conditions are the hypotheses D and D' introduced in Leadbetter (1974).

Under D and D' it was shown in Leadbetter (1974) and in Davis (1979), that the convergence in distribution of the maximum, properly normalized, is completely determined by the common distribution function of the sequence. That is, the maximum behaves asymptotically as though the underlying sequence is iid. This followed earlier work by Watson (1954), and Loynes (1965)

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and O'Brien (1974a), who proved analogous results under the more restrictive assumption of m-dependence and strong mixing, respectively.

In Davis (1979), there results were extended to the joint limiting distribution of the maximum and minimum under dependence conditions similar in nature to D and D'. Also, in that same paper, a 1-dependent sequence was constructed where the maximum and minimum are asymptotically dependent. In Sect. 4, we determine the class of all joint non-degenerate limiting distributions for the maximum and minimum from a stationary mixing sequence. The class of such limiting distributions turn out to be of the form $H(x,\infty)-H(x,-y)$ where $H(x,y)$ is a bivariate extreme value distribution (see Sect. 4 for definition).

A refinement of the asymptotic independence result in Davis (1979) is given in Sect. 3. In Sect. 2, further remarks are made concerning the limiting distribution of the maximum when the local dependence condition D' is no longer assumed.

Finally in Sect. 5, we construct a stationary sequence that satisfies D and D' but which is not weakly mixing. Also, for an arbitrary bivariate extreme value distribution, $H(x,y)$, a 2-dependent sequence is given with the property that the maximum and minimum has $H(x,\infty)-H(x,-y)$ as its limiting distribution. Further examples demonstrating various aspects of the results in earlier sections are also presented.

2. Limit Laws of the Maximum

Let $\{X_n\}$ be a stationary sequence of random variables with F and $F_{i_1\ldots i_p}(\cdot,\ldots,\cdot)$ denoting the common distribution function (df) and joint df of $X_{i_1},\ldots,X_{i_p}$, respectively. For a sequence of real numbers $\{u_n\}$ we shall say (cf. Leadbetter, 1974) that the condition $D(u_n)$ is satisfied by the sequence X_n if for any n, and any choice of integers $i_1<\ldots<i_p<j_1<\ldots<j_q$, $j_1-i_p>l$, we have

$$|F_{i_1\ldots i_p j_1\ldots j_q}(u_n,\ldots,u_n)-F_{i_1\ldots i_p}(u_n,\ldots,u_n)F_{j_1\ldots j_q}(u_n,\ldots,u_n)|\leqq\alpha_{n,l}$$

where $\alpha_{n,l}$ is nonincreasing in l and $\lim_{n\to\infty}\alpha_{n,l_n}=0$ for some sequence $l_n\to\infty$ with $l_n/n\to 0$. If we let $M(I)=\max_{i\in I}\{X_i\}$ for a set of integers I, then $D(u_n)$ requires the events $\{M(I)\leqq u_n\}$ and $\{M(J)\leqq u_n\}$, where $I=\{i_1,\ldots,i_p\}$ and $J=\{j_1,\ldots,j_q\}$, to become independent as the gap between the two sets goes to infinity. Notice that the gap, l_n, is coordinated with n and goes to infinity at a slower rate than n.

As in Loynes (1965), let $\{\hat{X}_n\}$ be the associated independent sequence of $\{X_n\}$ (i.e., $\{\hat{X}_n\}$ is an iid sequence with common df F). The main idea in the proof of the following theorem comes from Loynes.

Theorem 2.1. *Let $u_n=u_n(x)=a_n x+b_n$ where $a_n>0$, b_n are constants, and suppose $D(u_n)$ is satisfied by $\{X_n\}$ for all x. Further assume*

$$P(a_n^{-1}(\hat{M}_n-b_n)\leqq x)\to G(x) \quad \textit{and} \quad P(a_n^{-1}(M_n-b_n)\leqq x)\to H(x) \qquad (2.1)$$

for all x where $G(x)$ and $H(x)$ are non-degenerate df's *and* $\hat{M}_n = \max\{\hat{X}_1, \ldots, \hat{X}_n\}$. *Then $H(x)=G^{\beta}(x)$ where $0<\beta\leqq 1$. Since G is an extreme value* df, *this power relationship implies $H(x)=G(Ax+B)$ for some constants $A>0$ and B.*

Proof. We have by (2.1) and the $D(u_n)$ condition (cf. Leadbetter, 1974),

$$1-F(u_n(x)) = -\log G(x)/n + o(1/n), \tag{2.2}$$

and

$$P^k(M_n \leqq u_{nk}(x)) - P(M_n \leqq u_n(x)) \to 0 \quad \text{as} \quad n\to\infty \tag{2.3}$$

for every positive integer k. Clearly, $1-n((1-F(u_{nk}(x)) \leqq P(M_n \leqq u_{nk}(x))$ and upon letting $n\to\infty$ and using (2.1)-(2.3), we have $(1+\log G(x)/k)^k \leqq H(x)$. Now let $k\to\infty$ to obtain the inequality, $G(x)\leqq H(x)$ for all x with $G(x)>0$, which readily extends to all x.

Define $x_0=\inf\{x: G(x)>0\}$ and for $x>x_0$, let $w=-\log G(x)$ and $c_n(w) = u_n(x)$. Since $1-F(c_n(w)) = w/n + o(1/n)$,

$$\begin{aligned} &|P(M_n \leqq c_n(w/k)) - P(M_n \leqq c_{kn}(w))| \\ &\quad = P(c_n(w/k) < M_n \leqq c_{kn}(w)) + P(c_{kn}(w) < M_n \leqq c_n(w/k)) \\ &\quad \leqq n\,|F(c_{kn}(w)) - F(c_n(w/k))| \to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

Putting this together with (2.3) and letting $\delta(w)$ denote the limit of $P(M_n \leqq c_n(w))$, we see that δ must satisfy the functional equation $\delta^k(w/k)=\delta(w)$ for every positive integer k. As remarked in Loynes (1974), the only such limit function having this property is $\delta(w)=e^{-\beta w}$ where $0\leqq\beta\leqq\infty$. By the preceding paragraph, $G(x)\leqq H(x)=\delta(w)=G^{\beta}(x)$ for all $x>x_0$ which implies $\beta\leqq 1$. The result is complete once we show $\beta>0$ or, equivalently, prove that there exists an $x>x_0$ with $H(x)<1$.

Suppose, on the contrary, that $H(x)=1$ for all $x>x_0$. This necessarily implies $x_0>-\infty$ and $G(x)=\phi_{\gamma}(Ax+B)$ where $A>0$ and B are constants, $x_0 = -B/A$, and

$$\phi_{\gamma}(x) = \begin{cases} 0 & x\leqq 0 \\ e^{-x^{-\gamma}} & x>0, \end{cases} \qquad \gamma>0.$$

Choose $a_n' = \inf\{y: F(y) > 1-1/n\}$. Then, by Theorem 2.3.1 in de Haan (1970),

$$P(\hat{M}_n \leqq a_n' x) \to \phi_{\gamma}(x) \quad \text{and} \quad P(M_n \leqq a_n' x) \to H((x-B)/A)$$

by the convergence of types result Lemma 1, p. 253 in Feller (1971). In particular,

$$P(M_n \leqq 0) \to H(-B/A) = H(x_0) = 1 \tag{2.4}$$

since H is an extreme value distribution and hence continuous.

We now show that (2.4) cannot be true. First note that $F(0)<1$ for $0=\phi_{\gamma}(0) = \lim_{n\to\infty} F^n(0)$. Using an inequality of Chung and Erdos (1952),

$$P(M_n \leqq 0) \leqq 1 - \frac{n^2(1-F(0))^2}{n(1-F(0)) + 2S_{2,n}} \tag{2.5}$$

where $S_{2,n}=\sum_{1\leq i<j\leq n} P(X_i>0, X_j>0)$ which is bounded by $n^2(1-F(0))$. The right hand side of (2.5) is bounded by

$$1-\frac{n^2(1-F(0))^2}{n(1-F(0))+2n^2(1-F(0))}\to 1-\frac{(1-F(0))}{2}<1.$$

This contradicts (2.4) and thus confirms that $\beta>0$.

The last statement of the theorem can be proved by checking each of the three extreme value distributions.

Remarks. 1) By adjusting the normalizing constants in the above theorem, set $a_n'=a_n/A$ and $b_n'=(b_n-a_nB)/A$, we have $P(M_n\leqq a_n'x+b_n')\to H((x-B)/A)=G(x)$.

2) It is entirely possible for $P(\hat{M}_n\leqq a_nx+b_n)\to G(x)$ and $P(M_n\leqq c_nx+d_n)\to H(x)$ for some other choice of normalizing constants and where H and G are not of the same type. An example of this is given by O'Brien (1974b).

3) Under the additional assumption $D'(u_n)$, $\beta=1$.

4) For m-dependent sequences, $1\geqq\beta\geqq(m+1)^{-1}$, and a direct proof of the theorem can be given. For m-dependent sequences, β can easily be determined. The following proposition for 1-dependent sequences which follows from a theorem of Newell (1964) can be extended to m-dependence with some obvious modifications.

5) The above result appears to be a special case of a theorem in Chernick (1981). However, the proof supplied is inadequate even with the inclusion of the qualifier 'non-degenerate' in the statement of the theorem.

Proposition 2.2. *Let $\{X_n\}$ be a 1-dependent sequence and suppose $P(a_n^{-1}(\hat{M}_n-b_n)\leqq x)\to G(x)$, G non-degenerate. Then $P(a_n^{-1}(M_n-b_n)\leqq x)\to H(x)$, H non-degenerate if and only if*

$$nP(X_1>a_nx+b_n, X_2>a_nx+b_n)\to -(1-\beta)\log G(x) \tag{2.6}$$

as $n\to\infty$ for some x with $G(x)>0$. This last condition is equivalent to

$$P(X_1>y\,|\,X_2>y)\to 1-\beta \tag{2.7}$$

as $y\to y_0$ where $y_0=\sup\{x: F(x)<1\}$. Moreover, if this is the case, then (2.6) holds for all x with $G(x)>0$ and $H(x)=G^{\beta}(x)$.

Proof. We first show that (2.7) implies (2.6) for all x with $G(x)>0$. Write $u_n(x)=a_nx+b_n$, and note that $u_n(x)\to y_0$ and $nP(X_1>u_n(x))\to -\log G(x)$ for all x with $G(x)>0$. It follows, using (2.7), that

$$\begin{aligned} &nP(X_1>u_n(x), X_2>u_n(x))\\ &\quad =nP(X_1>u_n(x)\,|\,X_2>u_n(x))\,P(X_1>u_n(x))\to -(1-\beta)\log G(x). \end{aligned}$$

Now assume (2.6) holds for a fixed x and let y_j be an arbitrary sequence of numbers converging to y_0 from below. Define the sequence of integers $n_j=\sup\{k: u_k(x)\leqq y_j\}$ (this is well defined since $u_k(x)\to y_0$) and observe that $u_{n_j}(x)\leqq y_j<u_{n_j+1}(x)$. From this, the inequalities

$$P(X_1 > u_{n_{j+1}}(x), X_2 > u_{n_{j+1}}(x))(P(X_2 > u_{n_j}(x)))^{-1} \leqq P(X_1 > y_j | X_2 > y_j)$$
$$\leqq P(X_1 > u_{n_j}(x), X_2 > u_{n_j}(x))(P(X_2 > u_{n_{j+1}}(x)))^{-1}$$

are immediate. The two outside terms approach $1-\beta$ as $j \to \infty$ by (2.6). Since y_j was an arbitrary sequence, the equivalence of (2.6) and (2.7) is now complete.

The theorem in Newell (1964) gives $P(M_n \leqq u_n) e^{nP(X_1 > u_n) - nP(X_1 > u_n, X_2 > u_n)} \to 1$ as $n \to \infty$. Since $nP(X_1 > u_n(x)) \to -\log G(x)$, $P(M_n \leqq u_n) \to G^{\beta}(x)$ if and only if

$$nP(X_1 > u_n(x), X_2 > u_n(x)) \to -(1-\beta) \log G(x).$$

Invoking the preceding theorem completes the proof of the proposition. □

3. Asymptotic Independence of the Maximum and Minimum

Using the same notation as in the preceding section, define $W_n = \min\{X_1, \ldots, X_n\}$ and $\hat{W}_n = \min\{\hat{X}_1, \ldots, \hat{X}_n\}$. Again let $u_n(x) = a_n x + b_n$ and $v_n(y) = c_n y + d_n$ for constants $a_n > 0$, b_n, $c_n > 0$, and d_n. We shall say that the sequence $\{X_n\}$ satisfies C1 if for all x and y

$$P^k(M_{[\frac{n}{k}]} \leqq u_n, W_{[\frac{n}{k}]} > v_n) - P(M_n \leqq u_n, W_n > v_n) \to 0 \tag{C1}$$

as $n \to \infty$ for every integer k ($[s]$ = largest integer not greater than s).

Condition C1 is a type of mixing condition requiring that for every k, the k events

$$\{M_{[\frac{n}{k}]} \leqq u_n, W_{[\frac{n}{k}]} > v_n\},$$

$$\{M_{[\frac{n}{k}], [\frac{2n}{k}]} \leqq u_n, W_{[\frac{n}{k}], [\frac{2n}{k}]} > v_n\}, \ldots, \{M_{[\frac{(k-1)n}{k}], n} \leqq u_n, W_{[\frac{(k-1)n}{k}], n} > v_n\}$$

are asymptotically independent where $M_{s,t} = \max_{s<j\leqq t}\{X_j\}$, $W_{s,t} = \min_{s<j\leqq t}\{X_j\}$. C1 is easily seen to be implied by $D(v_n, u_n)$ in Davis (1979) and hence by strong mixing. In practice C1 may be harder to verify directly than $D(v_n, u_n)$ and is less appealing in that it already requires some knowledge of the maximum and minimum. However in what follows, C1 is the only mixing assumption necessary.

Proposition 3.1. *Assume that $P(M_n \leqq u(x)) \to G(x)$ and $P(W_n \leqq v_n(y)) \to H(y)$ where G and H are non-degenerate distribution functions. Also, in addition to* C1, *suppose $\{X_n\}$ satisfies the condition*

$$\limsup_{n\to\infty} n \sum_{j=1}^{[\frac{n}{k}]-1} \{P(X_1 > u_n, X_{j+1} \leqq v_n) + P(X_1 \leqq v_n, X_{j+1} > u_n)\} = o(1) \tag{C2}$$

as $k \to \infty$ for all x and y. Then

$$P(M_n \leqq u_n, W_n \leqq v_n) \to G(x) H(y).$$

Proof. First, we show that

$$P(M_{\left[\frac{n}{k}\right]} \leqq u_n) \to G^{1/k}(x) \quad \text{and} \quad P(W_{\left[\frac{n}{k}\right]} > v_n) \to [1-H(y)]^{1/k}.$$

Given $\varepsilon > 0$, choose y sufficiently small such that $H(y) < \varepsilon$. Then,

$$\begin{aligned} |P^k(M_{\left[\frac{n}{k}\right]} \leqq u_n) - P(M_n \leqq u_n)| &\leqq |P^k(M_{\left[\frac{n}{k}\right]} \leqq u_n) - P^k(M_{\left[\frac{n}{k}\right]} \leqq u_n, W_{\left[\frac{n}{k}\right]} > v_n)| \\ &+ |P^k(M_{\left[\frac{n}{k}\right]} \leqq u_n), W_{\left[\frac{n}{k}\right]} > v_n) - P(M_n \leqq u_n, W_n > v_n)| \\ &+ |P(M_n \leqq u_n, W_n > v_n) - P(M_n \leqq u_n)|. \end{aligned} \tag{3.1}$$

Applying the Mean Value Theorem, the first term on the right hand side of (3.1) is bounded by

$$\begin{aligned} &k(P(M_{\left[\frac{n}{k}\right]} \leqq u_n) - P(M_{\left[\frac{n}{k}\right]} \leqq u_n, W_{\left[\frac{n}{k}\right]} > v_n)) \\ &\leqq kP(W_{\left[\frac{n}{k}\right]} \leqq v_n) \leqq kP(W_n \leqq v_n) \to k(y) < k\varepsilon. \end{aligned}$$

Similarly, the $\limsup_{n\to\infty}$ of the third term is bounded by ε. The second term goes to zero by condition C1 and since ε is arbitrary, $P(M_{\left[\frac{n}{k}\right]} \leqq u_n) \to G^{1/k}(x)$ as claimed. The analogous result for the minimum can be proved in the same manner.

Now,

$$\begin{aligned} &1 - P(M_{\left[\frac{n}{k}\right]} \leqq u_n, W_{\left[\frac{n}{k}\right]} > v_n) \\ &= P(M_{\left[\frac{n}{k}\right]} > u_n) + P(W_{\left[\frac{n}{k}\right]} \leqq v_n) - P(M_{\left[\frac{n}{k}\right]} > u_n, W_{\left[\frac{n}{k}\right]} \leqq v_n), \end{aligned}$$

so that

$$\begin{aligned} &P(M_{\left[\frac{n}{k}\right]} \leqq u_n, W_{\left[\frac{n}{k}\right]} > v_n) \\ &= P(M_{\left[\frac{n}{k}\right]} \leqq u_n) - P(W_{\left[\frac{n}{k}\right]} \leqq v_n) + P(M_{\left[\frac{n}{k}\right]} > u_n, W_{\left[\frac{n}{k}\right]} \leqq v_n). \end{aligned}$$

Setting $\tau = -\log G(x)$ and $\beta = -\log(1 - H(y))$, the $\liminf_{n\to\infty}$ of the right hand side is greater than or equal to

$$e^{-\tau/k} - (1 - e^{-\beta/k}). \tag{3.2}$$

On the other hand, the $\limsup_{n\to\infty}$ of the same side is bounded above by

$$e^{-\tau/k} - (1 - e^{-\beta/k}) + \limsup_{n\to\infty} P(M_{\left[\frac{n}{k}\right]} > u_n, W_{\left[\frac{n}{k}\right]} \leqq v_n). \tag{3.3}$$

But

$$P(M_{\left[\frac{n}{k}\right]} > u_n, W_{\left[\frac{n}{k}\right]} \leqq v_n) = P\left\{\bigcup_{i=1}^{\left[\frac{n}{k}\right]} \{X_i > u_n\} \cap \bigcup_{j=1}^{\left[\frac{n}{k}\right]} \{X_j \leqq v_n\}\right\}$$

$$\leqq \left[\frac{n}{k}\right] \sum_{j=1}^{\left[\frac{n}{k}\right]-1} [P(X_1 > u_n, X_{j+1} \leqq v_n) + P(X_1 \leqq v_n, X_{j+1} > u_n)].$$

The $\limsup_{n\to\infty}$ of this last term is $o(1/k)$. Applying (3.2), (3.3), and Cl, we have

$$(e^{-\tau/k} - (1 - e^{-\beta/k}))^k \leqq \liminf_{n\to\infty} P(M_n \leqq u_n, W_n > v_n) \leqq \limsup_{n\to\infty} P(M_n \leqq u_n, W_n > v_n)$$

$$\leqq \left(e^{-\tau/k} - (1 - e^{-\beta/k}) + o\left(\frac{1}{k}\right)\right)^k.$$

Upon letting $k \to \infty$, the two sides of the inequality approach $e^{-\tau} e^{-\beta} = G(x)(1 - H(y))$, concluding the proof. ▯

Even under the strongest of mixing conditions C2 may note be satisfied as the 1-dependent example in Davis (1979) demonstrates. Under a slightly stronger condition that C2 ($D'(v_n, u_n)$ in Davis), the maximum and minimum are not only asymptotically independent but marginally have the same limiting distribution as the maximum and minimum of the associated independent sequence.

4. Class of Limiting Distributions of the Maximum and Minimum

Let (Y_n^1, Y_n^2) be an iid sequence of random vectors and define $M_n^1 = \max\{Y_1^1, \ldots, Y_n^1\}$, $M_n^2 = \max\{Y_1^2, \ldots, Y_n^2\}$ to be the respective component maximum. Suppose there exist constants $a_n > 0$, b_n, $c_n > 0$, d_n such that $P(M_n^1 \leqq a_n x + b_n, M_n^2 \leqq c_n y + d_n) \to H(x, y)$ where H is a nondegenerate distribution function. Such distribution functions H are called bivariate extreme value distributions (BEVD). A characterizing property for BEVD's is that there exist constants $A_k > 0$, B_k, $C_k > 0$, D_k such that $H^k(A_k x + B_k, C_k y + D_k) = H(x, y)$ for all positive integers k. This is in complete analogy with one dimensional extreme value distributions.

For a discussion of bivariate extreme value distributions see Chap. 5 in Galambos (1978).

One direction of the following theorem is immediate from the characterizing property of BEVD's.

Theorem 4.1. *Suppose the stationary sequence $\{X_n\}$ satisfies condition* C1. *Then the class of nondegenerate limiting distributions of $(a_n^{-1}(M_n - b_n), c_n^{-1}(W_n - d_n))$ is precisely $H(x, \infty) - H(x, -y)$ where H is a bivariate extreme value distribution.*

Proof. First assume $P(M_n \leqq u_n(x), W_n > v_n(-y)) \to H(x, y)$ where $u_n(x) = a_n x + b_n$, $v_n(y) = c_n y + d_n$, and $H(x, y)$ is a nondegenerate distribution. By the C1 assumption, $P(M_n \leqq u_{nk}(x),\ W_n > v_{nk}(-y)) \to H^{1/k}(x, y)$ for $k=1,2,\ldots$. Employing the multivariate analogue of the convergence of types result, there exist constants $A_k>0$, B_k, $C_k>0$, D_k such that $H(A_k x+B_k,\ C_k y+D_k)=H^{1/k}(x, y)$. Therefore, H is a bivariate extreme value distribution and

$$\begin{aligned} P(a_n^{-1}(M_n-b_n) \leqq x,\ c_n^{-1}(W_n-d_n) \leqq y) \\ = P(M_n \leqq u_n(x)) - P(M_n \leqq u_n(x),\ W_n > v_n(y)) \end{aligned}$$

converges to $H(x, \infty) - H(x, -y)$.

The proof is complete once we exhibit a stationary sequence satisfying C1 with $P(M_n \leqq u_n(x),\ W_n > v_n(-y))$ converging to an arbitrary bivariate extreme value distribution H. A basic property of BEVD's is that $H^2(x, y) = G(x, y)$ is also a BEVD and has the representation $G(x, y) = D_G(G_1(x),\ G_2(y))$ where $D_G(\cdot,\cdot)$ is the dependence function defined on p. 250 in Galambos (1978), and G_1 and G_2 are extreme value distributions. Let $F_1(x)$ and $F_2(x)$ be two df's such that $F_i(0)=0$ and F_i belongs to the domain of attraction of G_i, $i=1,2$. This implies the existence of constants $a_n>0$, b_n, $c_n>0$, d_n such that $F_1^n(a_n x + b_n) \to G_1(x)$ and $F_2^n(c_n y+d_n) \to G_2(y)$.

Now define the distribution function $F(x, y)$ to be equal to $D_G(F_1(x),\ F_2(y))$ and then

$$\begin{aligned} F^n(a_n x+b_n,\ c_n y+d_n) &= D_G^n(F_1(a_n x+b_n),\ F_2(c_n y+d_n)) \\ &= D_G(F_1^n(a_n x+b_n),\ F_2^n(c_n y+d_n)) \to G(x, y) \end{aligned}$$

follows from Theorems 5.21 and 5.22 in Galambos (1978). The stationary sequence $\{X_n\}$ can now be defined as follows. First let $\{(Y_n, Z_n)\}$ be an iid sequence with common df $F(x, y)$ defined above and let J be a Bernoulli random variable independent of (Y_n, Z_n) with $P(J=1)=P(J=0)=1/2$. Define the sequence $(X_1, X_2, \ldots,)$ to be $(Y_1, -Z_3, Y_3, -Z_5, Y_5, -Z_7, \ldots)$ if $J=1$ and $(-Z_2, Y_2, -Z_4, Y_4, -Z_6, Y_6, \ldots)$ if $J=0$.

It is clear that X_n is stationary (since $P(J=1)=P(J=0)$) and, moreover, for n sufficiently large, $P(M_n \leqq u_n,\ W_n > -v_n)$ is equal to

$$\left\{\begin{array}{ll} 1/2\, P^{\frac{n-1}{2}}(Y_1 \leqq u_n, Z_1 \leqq v_n)(P(Y_1 \leqq u_n) + P(Z_1 \leqq v_n)) & \text{if } n \text{ is odd} \\ 1/2\, P^{\frac{n-2}{2}}(Y_1 \leqq u_n, Z_1 \leqq v_n)(P(Y_1 \leqq u_n)\, P(Z_1 \leqq v_n) + P(Y_1 \leqq u_n, Z_1 \leqq v_n)) & \\ & \text{if } n \text{ is even,} \end{array}\right\} \tag{4.1}$$

where $u_n = u_n(x) = a_n x + b_n$ and $v_n = v_n(y) = c_n y + d_n$. Upon letting $n \to \infty$, we obtain $P(M_n \leqq u_n(x),\ W_n > -v_n(y)) \to G^{1/2}(x, y) = H(x, y)$. Finally property C1 is easily checked using (4.1). □

Examples similar to the one used in the above proof will be presented and discussed in the next section. We will also construct, for an arbitrary bivariate extreme value distributions $H(x, y)$, a 2-dependent stationary sequence with $P(M_n \leqq u_n(x),\ W_n > v_n(-y)) \to H(x, y)$. Thus, we have the following corollary.

Corollary 4.2. *The results of Theorem* 4.1 *remain true for strongly mixing sequences and processes satisfying the mixing condition* $D(v_n, u_n)$ *in Davis* (1979).

5. Examples

Let $\{Y_n\}$ be an iid sequence and let $\{J_n\}$ be a sequence of alternating zeros and ones, independent of the Y_n's, with $P(J_1=1, J_2=0, J_3=1, \ldots)=P(J_1=0, J_2=1, J_3=0, \ldots)=1/2$. Define the X_n sequence as follows:

$$X_n=\begin{cases} Y_n & \text{if } J_n=1 \\ Y_{n+1} & \text{if } J_n=0. \end{cases}$$

At first glance, one is tempted to conclude that X_n is a 1-dependent sequence for $(X_1, X_2, \ldots)=(Y_1, Y_3, Y_3, Y_5, Y_5, \ldots)$ and $(Y_2, Y_2, Y_4, Y_4, \ldots)$, each with probability 1/2. However, this is not the case.

Properties of the $\{X_n\}$ sequence.

1. X_n is stationary.
2. X_n is ergodic since the sequence $\{(Y_n, J_n)\}$ is ergodic.
3. X_n is not weakly mixing. If T is the shift operator, then a sequence of random variables is said to be weakly mixing if for any two events A and B, $P(A\cap T^{-j}B)-P(A)P(B)\to 0$ as $j\to\infty$. In the above example, take $A=\{X_1\leqq x_1\}\cap\{X_2\leqq x_2\}$ and $B=A$. Then, for $j\geqq 4$ and j even,

$$P(A\cap T^{-j}B)=1/2[F^2(x_1)F^2(x_2)+F^2(x_1\wedge x_2)]$$

where $s\wedge t=\min(s, t)$ and F is the df of Y_1. Yet,

$$P^2(A)=\tfrac{1}{4}[F(x_1)F(x_2)+F(x_1\wedge x_2)]^2$$

and so

$$P(A\cap T^{-j}B)-P(A)P(B)=\tfrac{1}{4}(F(x_1)F(x_2)-F(x_1\wedge x_2))^2\neq 0$$

unless $F(x_1)$ or $F(x_2)$ equals zero which establishes that X_n is not weakly mixing.

4. X_j is independent of $\{X_{j+2}, X_{j+3}, \ldots\}$. This can be seen by considering

$$\begin{aligned}&P(X_1\leqq x_1, X_3\leqq x_3, \ldots, X_m\leqq x_m)\\&=P(Y_1\leqq x_1, X_3\leqq x_3, \ldots, X_m\leqq x_m, J_1=1)\\&\quad+P(Y_2\leqq x_1, X_3\leqq x_3, \ldots, X_m\leqq x_m, J_1=0)\\&=P(Y_1\leqq x_1)P(X_3\leqq x_3, \ldots, X_m\leqq x_m)\end{aligned}$$

since Y_1 and Y_2 are independent of $(X_3, \ldots, X_m)$ and J. Note that (X_1, X_2) is not independent of $(X_4, X_5, \ldots)$ as demonstrated in 3. This can be interpreted as follows. From any two consecutive observations, we know the J_n sequence completely and, consequently, knowledge about any two consecutive observations in the future is also gained. For example, if $X_1=X_2$ then $X_n=X_{n+1}$ if n is odd and $X_n\neq X_{n+1}$ if n is even.

5. If the distribution F belongs to the domain of attraction of an extreme value distribution, $G(x)$, then

$$P(a_n^{-1}(M_n-b_n)\leqq x)=\begin{cases}1/2F^{\frac{n+2}{2}}(a_nx+b_n)+1/2F^{n/2}(a_nx+b_n) & \text{if } n \text{ is even}\\ F^{\frac{n+1}{2}}(a_nx+b_n) & \text{if } n \text{ is odd,}\end{cases}$$

$\to G^{1/2}(x)$ as $n\to\infty$. A quick calculation, $nP(X_1>a_nx+b_n, X_2>a_nx+b_n) \to -1/2\log G(x)$, gives the value of 1/2 to the β in Proposition 2.2. Thus, in terms of the maximum this sequence behaves exactly the same as a 1-dependent sequence.

There are some interesting modifications of this example. Let (Y_n, Z_n) be an iid sequence of random vectors with each component having the same marginal df. Define X_n as before only replacing Y_{n+1} by Z_{n+1}. Then

$$X_n=\begin{cases}Y_n & \text{if } J_n=1\\ Z_{n+1} & \text{if } J_n=0\end{cases}$$

and properties (1)-(4) above still hold for this sequence, provided Y_1 and Z_1 are not independent.

One special choice for the joint distribution of (Y_1, Z_1) is the following. If $F(x, y)$ denotes the df of (Y_1, Z_1) suppose $F(x, x)=F^2(x)$ for all x and $F(x, y) \neq F(x)F(y)$ for some $x\neq y$. Then $P(M_n\leqq x)=F^n(x)$ for all x so that condition D in Leadbetter (1974) is satisfied yet $\{X_n\}$ is not even weakly mixing.

Finally, let $H(x, y)$ be a bivariate extreme value df and, as in the proof of Theorem 4.1, set $G(x, y)=H^2(x, y)$. Then $G(x, y)=D_G(G_1(x), G_2(y))$ where $D_G(\cdot,\cdot)$ is the dependence function for G and G_1 and G_2 are extreme value distributions. Let $F_1(x)$ and $F_2(y)$ be two df's which are symmetric about the origin and assume F_i belongs to the domain of attraction of G_i, $i=1, 2$. Let (Y_n, Z_n) be an iid sequence with common df $D_G(F_1(x), F_2(y))$.

Now define the function g,

$$g(u, v, w)=\begin{cases}u & \text{if } u>0,\ v>0\\ v & \text{if } v<0,\ w>0\\ 0 & \text{otherwise}\end{cases}$$

and then define the sequence $X_n=g(Y_n, -Z_{n+1}, -Z_{n+2})$. It is clear that X_n is stationary and 2-dependent. We will show that $P(M_n\leqq u_n, W_n>-v_n)\to H(x, y)$ where $u_n=a_nx+b_n$, $v_n=c_ny+d_n$ are the normalizing constants associated with F_1 and F_2 respectively.

The following properties hold for the $\{X_n\}$ sequence:

(i) $P(X_1>u_n)=P(Y_1>u_n, Z_2<0)=-(\log G_1(x))/(2n)+o(1/n)$.

(ii) $P(X_1\leqq -v_n)=P(Z_2>v_n, Z_3<0)=-(\log G_2(y))/(2n)+o(1/n)$.

(iii) $P(X_1\leqq -v_n, X_2>u_n)=P(Z_2>v_n, Z_3<0, Y_2>u_n, Z_3<0)$
$=(\log h(x, y))/(2n)+o(1/n)$.

where $h(x, y)=G(x, y)/(G_1(x)G_2(y))$ (see Theorem 5.33 in Galambos (1978)).

Set $A_j^0=\{X_j>u_n\}$, $A_j^1=\{X_j\leqq -v_n\}$. After tedious, but elementary calculations, we have:

(iv) $\lim_{n\to\infty} \sum_{1\leq i_1<i_2\leq[\frac{n}{k}]} P(A_{i_1}^1 A_{i_2}^0)=\log h(x,y)/(2k)+o(1/k)$ and

$\lim_{n\to\infty} \sum_{1\leq i_1<i_2\leq[\frac{n}{k}]} P(A_{i_1}^{\varepsilon_1} A_{i_2}^{\varepsilon_2})=o(1/k)$ for all other choices of

$\varepsilon_1=0$ or 1 and $\varepsilon_2=0$ or 1.

(v) For any choice of $\varepsilon_i=0$ or 1, $i=1,2,3$,

$\limsup_{n\to\infty} \sum_{1\leq i_1<i_2<i_3\leq[\frac{n}{k}]} P(A_{i_1}^{\varepsilon_1} A_{i_2}^{\varepsilon_2} A_{i_3}^{\varepsilon_3})=o(1/k).$

Using a Bonferroni type inequality,

$$P(M_{[\frac{n}{k}]}\leq u_n,\ W_{[\frac{n}{k}]}>-v_n)=1-P\left(\bigcup_{\substack{i=1\\ j=1}}^{[\frac{n}{k}]} A_i^0 A_j^1\right)$$

is bounded above and below by

$$1-\left[\frac{n}{k}\right](P(A_1^0)+P(A_1^1))+\sum_{\varepsilon_1=0}^{1}\sum_{\varepsilon_2=0}^{1}\sum_{1\leq i_1<i_2\leq[\frac{n}{k}]} P(A_{i_1}^{\varepsilon_1} A_{i_2}^{\varepsilon_2})$$

and

$$1-\left[\frac{n}{k}\right](P(A_1^0)+P(A_1^1))+\sum_{\varepsilon_1=0}^{1}\sum_{\varepsilon_2=0}^{1}\sum_{1\leq i_1<i_2\leq[\frac{n}{k}]} P(A_{i_1}^{\varepsilon_1} A_{i_2}^{\varepsilon_2})$$
$$-\sum_{\varepsilon_1=0}^{1}\sum_{\varepsilon_2=0}^{1}\sum_{\varepsilon_3=0}^{1}\sum_{1\leq i_1<i_2<i_3\leq[\frac{n}{k}]} P(A_{i_1}^{\varepsilon_1} A_{i_2}^{\varepsilon_2} A_{i_3}^{\varepsilon_3}),$$

respectively. Using (i), (ii), (iv), and (v),

$$\begin{aligned}&(1+(\log G_1(x)+\log G_2(y)+\log h(x,y))/(2k)+o(1/k))^k\\ &\leq \liminf_n P^k(M_{[\frac{n}{k}]}\leq u_n,\ W_{[\frac{n}{k}]}>-v_n)\\ &\leq \limsup_n P^k(M_{[\frac{n}{k}]}\leq u_n,\ W_{[\frac{n}{k}]}>-v_n)\\ &\leq (1+(\log G_1(x)+\log G_2(y)+\log h(x,y))/(2k)+o(1/k))^k.\end{aligned}$$

A 2-dependent sequence certainly satisfies C1 so that $P^k(M_{[\frac{n}{k}]}\leq u_n, W_{[\frac{n}{k}]}>-v_n)$ can be replaced by $P(M_n\leq u_n, W_n>-v_n)$ in the above inequality. Now letting $k\to\infty$, the outside terms converge to $(G_1(x)G_2(y)h(x,y))^{1/2}=G^{1/2}(x,y)=H(x,y)$ so that $P(M_n\leq a_n x+b_n, W_n\leq c_n y-d_n)\to H(x,\infty)-H(x,-y)$ as desired.

Note that $P(\hat{M}_n\leq a_n x+b_n)\to H(x,\infty)$, and $P(\hat{W}_n\leq c_n y-d_n)\to 1-H(\infty,-y)$ where $\hat{M}_n$ and $\hat{W}_n$ are the maximum and minimum, respectively, of the associated independent sequence. The effect of the cross terms ((iii) above) on the asymptotic dependence structure of the maximum and minimum is clear in the above examples. This should be compared with the result of Proposition 3.1.

It is worth remarking that even under the most stringent of mixing conditions, m-dependence, the limiting behavior of the extremes may be markedly different than those for the associated independent sequence. In fact, most of the differences can be detected in one and two-dependent sequences. The above example and examples in O'Brien (1974b) and Mori (1976) illustrate this point.

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