Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 17–30 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Representing Last Exit Potentials as Potentials of Measures

Joseph Glover*

Mathematics, C-012, University of California, La Jolla, California 92093, USA

Summary. Let X_t be a transient Hunt process having a potential density $u(x,y)$ chosen in the usual way. *No duality hypotheses are assumed.* Let K be a closed set with $M=\sup\{t: X_t\in K\}<\infty$ almost surely. For f a bounded Borel function,

$$P^x(f(X_M); M>0)=\int u(x,y)(1_K f(y)\,\kappa(dy)+f(z)\,\nu_C(dy,dz)),$$

where κ is a measure on E, and ν_C is a measure on $E\times E$ so that $\nu_C((E-K)\times E)=0$. If X_t is simply a right process and K is closed,

$$P^x(1_K f(X_M); M>0)=\int u(x,y)\,1_K f(y)\,\kappa(dy).$$

If X_t is a right process and $K\subset E$ is closed in the Ray topology of X, then

$$P^x(f(X_M); M>0)=\int u(x,y)(1_K f(y)\,\kappa(dy)+f(z)\,\nu_C(dy,dz)).$$

If X_t is a diffusion, we obtain the representation of equilibrium potentials (for closed sets) due to Chung, Getoor-Sharpe, and Meyer without duality hypotheses.

0. Introduction

Let X_t be a transient Hunt process having a potential density $u(x,y)$. No duality hypotheses are assumed in this paper. Our main purpose is to show that if $u(x,y)$ is chosen in the "usual" way, then the following result is true.

Theorem. *Let K be a closed set so that $M=\sup\{t: X_t\in K\}<\infty$ almost surely. For any positive bounded Borel function f on the state space E,*

$$(*)\qquad P^x(f(X_M); M>0)=\int u(x,y)(1_K f(y)\,\kappa(dy)+f(z)\,\nu_C(dy,dz)),$$

* Research supported in part by NSF grant MCS-8002659 and a CNRS Fellowship while the author was visiting I.M.S.S., Universite de Grenoble II

0044-3719/82/0061/0017/$02.80

where κ is a measure on E and ν_C is a measure on $E \times E$. Moreover, $\nu_C((E-K) \times E)=0$.

In order to put this result into historical perspective, recall the evolution of representations such as (*). The left hand side is a bounded potential of an additive functional, and Hunt was the first to give general conditions implying that all such bounded potentials (and more!) can be represented as potentials of measures. (See Chap. VI of [1] for an exposition slightly different from Hunt's. In [1], it is assumed that X_t is a standard process in strong duality with a Feller process.) Subsequently, many weakenings and generalizations of Hunt's hypotheses have been given by various authors. These have all taken two general forms. First, regularity hypotheses have been given on X to insure that the bounded potentials of additive functionals can be represented as potentials of measures. For example, Sharpe [16] assumes that X_t is a standard process in strong duality with another standard process $\hat{X}_t$ (see also Sect. V of Revuz [15]). Chung [2] and Rao [14] assume the potential density $u(x, y)$ satisfies certain regularity hypotheses.

Meyer [12] assumes a condition which is a little bit weaker than duality. Second, various authors have obtained representations of bounded potentials of additive functionals and excessive functions by enlarging the original state space E and constructing a strong dual process for X on the enlarged state space $\bar{E}$. It must be emphasized that the representing measures so obtained charge $\bar{E}-E$ in general, so the potential kernel in (*) must be extended to $\bar{E} \times \bar{E}$ and the integral must be interpreted as being over $\bar{E}$. This approach probably first arose in the study of the Martin boundary (Doob, Dynkin, Hunt, Kunita-Watanabe, Meyer...) and was originally intended to obtain representations of extremal harmonic functions. Work of Garcia-Alvarez and Meyer [4] and Glover [9] followed, using Ray-type compactifications. As mentioned above, all of these approaches are designed to insure that essentially all potentials of additive functionals can be represented as potentials of measures, and, of course, a price is paid for this generality. Simple examples show that one needs some sort of duality hypotheses or an enlargement of the state space or some other auxiliary procedure.

Our point of view here is the following. We are given the process X without duality hypotheses, we do not extend $u(x, y)$ or the state space, and we single out an interesting class of potentials which can always be represented as potentials of measures on E. Notice that if f is identically 1, then the left hand side of (*) is the *equilibrium potential* of the closed set K. Moreover, if X is assumed to be a diffusion with infinite lifetime, then $X_M = X_{M-}$ and $\nu_C = 0$, so that (*) is an extension of the equilibrium formula of Chung [2], Getoor-Sharpe [7], and Meyer [12]. These authors prove (under various duality and regularity hypotheses) that

$$P^x(f(X_{M-}); M>0) = \int u(x, y) f(y) \mu(dy),$$

where μ is the equilibrium measure of K. They permit K to be any Borel set. Simple examples show that if X is a Hunt process with jumps, then $P^x(f(X_{M-}); M>0)$ cannot always be represented as the potential of a measure, even if K is chosen to be compact.

We can even say something if we assume X is simply a transient process satisfying the *hypothèses droites* (right hypotheses): if K is closed, then

$$P^x(1_K f(X_M); M>0)=\int u(x,y)\,1_K f(y)\kappa(dy).$$

Also, if K is any totally thin set (which need not be closed), then

$$P^x(f(X_M); M>0)=\int u(x,y) f(y)\,\kappa(dy).$$

If X is simply a right process, the representation theorem (*) given in the first paragraph is true if K is chosen to be a set in the state-space which is closed in a Ray-topology of X (see the end of Sect. 4 for a precise statement).

In brief, the plan of action is the following. In Sects. 1, 2 and 3, we construct a strong Markov dual $\hat{X}$ for X on an enlarged state space $\bar{E}_\Delta$. Then $P^x(f(X_M); M>0)$ can be represented as the potential of a measure μ on the enlarged space. In Sect. 4, we show that there is a measure v on E so that μ and v have the same potentials on E. The computations in Sect. 4 are actually rather delicate (the "most important" being the fact that $\hat{\mathbf{P}}^x_a(e^{-b\zeta})=a/(b+a)$ almost surely - see (17) and (18)), and call for some careful preparation. As some partial justification for this statement, we remark that (*) is false if K is taken to be either open or finely closed (or both), in general, and we leave the construction of an example to the interested reader (try that well-known counterexample in duality and time-reversal arguments - the "crotch").

We rely heavily on the theories of right processes and Ray processes; a good general reference is Getoor's set of notes [5]. We also rely heavily on a compactification procedure we developed in [9], which was based on the work of many people: we refer the reader to the bibliography of [9].

1. Moderate Duality

Let $X=(\Omega, \mathbf{F}, \mathbf{F}_t, X_t, \theta_t, P^x)$ be a right process on a Lusin topological space E_Δ together with its Borel field $\mathbf{E}_\Delta$ [5]. We let P_t be the semigroup of X, and we let $(U^a)_{a\geqq 0}$ be the resolvent of X. As usual, we assume that Δ acts as a trap for X, and we define the lifetime of X to be $\zeta(\omega)=\inf\{t: X_t(\omega)=\Delta\}$. When we refer to a "Borel function f on E_Δ," f is assumed to be zero at Δ, and the resolvent is not considered to charge Δ.

We make two other assumptions which will be in force throughout the paper. First, X is *transient*: there exists a Borel function h bounded by 1 which is strictly positive on $E=E_\Delta-\{\Delta\}$ so that $Uh\leqq 1$. Second, we assume μ is a *reference probability measure* on E so that $U^a(x,\cdot)\ll\mu$ for all x in E and for all $a\geqq 0$. We assume that P_t is Borel measurable. In Sect. 4, we shall *occasionally* assume that X is a *Hunt* process. This means that X is quasi-left continuous: if (T_n) is a sequence of $(\mathbf{F}_t)$-optional times increasing to a limit T, then $X(T_n)\to X(T)$ almost surely on $\{T<\infty\}$. We shall indicate explicitly when this hypothesis is being used.

Throughout this paper, we shall work with an extension of X which we now describe. Let (Γ_n) be a countable sequence of points not in E_Δ which we adjoin as isolated points to E_Δ: $\tilde{E}_\Delta=E_\Delta\cup\bigcup_n\{\Gamma_n\}$. Let (S_n) be a sequence of independent random variables which are independent of X and which are exponentially distributed with parameter 1. Set

$$\begin{aligned} \tilde{X}_t &= X_t \quad \text{if } t<\zeta \\ &= \Gamma_j \quad \text{if } \zeta + \sum_{k=1}^{j-1} S_k \leqq t < \zeta + \sum_{k=1}^{j} S_k \quad (j \geqq 1). \end{aligned}$$

Then $\tilde{X}_t$ (together with the appropriate completion of its natural filtration) is a right process which enjoys most of the properties of X. That is, there is a function $\tilde{h} \leqq 1$ so that if $\tilde{U}^a$ is the resolvent of $\tilde{X}$, then $\tilde{U}\tilde{h} \leqq 1$. There is a reference probability measure $\tilde{\mu}$ on E. It will be left to the reader later to check that if $\tilde{u}(x, y)$ is the "appropriately regularized" potential density of $\tilde{X}$, then the restriction of $\tilde{u}(x, y)$ to $E \times E$ is the "appropriately regularized" potential density for X. If X is a Hunt process, then $\tilde{X}$ is a Hunt process. If K is any set in E with $L = \sup\{t: X_t \in K\}$, then $L = \sup\{t: \tilde{X}_t \in K\}$. Thus we may use $\tilde{X}_t$ instead of X_t. The main advantage in using $\tilde{X}_t$ is that it has infinite lifetime. Thus we replace X with $\tilde{X}$, although *we conserve the notation of* X, so that $\zeta = \infty$ almost surely, and the Γ_n are now points in E_Δ.

Let $A_t = \int_0^t h(X_s)\,ds$, and let $T_t = \inf\{s: A_s > t\}$. If we set $Y_t = X_{T(t)}$, $\mathbf{G} = \mathbf{F}$, $\mathbf{G}_t = \mathbf{F}_{T(t)}$, and $\Theta_t = \theta_{T(t)}$, then it is a standard fact that $Y = (\Omega, \mathbf{G}, \mathbf{G}_t, Y_t, \Theta_t, P^x)$ is a Borel right process on $(E_\Delta, \mathbf{E}_\Delta)$. We let $(V^a)_{a \geqq 0}$ be its resolvent and note that $V1 = Uh \leqq 1$ on E. Thus, if we let $z(\omega) = \inf\{t: Y_t(\omega) = \Delta\}$, $z < \infty$ almost surely. Define the reverse of Y by setting

$$\begin{aligned} \hat{Y}_t(\omega) &= Y_{z(\omega)-t}(\omega) \quad \text{if } 0 < t \leqq z(\omega) \\ &= \Delta \quad \text{if } t > z(\omega). \end{aligned}$$

Then $(\hat{Y}_t, P^\mu)$ is a left-continuous moderate Markov process with a Borel semigroup $(\hat{Q}_t)_{t>0}$ and resolvent $(\hat{V}^a)_{a \geqq 0}$ ([3], [13]). Set $\rho = \mu V$, and let $\rho(f) = \int f(x)\,\rho(dx)$ for all positive Borel functions f on E. For each $a \geqq 0$, V^a and $\hat{V}^a$ satisfy $\rho(f \cdot V^a g) = \rho(f \hat{V}^a \cdot g)$ for all positive Borel functions f and g on E. Here, we shall use the convention that coresolvents and cosemigroups such as $\hat{V}^a$ and $\hat{Q}_t$ act on functions on the left (see, for example, Chap. VI of [1]).

Thus Y is a right process in *weak duality* with a left continuous moderate Markov process $\hat{Y}$ with respect to a finite excessive reference measure ρ. The dual $\hat{Y}$ has two defects (which we "remedy" in the next section): $\hat{Y}$ may not have a right continuous strong Markov version on E_Δ, and $\hat{V}^a(\cdot, x)$ may not be absolutely continuous with respect to ρ for some x in E.

2. The Double-Ray Compactification

We developed in [9] a procedure to produce a "strong dual" process for Y starting from $\hat{Y}$ by enlarging the space E_Δ. We shall need those results and some complements to those results, which we describe below.

We can construct a compact metric space $\bar{E}_\Delta$ with Borel field $\bar{\mathbf{E}}_\Delta$ and two Ray resolvents $(\bar{V}^a)_{a>0}$ and $(\hat{\bar{V}})_{a>0}$ on $\bar{E}_\Delta$ so that:

(1) $E_\Delta \subset \bar{E}_\Delta$, $E_\Delta \in \bar{\mathbf{E}}_\Delta$, and E_Δ is dense in $\bar{E}_\Delta$.

(2) For each x in E, $\bar{V}^a(x,\cdot)=V^a(x,\cdot)$ and $\hat{\bar{V}}^a(\cdot,x)=\hat{V}^a(\cdot,x)$.

(Note: (2) is not stated explicitly in [9], but follows quickly from the contruction of $\bar{V}^a$ and $\hat{\bar{V}}^a$. Using the notation in [9], $\bar{V}^a$ is defined by setting $\bar{V}^a\bar{f} = \overline{V^a f}$, where $f=\bar{f}|_E$ and f is an element of $\mathfrak{R}$, the double Ray cone. For x in E, this says that $\bar{V}^a\bar{f}(x)=V^a(\bar{f}\,1_E)(x)$. Since $\bar{\mathfrak{R}}-\bar{\mathfrak{R}}$ is uniformly dense in the continuous functions on $\bar{E}_\Delta$ and $V^a(x,\cdot)$ is carried by E, we have $V^a(x,\cdot) = \bar{V}^a(x,\cdot)$. The argument for $\hat{\bar{V}}^a$ is the same.)

Let $(\bar{Q}_t)_{t\geq 0}$ and $(\hat{\bar{Q}}_t)_{t\geq 0}$ be the right-continuous Ray semigroups with resolvents $(\bar{V}^a)_{a>0}$ and $(\hat{\bar{V}}^a)_{a>0}$, respectively. Let $\bar{Q}_t^m=\text{vague-limit}_{s\uparrow\uparrow t}\bar{Q}_s$, and let $\hat{\bar{Q}}_t^m=\text{vague-limit}_{s\uparrow\uparrow t}\hat{\bar{Q}}_s$: then $(\bar{Q}_t^m)_{t>0}$ and $(\hat{\bar{Q}}_t^m)_{t>0}$ are left-continuous (moderate Markov) Ray semigroups with resolvents $(\bar{V}^a)_{a>0}$ and $(\hat{\bar{V}}^a)_{a>0}$ [19].

Let $\bar{\Omega}=\{\bar{\omega}: R^+\to\bar{E}_\Delta$ so that $\bar{\omega}(t)$ is right continuous and has left limits in the topology of $\bar{E}_\Delta$, and so that Δ is a trap$\}$. Let $\bar{Y}_t(\bar{\omega})=\hat{\bar{Y}}_t(\bar{\omega})=\bar{\omega}(t)$, $\bar{\mathbf{I}}_t^0 =\sigma(\bar{Y}_s: s\leq t)$, and let $\bar{\Theta}_t$ be the shift on $\bar{\Omega}$. Given a probability measure κ on $\bar{E}_\Delta$, let $\bar{P}^\kappa$ (resp. $\hat{\bar{P}}^\kappa$) be the measure on $(\bar{\Omega},\bar{\mathbf{I}}^0)$ so that $\bar{Y}_t$ (resp. $\hat{\bar{Y}}_t$) is a strong Markov process with semigroup $\bar{Q}_t$ (resp. $\hat{\bar{Q}}_t$) and initial distribution $\kappa\bar{Q}_0$ (resp. $\hat{\bar{Q}}_0\,\kappa$).

(3) There is a set $N\subset E$ which is polar for the process Y so that if κ is any probability measure on $E-N$, then $P^\kappa(Y_t$ is right continuous with left limits in the topology of $\bar{E}_\Delta)=1$. Let $\tilde{P}^\kappa$ (resp. $\mathscr{P}^\kappa$) be the measure constructed on $(\bar{\Omega},\bar{\mathbf{I}}^0)$ so that $\bar{Y}_t$ is Markov with semigroup Q_t (resp. $\bar{Q}_t$) and initial distribution κ (see Theorem 11.8 in [4]). Then $\tilde{P}^\kappa=\hat{\mathscr{P}}^\kappa$.

(4) $P^\rho(\hat{Y}(t)$ is left continuous with right limits in the topology of $\bar{E}_\Delta)=\rho(E)$. Let $\hat{Y}_{t+}$ be the right limit of $\hat{Y}_t$, taken in the topology of $\bar{E}_\Delta$, and let κ be the P^ρ-distribution of $\hat{Y}_{0+}$. Let v_t be the P^ρ-distribution of $\hat{Y}_t$: (v_t) is an entrance law for $\hat{Q}_t$. Let $\hat{\tilde{P}}$ (resp. $\hat{\mathscr{P}}$) be the measure constructed on $(\bar{\Omega},\bar{\mathbf{I}}^0)$ so that $\hat{\bar{Y}}_{t-}$ is moderate Markov with semigroup $\hat{Q}_t$ (resp. $\hat{\bar{Q}}_t^m$) and entrance law v_t (resp. initial distribution κ). Then $\hat{\tilde{P}}=\hat{\mathscr{P}}$.

In view of (3) and (4), we shall be content to work with $\bar{Y}$ and $\hat{\bar{Y}}$ instead of Y and $\hat{Y}$. The exceptional polar set N in (3) will cause no problems. By (2), we have $\rho(f\cdot\bar{V}^a g)=\rho(f\,\hat{\bar{V}}^a\cdot g)$ for all positive Borel functions f and g on $\bar{E}_\Delta$. To achieve duality, we simply need $\bar{V}^a(x,\cdot)\ll\rho$ and $\hat{\bar{V}}^a(\cdot,x)\ll\rho$. But first, we recall an important property of $\hat{\bar{Y}}_t$, which every Ray process has [5].

(5) Fix an initial measure κ on $\bar{E}_\Delta$, and let (T_n) be an increasing sequence of $(\hat{\bar{\mathbf{I}}}_t^k)$-optional times. Let $T=\lim_{n\to\infty} T_n$, and let $\Lambda=\{T<\infty;\ T_n<T$ for all $n\}$. If f is a bounded Borel function on $\bar{E}_\Delta$, then

$$\hat{\bar{P}}^\kappa(f(\hat{\bar{Y}}_T)\,1_{\{T<\infty\}}\,|\,\bigvee_n \hat{\bar{\mathbf{I}}}^\kappa_{T_n})=f(\hat{\bar{Y}}_T)\,1_{\{T<\infty\}}\,1_{\Lambda^c}+f\hat{\bar{Q}}_0(\hat{\bar{Y}}_{T-})\,1_\Lambda.$$

(6) Let $\hat{J}=\{x\in\bar{E}_\Delta:\ \hat{\bar{V}}^a(\cdot,x)\ll\rho\}$. Then $\rho(\bar{E}_\Delta-\hat{J})=0$ and $\hat{\bar{P}}^x(\hat{\bar{Y}}_t\in\bar{E}_\Delta-\hat{J}$ for some $t\geq 0)=0$ for every x in $\hat{J}$. What is also true (although not mentioned in [9]) is

that $\hat{\hat{P}}^x(\hat{\hat{Y}}_{t-}\in\hat{\hat{E}}_\Delta-\hat{J}$ for some $t>0)=0$ for every x in $\hat{J}$. To see this, let $\Lambda=\{t>0: \hat{\hat{Y}}_{t-}\in\hat{\hat{E}}_\Delta-\hat{J}\}$: Λ is a predictable set. Fix x in $\hat{J}$. If Λ is not $\hat{\hat{P}}^x$-evanescent, let T be a predictable time with $[T]\subset\Lambda$ and $\hat{\hat{P}}^x(T<\infty)>0$. Then $\hat{\hat{Y}}_{T-}\in\bar{E}_\Delta-\hat{J}$ and $\hat{\hat{Y}}_T\in\hat{J}$ on $\{T<\infty\}$. By (5),

$$\hat{\hat{P}}^x(T<\infty)=\hat{\hat{P}}^x(\hat{\hat{Y}}_T\in\hat{J};\, T<\infty)=\hat{\hat{P}}^x(1_{\hat{J}}\hat{\hat{Q}}_0(\hat{\hat{Y}}_{T-});\, T<\infty).$$

Therefore, $\hat{\hat{Y}}_{T-}\in\{y: 1_{\hat{J}}\hat{\hat{Q}}_0(y)=1\}$ on $\{T<\infty\}$ almost surely $(\hat{\hat{P}}^x)$. We claim that $\hat{J}=\{y: 1_{\hat{J}}\hat{\hat{Q}}_0(y)=1\}$. For if $y\in\hat{J}$, $\hat{\hat{P}}^y(\hat{\hat{Y}}_t\in\bar{E}_\Delta-\hat{J}$ for some $t\geqq 0)=0$ as mentioned above, so $1_{\hat{J}}\hat{\hat{Q}}_0(y)=1$. If $1_{\hat{J}}\hat{\hat{Q}}_0(y)=1$, then

$$\hat{\hat{V}}^a(\cdot,y)=\hat{\hat{V}}^a\hat{\hat{Q}}_0(\cdot,y)=\int\hat{\hat{V}}^a(\cdot,x)\,1_{\hat{J}(x)}\hat{\hat{Q}}_0(dx,y)\ll\rho.$$

Thus we conclude that in fact $\hat{\hat{Y}}_{T-}\in\hat{J}$ on $\{T<\infty\}$ almost surely $(\hat{\hat{P}}^x)$.

For each $a\geqq 0$, we define

$$\begin{aligned} \bar{W}^a f(x)&=\bar{V}^a f(x) && \text{if } x\in E\\ &=0 && \text{if } x\in\bar{E}_\Delta-E.\\ f\hat{\hat{W}}^a(x)&=f\hat{\hat{V}}^a(x) && \text{if } x\in\hat{J}\\ &=0 && \text{if } x\in\bar{E}_\Delta-\hat{J}. \end{aligned}$$

(By the observations made in (6), $(\bar{W}^a)_{a\geqq 0}$ and $(\hat{\hat{W}}^a)_{a\geqq 0}$ are resolvents on $\bar{E}_\Delta$.) This corresponds to replacing the measure $\bar{P}^x$ with ε_Δ if x is in $\bar{E}_\Delta-E$ and replacing $\hat{\hat{P}}^x$ with ε_Δ if x is in $\bar{E}_\Delta-\hat{J}$. With this modification made, note that (5) remains true if we restrict our attention to measures κ which are concentrated on $\hat{J}$.

We now have $(\bar{W}^a)$ and $(\hat{\hat{W}}^a)$ the resolvents of two strong Markov processes on $\bar{E}_\Delta$ (which may have branch points - even degenerate branch points!) and which are in duality with respect to ρ: $\rho(f\cdot\bar{W}^a g)=\rho(f\hat{\hat{W}}^a\cdot g)$, $\bar{W}^a(x,\cdot)\ll\rho$, and $\hat{\hat{W}}^a(\cdot,x)\ll\rho$. By the procedure given in Chap. VI of [1], we may construct for each $a\geqq 0$ an a-potential density $\bar{w}^a(x,y)$ so that:

(i) $x\to\bar{w}^a(x,y)$ is a-excessive for $(\bar{W}^b)_{b\geqq 0}$.

(ii) $y\to\bar{w}^a(x,y)$ is a-excessive for $(\hat{\hat{W}}^b)_{b\geqq 0}$.

(iii) $\bar{W}^a f(x)=\int_{\bar{E}}\bar{w}^a(x,y)f(y)\,\rho(dy), f\in\bar{\mathbf{E}}_\Delta^+$.

(iv) $f\hat{\hat{W}}^a(y)=\int_{\bar{E}}\bar{w}^a(x,y)f(x)\,\rho(dx), f\in\bar{\mathbf{E}}_\Delta^+$.

Since ρ is concentrated on E, these integrals may be interpreted as being on E. If we let $w^a(x,y)$ be the restriction of $\bar{w}^a(x,y)$ to $E\times E$, then $w^a(x,y)$ is the potential density for Y on E with respect to the resolvents $V^b(x,\cdot)$ and $\hat{V}^b(\cdot,y)\,1_{\hat{J}\cap E}(y)$.

(7) One of the main objectives of [9] was to obtain the representation theory of Revuz [15] as extended by Sharpe [16] by considering the process Y on the enlarged space $\bar{E}_\Delta$. Let A_t be an additive functional of the process $\bar{Y}$ so that $\bar{P}^x(A_\infty)<\infty$ almost surely (ρ). For f a bounded continuous function on $\bar{E}_\Delta\times\bar{E}_\Delta$, we define $\nu_A(f)=\lim_{a\to\infty} a\bar{P}^\rho\int e^{-at}f(\bar{Y}_{t-},\bar{Y}_t)\,dA_t$. The limit exists, and ν_A is a

measure. Moreover,

$$\bar{P}^x \int e^{-at} f(\bar{Y}_{t-}, \bar{Y}_t)\, dA_t = \iint \bar{w}^a(x, y) f(y, z)\, \nu_A(dy, dz).$$

Notice that ν_A will charge $B \times E$, in general, where $B = \{x \in \bar{E}_\Delta \colon \bar{Q}_0(x, \cdot) \neq \varepsilon_x(\cdot)\}$, the set of branch points of $\bar{Y}$.

3. Undoing the Time Change

Set $\bar{A}_t = \int_0^t (h^{-1} + 1_{\bar{E}_\Delta - E_\Delta})(\bar{Y}_s)\, ds$. Since $\bar{Y}$ has the same law as Y if we start the two processes from any point in $E - N$, we have that $\bar{A}_t$ is continuous and strictly increasing on $[0, \inf\{t \colon \bar{Y}_t = \Delta\})$, almost surely $(\bar{P}^\kappa)$ for any measure κ on $\bar{E}_\Delta - N$. Let $S_t = \inf\{s \colon \bar{A}_s > t\}$: S_t is strictly increasing and continuous on $[0, \inf\{t \colon \bar{Y}_t = \Delta\})$ almost surely $(\bar{P}^\kappa)$ for κ on $\bar{E}_\Delta - N$. As a process on $\bar{E}_\Delta - N$, we may time-change $\bar{Y}_t$ by setting $Z_t = \bar{Y}_{S(t)}$, $\mathbf{H}_t = \bar{\mathbf{I}}_{S(t)}$, $\theta_t = \bar{\Theta}_{S(t)}$, $\mathbf{P}^x = \bar{P}^x$. If κ is any probability measure on $E - N$, $(Z_t, \mathbf{P}^\kappa)$ has the same law as our original process (X_t, P^κ). We shall find it convenient to work with $(Z_t, \mathbf{P}^\kappa)$ most of the time. In particular, if we let $\zeta = \inf\{t \colon Z_t = \Delta\}$, then $\zeta = \infty$ almost surely $(\mathbf{P}^x)$ for all x in $E - N$. It is worth emphasizing that we always consider Z_t as a process on the enlarged space $\bar{E}_\Delta$. Of course, Z_t is in E_Δ, but Z_{t-} may be in $\bar{E}_\Delta - E_\Delta$. From (2), it is immediate that the resolvent of Z_t is an extension to $\bar{E}_\Delta$ of the resolvent of X_t, so we use the same notation $(U^a)_{a \geq 0}$ to denote the extension. Notice that $Uf(x) = \bar{W} f h^{-1}(x)$. Let $\lambda(dx) = h^{-1}(x)\, \rho(dx)$.

We would like to time-change $\hat{\bar{Y}}$ by the inverse of $\int_0^t h^{-1}(\hat{\bar{Y}}_s)\, ds$. We verify directly that this process does not jump to infinity before the death time of $\hat{\bar{Y}}$. We may assume that h is of the form $U^1 U^1 k$, where k is a strictly positive bounded Borel function on E (see p. 401 of [6]), and we may assume that $U^1 U^1 k$ and $U^1 k$ were put into the double Ray cone $\mathfrak{R}$, so that h and $U^1 k$ may be extended to continuous functions on $\bar{E}_\Delta$ (which extensions we denote by h and $U^1 k$ again). Let $G = \{x \colon h(x) = 0\}$; this set is closed in $\bar{E}_\Delta$. Since $G \cap E = \emptyset$, we have by time-reversal that $\hat{\bar{P}}^\rho(\hat{\bar{Y}}_{t-} \in G$ for some $t > 0) = 0$. Since $g(x) = \hat{\bar{P}}^x(\hat{\bar{Y}}_{t-} \in G$ for some $t > 0)$ is excessive, $g(x)$ is zero everywhere. Again, by time-reversal, the set $\hat{\Gamma} = \{t \colon 0 < t < \zeta,\ \hat{\bar{Y}}_t \in G\}$ is $\hat{\bar{P}}^\rho$-evanescent if and only if the set $\Gamma = \{t \colon 0 < t < \zeta,\ \bar{Y}_{t-} \in G\}$ is $\bar{P}^\rho$-evanescent. Suppose T is a predictable time with $[T] \subset \Gamma$ and $\bar{P}^\rho(T < \infty) > 0$. By (5),

$$\begin{aligned} 0 &= \bar{P}^\rho(h(\bar{Y}_{T-});\, T < \infty) = \bar{P}^\rho((\bar{Q}_0\, U^1 U^1 k)(\bar{Y}_{T-});\, T < \infty) \\ &= \bar{P}^\rho(U^1 U^1 k(\bar{Y}_T);\, T < \infty) = \bar{P}^\rho\left(\int_T^\infty U^1 k(\bar{Y}_s)\, ds;\, T < \infty\right). \end{aligned}$$

Since $U^1 k > 0$ on E, $T \geq \zeta$ almost surely $(\bar{P}^\rho)$. Therefore, $\hat{\bar{P}}^\rho(\hat{\bar{Y}}_t \in G$ for some $0 < t < \zeta) = 0$, and this implies that $\hat{\bar{P}}^x(\hat{\bar{Y}}_t \in G$ for some $0 < t < \zeta) = 0$ for all x. Let $\Phi = \{\omega \in \bar{\Omega} \colon \hat{\bar{Y}}_{t-} \notin G$ for all $t > 0$ and $\hat{\bar{Y}}_t \notin G$ for $0 \leq t < \zeta\}$. Then $\hat{\bar{P}}^x(\Phi) = 1$ for all x in $(\bar{E}_\Delta - \{x \colon 1_G \hat{\bar{Q}}_0(x) > 0\})$. For $\omega \in \Phi$, consider $h^{-1}(\hat{\bar{Y}}_t(\omega))$ as a function from a compact subinterval $[0,t]$ of $[0,\zeta(\omega))$: this function is *bounded* and right continuous. For the only way it can be unbounded is if either $\hat{\bar{Y}}_s(\omega)\in G$ for some $s\leqq t$ or $\hat{\bar{Y}}_{s-}\in G$ for some $s\leqq t$, and this cannot happen. Therefore, if we set $\hat{\bar{A}}_t=\int_0^t h^{-1}(\hat{\bar{Y}}_s)\,ds$, $\hat{\bar{A}}_t$ is strictly increasing and continuous on $[0, \inf\{t: \hat{\bar{Y}}_t=\Delta\})$ almost surely $(\hat{\bar{P}}^x)$ for all x in $(\bar{E}_\Delta-\{x: 1_G\hat{\bar{Q}}_0(x)>0\})$. (We can henceforth ignore the set $\{x: 1_G\hat{\bar{Q}}_0(x)>0\}$: if you like you can adjust $\hat{\bar{Y}}$ by replacing $\hat{\bar{P}}^x$ with ε_Δ for every x in $\{x: 1_G\hat{\bar{Q}}_0(x)>0\}$.) Set $\hat{S}_t=\inf\{s: \hat{\bar{A}}_s>t\}$ and set $\hat{Z}_t=\hat{\bar{Y}}(\hat{S}_t)$, $\hat{\mathbf{H}}_t=\hat{\bar{\mathbf{I}}}(\hat{S}_t)$, $\hat{\theta}_t=\hat{\bar{\Theta}}(\hat{S}_t)$, $\hat{\mathbf{P}}^x=\hat{\bar{P}}^x$. Let $(\hat{U}^a)_{a\geqq 0}$ denote the resolvent of $\hat{Z}$. Then Z and $\hat{Z}$ are in duality with respect to the measure λ: this follows as in Revuz [15].

Now $\zeta=\infty$ almost surely $(\mathbf{P}^x)$ for all x in $E-N$. Therefore, if we set $\sigma=\inf\{t: \bar{Y}_t=\Delta\}$, $\bar{A}_{\sigma-}=\infty$ almost surely. By time reversal, we have that $\hat{\bar{P}}^\rho(\hat{\bar{A}}_{\sigma-}<\infty)=0$. If $f(x)=x(1+x)^{-1}$ and $f(\infty)=1$, then $g(x)=\hat{\bar{E}}^x(f(\hat{\bar{A}}_\infty))$ is excessive for $\hat{\bar{Y}}_t$, and $\rho\{x: g(x)\neq 1\}=0$. Therefore, $g(x)$ is 1 everywhere on the non-branch points of $\hat{\bar{Y}}_t$, and it follows that $\hat{\zeta}=\inf\{t: \hat{Z}_t=\Delta\}=\infty$ almost surely $(\hat{\mathbf{P}}^x)$ for all x in the non-branch points of $\hat{Z}_t$. Thus, $aU^a 1=1$ and $a1\hat{U}^a=1$ almost surely (λ) for $a>0$. By duality, $\lambda(ag\hat{U}^a)=\lambda(g)$ and $\lambda(aU^a g)=\lambda(g)$, so λ is an *invariant* measure for the resolvents (U^a) and $(\hat{U}^a)$. This is extremely important for this section and the next.

For each $a\geqq 0$, we may construct an a-potential density as in [1] having the following properties:

(i) $x\to u^a(x,y)$ is a-excessive for $(U^b)_{b\geqq 0}$

(ii) $y\to u^a(x,y)$ is a-excessive for $(\hat{U}^b)_{b\geqq 0}$

(iii) $U^a f(x)=\int u^a(x,y)f(y)\,\lambda(dy)$, $f\in\bar{\mathbf{E}}_\Delta^+$

(iv) $f\hat{U}^a(y)=\int f(x)\,u^a(x,y)\,\lambda(dx)$, $f\in\bar{\mathbf{E}}_\Delta^+$.

The integrals in (iii) and (iv) may be taken over $\bar{E}_\Delta$ or E_Δ since $\lambda(\bar{E}_\Delta-E_\Delta)=0$. It follows from (2) that the restriction of $u^a(x,y)$ to $E\times E$ is exactly the regularized potential density of the resolvents (U^a) and $\hat{U}^a)$ restricted to be resolvents on E: this is the "good" version of the potential density of X, and the representation theory holds here. That is, if A_t is any additive functional of Z so that $\mathbf{P}^x(A_\infty)<\infty$ almost surely (λ), then

$$v_A(f)=\lim_{a\to\infty} a\mathbf{P}^\lambda\int e^{-at}f(Z_{t-},Z_t)\,dA_t \tag{8}$$

defines a measure on $\bar{E}_\Delta\times\bar{E}_\Delta$, and

$$\mathbf{P}^x\int e^{-at}f(Z_{t-},Z_t)\,dA_t=\iint u^a(x,y)f(y,z)\,v_A(dy,dz). \tag{9}$$

Call the semigroups of Z_t and $\hat{Z}_t$, P_t and $\hat{P}_t$, respectively. Note that $\hat{P}_0(\cdot,x)=\hat{\bar{Q}}_0(\cdot,x)$ for all x in $\hat{J}$. It is simple to check that the analogue of (5) holds for $\hat{Z}_t$. That is, let (T_n) be an increasing sequence of $(\hat{\mathbf{H}}_t)$-optional times, let $T=\lim_{n\to\infty}T_n$, and let $\Lambda=\{T<\infty,\ T_n<T \text{ for all } n\}$. If κ is a measure on $\hat{J}$ and f is a bounded Borel function on $\bar{E}_\Delta$, then

$$\hat{\mathbf{P}}^\kappa(f(\hat{Z}_T)1_{\{T<\infty\}}|\bigvee_n \hat{\mathbf{H}}^\kappa_{T_n})=f(\hat{Z}_T)1_{\{T<\infty\}}1_{\Lambda^c}+f\hat{P}_0(\hat{Z}_{T-})1_\Lambda. \tag{10}$$

We shall use another time-reversal argument as a key step in Sect. 4, and we now prepare for this. If Z and $\hat{Z}$ were standard processes, then the result we now prove would be a special case of Nagasawa's theorem [18]. However, Z and $\hat{Z}$ are not standard, and it is not difficult to prove what we need directly since λ is invariant. If we set $P_t^m = \text{vague-limit}_{s\uparrow\uparrow t} P_s$ and $\hat{P}_t^m = \text{vague-limit}_{s\uparrow\uparrow t} \hat{P}_s$, then P_t^m and $\hat{P}_t^m$ are left continuous moderate Markov semigroups (they are the "time-changes" of the semigroups $\bar{Q}_t^m$ and $\hat{\bar{Q}}_t^m$). Since $\bar{Y}_{t-}$ and $\hat{\bar{Y}}_{t-}$ are moderate Markov with semigroups $\bar{Q}_t^m$ and $\hat{\bar{Q}}_t^m$ [19], Z_{t-} and $\hat{Z}_{t-}$ are moderate Markov with semigroups P_t^m and $\hat{P}_t^m$. Let R be a random time which is exponentially distributed with parameter $a>0$ and which is independent of Z. Define $\mathscr{Z}_t$ by setting

$$\begin{aligned}\mathscr{Z}_t &= Z_{(R-t)-} \quad &&\text{if } 0 \leqq t < R,\\ &= \Delta &&\text{if } t \geqq R.\end{aligned}$$

We claim that $(\mathscr{Z}_t, \mathbf{P}^\lambda)$ is strong Markov with semigroup $e^{-at}\hat{P}_t$.

To see this, let $f_1, \ldots, f_n$ be positive continuous functions on $\bar{E}_\Delta$, let $t_1 < t_2 < \ldots < t_n$, and compute

$$\begin{aligned}\mathbf{P}^\lambda\left(\prod_{i=1}^n f_i(\mathscr{Z}_{t_i})\right) &= \mathbf{P}^\lambda\left(\prod_{i=1}^n f_i(Z_{(R-t_i)-});\, t_n < R\right)\\ &= \mathbf{P}^\lambda \int a e^{-au} \prod_{i=1}^n f_i(Z_{u-t_i-})\, 1_{\{t_n<u\}}\, du\\ &= \int a e^{-au} \lambda(P_{u-t_n}^m f_n P_{t_n-t_{n-1}}^m f_{n-1} \ldots P_{t_2-t_1}^m f_1)\, 1_{\{t_n<u\}}\, du. \quad (11)\end{aligned}$$

If g is a continuous function on $\bar{E}_\Delta$, $\lambda P_r^m g = \lambda$ ((vague-limit$_{s\uparrow\uparrow r} P_s$) g) $= \lim_{s\uparrow\uparrow r} \lambda P_s g = \lambda(g)$, so $\lambda P_r^m = \lambda$. Also, $\lambda(f \cdot b U^b g) = \lambda(b f \hat{U}^b \cdot g)$ for all continuous functions f and g implies that $\lambda(f P_t^m g) = \lambda(f \hat{P}_t^m g)$ for almost all t. Since both sides are left continuous functions of t, they are equal for all t. Using these two facts, (11) becomes

$$\begin{aligned}&\int a e^{-au} \lambda(P_{u-t_{n-1}}^m (f_n \hat{P}_{t_n-t_{n-1}}^m) f_{n-1} P_{t_{n-1}-t_{n-2}}^m \ldots P_{t_2-t_1}^m f_1)\, 1_{\{t_n<u\}}\, du\\ &\quad = \mathbf{P}^\lambda \int a e^{-au} (f_n \hat{P}_{t_n-t_{n-1}}^m f_{n-1})(Z_{u-t_{n-1}-}) \prod_{i=1}^{n-1} f_i(Z_{u-t_i-})\, 1_{\{t_n<u\}}\, du.\end{aligned}$$

If we let $u = v + t_n - t_{n-1}$ and if we let $c = \exp(-a(t_n - t_{n-1}))$, we obtain

$$\begin{aligned}&\mathbf{P}^\lambda \int a e^{-au} (c f_n \hat{P}_{t_n-t_{n-1}}^m)(Z_{u-t_{n-1}-}) \prod_{i=1}^{n-1} f_i(Z_{u-t_i-})\, 1_{\{t_{n-1}<u\}}\, du\\ &\quad = \mathbf{P}^\lambda (c f_n \hat{P}_{t_n-t_{n-1}}^m (\mathscr{Z}_{t_{n-1}}) \prod_{i=1}^{n-1} f_i(\mathscr{Z}_{t_i})\, 1_{\{t_{n-1}<R\}}). \quad (12)\end{aligned}$$

Now let t_n decrease to $s > t_{n-1}$. Then (11) converges to $\mathbf{P}^\lambda\left(f_n(\mathscr{Z}_s) \prod_{i=1}^{n-1} f_i(\mathscr{Z}_{t_i})\right)$, while (12) converges to $\mathbf{P}^\lambda(c f_n \hat{P}_{s-t_{n-1}}(\mathscr{Z}_{t_{n-1}}) \prod_{i=1}^{n-1} f_i(\mathscr{Z}_{t_i}))$ since $P_v = \text{vague-limit}_{u\downarrow\downarrow v} P_u^m$. This proves that $(\mathscr{Z}_t, \mathbf{P}^\lambda)$ is simple Markov with with semigroup

$e^{-at}\hat{P}_t$. Then $(\mathscr{X}_t, \mathbf{P}^\lambda)$ is strong Markov since $\mathscr{X}_t$ is right continuous and $e^{-at}\hat{P}_t$ is a strong Markov semigroup. Notice that the initial distribution of $\mathscr{X}$ is λ:

$$\mathbf{P}^\lambda(f(\mathscr{X}_0)) = \mathbf{P}^\lambda(f(Z_{R-})) = \mathbf{P}^\lambda \int a e^{-at} f(Z_{t-})\, dt = \lambda(f).$$

4. Equilibrium Formulae

Let K be a Borel set contained in E, and let $M = \sup\{t: X_t \in K\}$. We assume always that K is *transient*: $P^\lambda(M=\infty)=0$. Since $P^x(M=\infty)$ is an invariant function for X, $P^x(M=\infty)=0$ for all x in E if K is transient. Let $L = \sup\{t: Z_t \in K\}$. If K is transient, then $\mathbf{P}^\lambda(L=\infty)=0$, and so $\mathbf{P}^x(L=\infty)=0$ for all x in $\bar{E}_\Delta$.

For the purposes of time-reversal, we shall need the a-subprocesses of Z, so we fix $(R(a))_{a>0}$ a decreasing collection of random variables independent of X and Z, and so that $R(a)$ is exponentially distributed with parameter a. We set $R(0)=\infty$. Let $\mathbf{H}_t^a = \sigma\{H \cap \{t < R(a)\}: H \in \mathbf{H}_t\}$, and define Z^a by setting

$$\begin{aligned} Z_t^a &= Z_t \quad \text{if } t < R(a) \\ &= \Delta \quad \text{if } t \geqq R(a). \end{aligned}$$

Then Z_t^a is Markov with respect to the filtration $(\mathbf{H}_t^a)$.

Let $L(a) = \sup\{t < R(a): Z_t \in K\}$, $M(a) = \sup\{t < R(a): X_t \in K\}$. We let A_t^a denote the $(\mathbf{H}_t^a)$-dual optional projection of $B_t^a = 1_{\{0 < L(a) \leqq t\}} 1_{\{t < R(a)\}}$. Since B_t^a is a raw additive functional of Z_t^a, A_t^a is an (adapted) additive functional of Z_t^a ([10], [17]).

We shall need two auxiliary results, which we discuss now.

(13) **Lemma.** *There is a Borel function j on $\bar{E}_\Delta$ so that $0<j<1$ and $j\hat{U}(x)$ is bounded on $\bar{E}_\Delta$.*

Proof. The process $\hat{Z}$ restricted to its set of non-branch points $\hat{D}$ is a right process, and $\lambda(Uh) = \lambda(1\hat{U} \cdot h) < \infty$ implies that $1\hat{U}(x) < \infty$ except perhaps on some polar set $\hat{\Gamma}$. Therefore, $\hat{Z}$ restricted to $\hat{D} - \hat{\Gamma}$ has $1\hat{U} < \infty$. By Proposition (2.2) of [6], there is a positive bounded Borel function j on $\hat{D} - \hat{\Gamma}$ so that $j\hat{U}$ is bounded. Therefore, $j\hat{U}$ is bounded on $\hat{D}$. Extend j to be 1 on $\bar{E}_\Delta - (\hat{D} - \hat{\Gamma})$ $(j(\Delta) = 0)$. Then $j\hat{U}(x)$ is bounded for all x in $\bar{E}_\Delta$. Q.E.D.

(14) **Proposition.** *Let F be a bounded positive $\mathbf{H}_\infty$-measurable random variable. There is a positive Borel-measurable function Φ^F on $\bar{E}_\Delta$ so that for every positive $(\mathbf{H}_t^a)$-optional process W_t,*

$$\mathbf{P}^x(W_{L(a)} F \circ \theta_{L(a)} 1_E(Z_{L(a)}^a); L(a) < \infty) = \mathbf{P}^x(W_{L(a)} \Phi^F(Z_{L(a)}^a); L(a) < \infty)$$

for all x in $\bar{E}_\Delta$.

Remarks on the Proof. This was proved in an only slightly-more restrictive setting by Getoor and Sharpe [8]. Note that $L(a)$ is a coterminal time for the process Z^a. Their proof applies here without change.

(15) **Theorem.** *Let K be closed in E. Then for all positive bounded Borel functions f on E_Δ, $P^x(1_K f(X_M);\ M>0)=\int u(x,y)\ 1_K f(y)\,\kappa(dy)$, where κ is a measure on E.*

Proof. Fix $a>0$, and let $R=R(a)$. Let $g=\bar{g}j$, where $\bar{g}$ is chosen to be any positive bounded Borel function on $\bar{E}_\Delta$ so that $\int g(x)\,\mathbf{P}^x(L>0)\,\lambda(dx)<\infty$. Let f be any positive bounded Borel function on $\bar{E}_\Delta$, and let v_a be the Revuz measure of A^a; thus

$$\iiint g(x)\,u^a(x,y)\,f(z)\,v_a(dy,dz)\,\lambda(dx)$$
$$=\lim_{b\to\infty} b\mathbf{P}^\lambda \int e^{-bt}\,g\,\hat{U}^a(Z_{t-})\,f(Z_t)\,dA_t^a$$
$$=\lim_{b\to\infty} b\mathbf{P}^\lambda(e^{-bL(a)}\,g\,\hat{U}^a(Z_{L(a)-})\,f(Z_{L(a)});\ 0<L(a)<R). \tag{16}$$

Now we obtain the right continuous process $\mathscr{Z}_t$ as in Sect. 3 by reversing at R. Let $T=\inf\{t>0\colon \mathscr{Z}_{t-}\in K\}$. We rewrite (16) as

$$\lim_{b\to\infty} b\mathbf{P}^\lambda(e^{-b(R-T)}\,g\,\hat{U}^a(\mathscr{Z}_T)\,f(\mathscr{Z}_{T-});\ 0<T<\infty). \tag{17}$$

Let $\mathscr{H}_t$ be $\sigma\{\mathscr{Z}_s\colon s\leq t\}$ augmented with the $\mathbf{P}^\lambda$-null sets in $\mathbf{H}^\lambda$. Let $\hat{\mathbf{P}}_a^x$ be the measure on $(\bar{\Omega},\bar{\mathbf{I}}^0)$ constructed from the semigroup $e^{-at}\hat{P}_t$. Let

$$\zeta(\bar{\omega})=\inf\{t\colon \bar{\omega}(t)=\Delta\}.$$

We leave it to the reader to check (using the fact that $(\mathscr{Z}_t,\mathbf{P}^\lambda)$ is strong Markov with semigroup $e^{-at}\hat{P}_t$) that $\mathbf{P}^\lambda(e^{-b(R-T)}|\mathscr{H}_T)=\hat{\mathbf{P}}_a^{\mathscr{Z}(T)}(e^{-b\zeta})$ almost surely $(\mathbf{P}^\lambda)$. But $\lambda\{x\colon 1\hat{P}_t(x)\neq 1\}=0$ for all t, so $\hat{\mathbf{P}}_a^x(e^{-b\zeta})=a/(b+a)$ almost surely (λ). Since $\hat{\mathbf{P}}_a^x(e^{-b\zeta})$ is b-excessive, $t\to\hat{\mathbf{P}}_a^{\mathscr{Z}(t)}(e^{-b\zeta})$ is right continuous almost surely $(\mathbf{P}^\lambda)$, so $\hat{\mathbf{P}}_a^{\mathscr{Z}(T)}(e^{-b\zeta})=a/(b+a)$ almost surely $(\mathbf{P}^\lambda)$. Thus (17) becomes

$$a\,\mathbf{P}^\lambda(g\,\hat{U}^a(\mathscr{Z}_T)\,f(\mathscr{Z}_{T-});\ 0<T<\infty). \tag{18}$$

Let $T_n=\inf\{t>0\colon d(\mathscr{Z}_{t-},K)<1/n\}$, where d is the original metric on E (recall that $\mathscr{Z}_{t-}\in E$). Since K is closed, T_n increases to T. On $\{\mathscr{Z}_{T-}\in K\}\cap\{0<T<\infty\}$, $T_n<T$, and on $\{\mathscr{Z}_{T-}\notin K\}\cap\{0<T<\infty\}$, $T_n=T$ for n sufficiently large. Applying (10), (18) becomes

$$a\mathbf{P}^\lambda(g\,\hat{U}^a(\mathscr{Z}_T)\,f(\mathscr{Z}_{T-});\ \mathscr{Z}_{T-}\notin K,\ 0<T<\infty)$$
$$+a\mathbf{P}^\lambda(g\,\hat{U}^a\,\hat{P}_0(\mathscr{Z}_{T-})\,f(\mathscr{Z}_{T-});\ \mathscr{Z}_{T-}\in K, 0<T<\infty). \tag{19}$$

Replacing f with $f1_K$ and observing that $g\hat{U}^a\hat{P}_0=g\hat{U}^a$, we obtain from the representation theorem (9), (16) and (19) that

$$\int g(x)\,\mathbf{P}^x(1_K f(Z_{L(a)});\ 0<L(a)<R(a))\,\lambda(dx)$$
$$=a\mathbf{P}^\lambda(g\,\hat{U}^a(Z_{L(a)})1_K f(Z_{L(a)});\ 0<L(a)<R(a)). \tag{20}$$

In Proposition (14), let $F=1_{\{L=0\}}$, and replace f in (20) with $\Phi^F f$. If we apply the result of (14), then (20) becomes

$$\int g(x)\,\mathbf{P}^x(1_K f(Z_L);\,0<L<R(a),\,L=L(a))\,\lambda(dx)$$
$$=a\mathbf{P}^\lambda(g\,\hat{U}^a(Z_L)\,1_K f(Z_L);\,0<L<R(a),\,L=L(a)). \tag{21}$$

Now the right hand side of (21) is

$$\begin{aligned}a\mathbf{P}^\lambda(g\hat{U}^a(Z_L)1_K f(Z_L);\,0<L<R(a)) &= a\mathbf{P}^\lambda(g\hat{U}^a(Z_L)1_K f(Z_L)e^{-aL};\,L>0)\\ &= a\mathbf{P}^\lambda \int e^{-at}\,g\,\hat{U}^a(Z_t)\,1_K f(Z_t)\,dA_t^0\\ &= a\int \lambda(dx)\iint u^a(x,y)\,g\,\hat{U}^a(z)\,1_K f(z)\,\nu_0(dy,dz)\\ &= \iint g\,\hat{U}^a(z)\,1_K f(z)\,\nu_0(dy,dz).\end{aligned}$$

Define κ by setting $\int \phi(z)\,\kappa(dz)=\int \phi(z)\,\nu_0(dy,dz)$: κ is a measure concentrated on E, and we now have

$$\int g(x)\,\mathbf{P}^x(1_K f(Z_L);\,0<L<R(a))\,\lambda(dx)=\int g(x)\,u^a(x,z)\,1_K f(z)\,\kappa(dz)\,\lambda(dx).$$

The monotone convergence theorem applies, and we let a decrease to 0 to obtain

$$\int g(x)\,\mathbf{P}^x(1_K f(Z_L);\;0<L)\,\lambda(dx)=\int g(x)\,u(x,z)\,1_K f(z)\,\kappa(dz)\,\lambda(dx).$$

for all functions g as described at the beginning of the proof. Therefore,

$$\mathbf{P}^x(1_K f(Z_L);\,0<L)=\int u(x,z)\,1_K f(z)\,\kappa(dz)$$

almost everywhere (λ) and hence everywhere since both functions are excessive. Since $\mathbf{P}^x(1_K f(Z_L);0<L)=P^x(1_K f(X_M);0<M)$ for all x in $E-N$, we have $P^x(1_K f(X_M);0<M)=\int u(x,z)\,1_K f(z)\,\kappa(dz)$ for all x in E. Q.E.D.

(22) **Theorem.** *Let K be a totally thin set for X_t. Then for all positive bounded Borel functions f on E_Δ, $P^x(f(X_M);\;M>0)=\int u(x,y)f(y)\,\kappa(dy)$, where κ is a measure on E.*

Proof. Following the proof of the preceding theorem through (18), we then let $\Gamma=\{t>0\colon \mathscr{Z}_{t-}\in K\}$; Γ is an $(\mathscr{H}_t)$-predictable set. Since K is totally thin, Γ is discrete, so the début of Γ, T, is in Γ on $\{T<\infty\}$, and is therefore a predictable time. Therefore, (19) becomes $\mathbf{P}^\lambda(g\,\hat{U}^a\hat{P}_0(\mathscr{Z}_{T-})f(\mathscr{Z}_{T-});\;T>0)$, and the rest of the proof follows as before. Q.E.D.

(23) **Theorem.** *Let X be a Hunt process, and let K be closed. There is a measure ν_C on $E_\Delta\times E_\Delta$ so that for all positive bounded Borel functions f on E,*

$$P^x(f(X_M);\,M>0)=\int u(x,y)\,(1_K f(y)\,\kappa(dy)+f(z)\,\nu_C(dy,dz)).$$

Moreover, $\nu_C((E_\Delta-K)\times E_\Delta)=0$, and κ is a measure concentrated on E.

Proof. From (15), we have that $P^x(1_K f(X_M);\,M>0)=\int u(x,y)\,1_K f(y)\,\kappa(dy)$. Thus we need only consider $P^x(f(X_M);\,X_M\notin K,\,M>0)$. Since K is closed, $X_M\notin K$ only if $X_{M-}\neq X_M$ on $\{M>0\}$. Therefore, the dual predictable projection of $1_{\{X(M)\notin K\}}\,1_{\{0<M\leq t\}}$ is a continuous additive functional of X. So if we let C_t be the dual predictable projection of $1_{\{Z(L)\notin K\}}\,1_{\{0<L\leq t\}}$, then C_t is a continuous additive functional of Z_t. If $e(y)$ is any positive bounded Borel function on $\bar{E}_\Delta$,

then

$$\mathbf{P}^x(e(Z_{L-})f(Z_L);\, Z_L \notin K, L>0) = \int u(x,y)\, e(y) f(z)\, \nu_C(dy, dz), \tag{24}$$

where ν_C is the Revuz measure of C_t. Let ϕ be a continuous function on E (in the original topology of E), and extend ϕ to be 1 on $\bar{E}_\Delta - E$. Then $\{t: \phi(Z_{t-}) \neq \phi(Z_t)\}$ is countable, so

$$\nu_C(\phi \times 1) = \lim_{b \to \infty} b\mathbf{P}^\lambda \int e^{-bt} \phi(Z_t)\, dC_t$$

since C_t is continuous. Also, $\{t: \phi(Z_t)_-$ does not exist or is not equal to $\phi(Z_t)\}$ is countable. Therefore,

$$\nu_C(\phi \times 1) = \lim_{b \to \infty} b\mathbf{P}^\lambda \int e^{-bt} \phi(Z_t)_-\, dC_t = \lim_{b \to \infty} b\mathbf{P}^\lambda (e^{-bL} \phi(Z_L)_-;\, L>0).$$

Now if the support of ϕ is contained in K^c, we see that $\nu_C(\phi \times 1) = 0$. Therefore, $\nu_C((\bar{E}_\Delta - K) \times \bar{E}_\Delta) = 0$. Combining (15) and (24), we have

$$P^x(f(X_M);\, M>0) = \int u(x,y)\,(1_K f(y)\, \kappa(dy) + f(z)\, \nu_C(dy, dz)). \quad \text{Q.E.D.}$$

It is worth observing the following corollary, which extends the results of Chung, Getoor-Sharpe and Meyer.

(25) **Corollary.** *Let X be a diffusion with infinite lifetime. If K is closed in the original topology of E, then $P^x(f(X_{M-});\, M>0) = \int u(x,y) f(y)\, \kappa(dy)$.*

Proof. Since $X_M = X_{M-}$, $\nu_C = 0$. Q.E.D.

Thus in the case of a diffusion, κ is the *equilibrium measure* of the set K. (The reader is referred to Theorem (2.1) of [11]. Corollary (25) allows us to drop the duality hypothesis there if X and Y are diffusions.)

(26) *Definition.* Let X_t be a right process. We say that X is *locally Hunt* if we can find a sequence (E_n) of open sets with compact closures increasing to E so that the following is true: whenever (T_k) is an increasing sequence of $(\mathbf{F}_t)$-optional times with limit T so that for some m, $X(T_k) \in E_m \cup \{\Delta\}$ for all k, then $X(T_k)$ converges to $X(T)$ on $\{T < \infty\}$.

We leave it to the reader to check that Theorem (23) remains true if X is assumed to be a locally Hunt process and K is assumed to be compact.

Now, is there an analogue of Theorem (23) if X_t is simply a transient right process? The usual Ray-Knight procedure [5] produces a compact metric space F with metric ∂ so that $E \subset F$. Thus we may consider E as a metric space with new metric ∂, and X is a right process on (E, ∂). We say that a set $K \subset E$ is Ray-closed if K is closed in F.

(27) **Theorem.** *Let X be a right process, and let $K \subset E$ be Ray-closed. There is a measure ν_C on $E_\Delta \times E_\Delta$ so that for all positive bounded functions f on E_Δ,*

$$P^x(f(X_M);\, M>0) = \int u(x,y)\,(1_K f(y)\, \kappa(dy) + f(z)\, \nu_C(dy, dz)).$$

Moreover, $\nu_C((E_\Delta - K) \times E_\Delta) = 0$, and κ is a measure concentrated on E.

Proof. Theorem (15) clearly applies here. We may recopy the proof of Theorem (23) word for word once we verify that the dual predictable projection of $1_{\{X(M)\notin K\}} 1_{\{0<M\leqq t\}}$ is a continuous additive functional. Since K is Ray-closed, this amounts to checking that $P^x(0<M=T, X_{T-}\neq X_T)=0$ for all predictable times T. But $X_{T-}\neq X_T$ (the left limit taken in F, of course) at a predictable time T if and only if X_{T-} is in the branch points of X_t. Since $K\subset E$ is Ray-closed, this cannot happen. Q.E.D.

References

1. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
2. Chung, K.L.: Probabilistic approach in potential theory to the equilibrium problem. Ann. Inst. Fourier **23**, 313-322 (1973)
3. Chung, K.L., Walsh, J.B.: To reverse a Markov process. Acta Math. **123**, 225-251 (1970)
4. Garcia-Alvarez, M.A., Meyer, P.A.: Une théorie de la dualité a ensembles polaires près I. Ann. Probab. **1**, 207-222 (1972)
5. Getoor, R.K.: Markov Processes: Ray Processes and Right Processes. Lecture Notes in Mathematics **440**. Berlin-Heidelberg-New York: Springer 1975
6. Getoor, R.K.: Transience and recurrence of Markov processes. Lecture Notes in Mathematics **784**, 397-409. Berlin-Heidelberg-New York: Springer 1980
7. Getoor, R.K., Sharpe, M.J.: Last exit times and additive functionals. Ann. Probab. **1**, 550-569 (1973)
8. Getoor, R.K., Sharpe, M.J.: The Markov property at cooptional times. Z. Wahrscheinlichkeitstheorie verw. Gebiete **48**, 201-211 (1979)
9. Glover, J.: Compactifications for dual processes. Ann. Probab. **8**, 1119-1134 (1980)
10. Glover, J.: Intrinsically homogeneous sets, splitting times, and the big shift. Z. Wahrscheinlichkeitstheorie verw. Gebiete **56**, 133-144 (1981)
11. Glover, J.: Markov processes with identical last exit distributions. Zeitschrift für Wahrscheinlichkeitstheorie verw. Gebiete **59**, 67-75 (1982)
12. Meyer, P.A.: Note sur l'interprétation des mesures d'équilibre. Lecture Notes in Mathematics **321**, 210-216. Berlin-Heidelberg-New York: Springer 1973
13. Meyer, P.A.: Le retournement du temps d'après Chung et Walsh. Lecture Notes in Mathematics **191**, 213-236. Berlin-Heidelberg-New York: Springer 1971
14. Rao, K.M.: A note on Revuz measure. Lecture Notes in Mathematics **784**, 418-436. Berlin-Heidelberg-New York: Springer 1980
15. Revuz, D.: Mesures associés aux fonctionnelles additives de Markov I. Trans. Amer. Math. Soc. **148**, 501-531 (1970)
16. Sharpe, M.J.: Discontinuous additive functionals of dual processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **21**, 81-95 (1972)
17. Sharpe, M.J.: General Theory of Markov Processes. (forthcoming book)
18. Sharpe. M.J.: Some transformations of diffusions by time reversal. Ann. Probab. **8**, 1157-1162 (1980)
19. Walsh, J.B.: Two footnotes to a theorem of Ray. Lecture Notes in Mathematics **191** 283-289. Berlin-Heidelberg-New York: Springer 1971

Received June 2, 1981; in revised form February 15, 1982