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On Large Deviations

V. A. Statulevičius

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A well-known lemma by K. Esseen ([1], § 39, Bd. 1) and its sharpenings obtained by A. C. Berry [2] and V. M. Zolotarev (cf. (25)) allow to estimate the distance $\sup |F(x) - G(x)|$ of two distribution functions F and G in terms of the closeness of their characteristic functions. Similarly, using the Laplace transform

$$\int_{-\infty}^{\infty} e^{zx} d(F(x) - G(x)), \qquad |z| \leqq \Delta$$

one might try to estimate the ratios $\frac{1 F(-x)}{1 - G(x)}$ and $\frac{F(-x)}{G(-x)}$ in the interval $0 \leqq x \leqq \delta \Delta$ and in this way simplify the derivation of limit theorems including the calculation of large deviations. We consider here the case where G is the normal law $\Phi = N(0, 1)$.

Lemma. *Let ξ be a random variable with distribution function F, mean $m = M\xi$, dispersion $\sigma^2 = D\xi$ and finite moments of any order $M|\xi|^k < \infty, k = 1, 2, \ldots$. Let $\mathfrak{S}_k$ be the k-th cumulant of ξ and*

$$\Delta = \sigma \inf \left(\frac{k! H \sigma^2}{|\mathfrak{S}_k|} \right)^{\frac{1}{k-2}} \tag{1}$$

where $H > 0$. Then in the interval

$$1 \leqq x \leqq \bar{\delta} \Delta, \bar{\delta} < \bar{\delta}_H$$

the following relations hold:

$$\frac{1 - F(m + x\sigma)}{1 - \Phi(x)} = e^{\frac{x^3}{\Delta} \lambda \left(\frac{x}{\Delta}\right)} \left(1 + f_1(\bar{\delta}, H) \frac{x}{\Delta}\right), \tag{2}$$

$$\frac{F(m - x\sigma)}{\Phi(-x)} = e^{-\frac{x^3}{\Delta} \lambda \left(-\frac{x}{\Delta}\right)} \left(1 + f_2(\bar{\delta}, H) \frac{x}{\Delta}\right). \tag{2'}$$

Here,

$$|f_i(\bar{\delta}, H)| < \frac{8H\{1 + 7{,}2(1 + 2\delta + \min\{\frac{1}{3}(1 - \bar{\delta})^3 H^{-1}, \frac{1}{2}H - \frac{1}{4}\})^4\}}{(1 - \delta)^4 (1 - \varrho)^{\frac{3}{2}}}. \tag{3}$$

$i = 1, 2,$ *the number δ and δ_H with $0 < \delta < \delta_H$ are determined by the equations*

$$\bar{\delta} = \frac{\delta(1 + \delta)}{2}, \quad \varrho = \frac{6H\delta}{(1 - \delta)^3}, \quad \bar{\delta}_H = \frac{\delta_H(1 + \delta_H)}{2}, \tag{4}$$

δ_H is a real root of the equation $\varrho = 1$ and $\lambda(t) = \sum_{k=0}^{\infty} \lambda_k t^k$ is the power series of Cramèr which converges for $|t| < \bar{\delta}_H$, where

$$|\lambda_k| \leqq \frac{\delta_H}{(k + 3)\bar{\delta}_H^{k+2}}, k = 0, 1, \ldots. \tag{5}$$

We remark that

$$\bar{\delta}_H > \frac{1}{1 + 14{,}55 \max\{H, H^{\frac{1}{3}}\}} \tag{6}$$

A more precise estimate for $h(\bar{\delta}, H)$ is given in (38) and (39). The coefficients λ_k are determined by the first $k+3$ cumulants (cf. (42), (43)). We always set $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$. Moreover, θ always denotes some quantity not exceeding 1 in absolute value.

Instead of the formular (1) in terms of semiinvariants, which is equivalent to

$$|\mathfrak{S}_k| \leqq \frac{k!\, H \sigma^k}{\Delta^{k-2}}, \quad k = 3, 4, \ldots \tag{1'}$$

it is sometimes more convenient to use immediately the Laplace transform

$$\varphi(z) = \int_{-\infty}^{+\infty} e^{zx} dF(x+m)$$

in order to define H and Δ.

Remark. If there exist $H < \infty$ and $\Delta < \infty$ satisfying

$$|\ln \varphi(z)|_{|z| = \frac{\Delta}{\sigma}} \leqq H \Delta^2, \tag{1''}$$

then the assumptions of the lemma are satisfied and (2) and (2′) hold with these H and Δ.

Here we take the principle value of logarithm. Before entering the proof of the lemma we obtain with its help some theorems on probabilities of large deviations.

Let $\{\xi(t), 0 \leqq t < \infty\}$ be a measurable stochastic process. We say that $\xi(t)$ belongs to the class $T^{(k)}$ if

$$M|\xi(t)|^k \leqq C_k < \infty.$$

Consider the random variable

$$\zeta_T = \int_0^T \xi(t)\, dt$$

and let

$$m_T = M\zeta_T, \quad \sigma_T^2 = \mathfrak{D}\zeta_T$$

and $s_\xi^{(k)}(t_1, \ldots, t_k)$ be the correlation function of k-th order of the process $\xi(t)$, which is simply the semiinvariant of the random vector $(\xi(t_1), \ldots, \xi(t_k))$.

Theorem 1. *If* $\xi(t) \in T^{(\infty)}$ *and*

$$\sigma_T^{-k} \left| \int_0^T \cdots \int_0^T s_\xi^{(k)}(t_1, \ldots, t_k)\, dt_1 \cdots dt_k \right| \leqq \frac{k!\, H_1}{\Delta_T^{k-2}} \tag{7}$$

for all $k \geqq 3$ *then for* $F(x) = P\{\zeta_T < x\}$ *the relations* (2) *and* (2′) *hold with* $m = m_T$, $\sigma = \sigma_T$, $H = H_1$, $\Delta = \Delta_T$ *and*

$$\mathfrak{S}_k = \int_0^T \cdots \int_0^T s_\xi^{(k)}(t_1, \ldots, t_k)\, dt_1 \ldots dt_k. \tag{8}$$

In fact, $\mathfrak{S}_k$ as defined by the equality (8) is the cumulant of order k of the random variable ζ_T (cf. [*9*], (1. 14)), and by comparing (7) with (1′) we obtain the proof of the theorem.

Let

$$\xi_1, \ldots, \xi_n, \quad n \geqq 1 \tag{9}$$

be independent random variables with $M\xi_j = 0$, $j = 1, 2, \ldots, n$,

$$S_n = \sum_{j=1}^{n} \xi_j, \quad \sigma_j^2 = \mathfrak{D}\xi_j, \quad B_n^2 = \sum_{j=1}^{n} \sigma_j^2$$

and Γ_{kn} the cumulant of S_n of order k.

Theorem 2. *If the condition of* S. N. Bernstein:

$$|M\xi_j^k| \leqq k!\, H_2 K^{k-2} \mathfrak{D}\xi_j, \qquad j = 1, \ldots, n \tag{10}$$

is satisfied for all $k \geqq 3$, where H_2 and K are some positive numbers, then for

$$F(x) = P\{S_n < x\}$$

the relations (2) *and* (2′) *hold with $m = 0$, $\sigma = B_n$, $H = \frac{3}{2}$,*

$$\Delta = \frac{B_n}{\max\{K(1 + 2H_2), \sqrt{2} \max\limits_{1 \leqq j \leqq n} \sigma_j\}}$$

and $\mathfrak{S}_k = \Gamma_{kn}$.

We set

$$z_0 = (\max\{K(1 + 2H_2), \sqrt{2} \max_{1 \leqq j \leqq n} \sigma_j\})^{-1}.$$

Then, taking into account (10), we obtain in the circle $|z| \leqq z_0$

$$\varphi_{\xi_j}(z) = M e^{z\,\xi_j} = 1 + \sum_{k=2}^{\infty} \frac{z^k}{k!} M\xi_j^k = 1 + \theta |z|^2 \sigma_j^2.$$

Moreover, $\ln(1 + w) = w + \theta |w|^2$ for $|w| \leqq \frac{1}{2}$, and since $|z^2 \sigma_j^2| \leqq \frac{1}{2}$, we have

$$\ln \varphi_{S_n}(z) = \sum_{j=1}^{n} \ln \varphi_{\xi_j}(z) = \theta |z|^2 B_n^2 + \\ + \theta |z|^4 B_n^2 \max_{1 \leqq j \leqq n} \sigma_j^2 = \tfrac{3}{2} \theta |z|^2 B_n^2$$

or

$$|\ln \varphi_{S_n}(z)|_{|z| = z_0} \leqq \tfrac{3}{2} z_0^2 B_n^2.$$

The inequality thus obtained shows that (1″) holds for $\sigma = B_n$, $H = \frac{3}{2}$, $\Delta = z_0 B_n$. This proves the theorem.

Limit theorems of Cramèr's type with calculation of large deviations for sums of independent random variables (9) had been proved by various authors (cf. H. Cramèr [*4*], V. V. Petrov [*5*], [*6*], W. Richter [*7*]) under the condition of Cramèr and Petrov: there exist positive numbers A, L and l such that

$$l \leqq |M e^{z\xi_j}| \leqq L \quad \text{for} \quad |z| \leqq A, \quad j = 1, \ldots, n.$$

In addition these authors assumed that $B_n^2 \geqq c^2 n$ with some $c > 0$. We remark that in this case we can choose

$$H = \frac{\mathscr{E}}{A^2 c^2}, \quad \Delta = A\, B_n \geqq A\, c \sqrt{n},$$

where $\mathscr{E} = 2\pi + \max\{|\ln l|, |\ln L|\}$ since

$$|\ln \varphi S_n(z)|_{|z|=\Delta} \leqq \sum_{j=1}^{n} \{|\ln|\varphi_{\xi_j}(z)||_{|z|=\Delta} + 2\pi\} \leqq n\mathscr{E}.$$

Consider random variables $X_1, \ldots, X_n$, forming a Markov chain with n moments of time, transition probabilities $P_j(w, A)$ from the state w at time j into the measurable set of states A at time $j+1$, initial probability distribution $P_0(A)$ and coefficients of ergodicity

$$\alpha_n = 1 - \max_{0 \leqq j < n} \sup_{\omega, \tilde{\omega}, A} (P_j(\omega, A) - P_j(\tilde{\omega}, A)|.$$

Set

$$S_n = \sum_{j=1}^{n} X_j, \quad B_n^2 = \mathfrak{D} S_n.$$

Theorem 3. *If with probability 1*

$$|X_j| \leqq C_n < \infty, \quad j = 1, 2, \ldots, n, \quad \alpha_n > 0,$$

then there exists an absolute constant $H_3 > 0$, $H_4 > 0$ *such that for* $F(x) = P\{S_n < x\}$ *the relations* (2) *and* (2′) *hold with*

$$m = M S_n, \quad \sigma = B_n, \quad H = H_3, \quad \Delta = \frac{\alpha_n B_n}{C_n H_4}.$$

The *proof* of this theorem follows from the fact that the k-th cumulant $\mathfrak{S}_k$ of the sum S_n can be estimated in the following way:

$$|\mathfrak{S}_k| \leqq \frac{k!\, H_3 C_n^{k-2} B_n^2 H_4^{k-2}}{\alpha_n^{k-2}}, \quad k = 3, 4, \ldots. \tag{11}$$

Let $\zeta(s, t)$ be a random function which is an additive function of the interval (s, t), that is,

$$\zeta(s, u) + \zeta(u, t) = \zeta(s, t)$$

with probability 1 for all $0 \leqq s < u < t \leqq T$

$$m(s, t) = M\zeta(s, t), \quad \sigma^2(s, t) = \mathfrak{D}\zeta(s, t),$$

e. g.

$$\zeta(s, t) = \int_s^t \xi(u)\, du \quad \text{or} \quad \zeta(s, t) = \sum_{s \leqq k \leqq t} \xi(k)$$

where $\xi(t)$ is some stochastic process. Let $\mathscr{F}^t$ denote the σ-algebra generated by the events $\{\zeta(u, v) < x\}$, $s \leqq u < v < t$.

Theorem 4. *Assume that* $\zeta(s, t)$ *satisfies the following strong mixing condition of* M. ROSENBLATT

$$\sup_{0 \leqq t \leqq T-\tau} \sup_{\substack{A \in \mathfrak{X}_0^t \\ B \in \mathfrak{X}_{t+\tau}^T}} |P(AB) - P(A)P(B)| \leqq e^{-\alpha_T \tau}$$

where $\alpha_T > 0$. *In addition, assume that there are*

$$1 \leqq T_0 \leqq \left[\frac{1}{\alpha_T}\right] + 1 \quad \textit{and} \quad C_{T_0, T} < \infty$$

such that

$$\frac{|\zeta(s, s+T_0)|}{T_0} \leqq C_{T_0, T}$$

with probability 1 for all $0 \leqq s \leqq T - T_0$. *Then there exists an absolute constant* $H_5 > 0$, $H_6 > 0$ *such that for* $F(x) = P\{\zeta(0, T) < x\}$ *the relations* (2) *and* (2′) *hold with* $m = m(0, T)$, $\sigma = \sigma(0, T)$, $H = H_5$ *and*

$$\Delta = \frac{\alpha_T^3 \sigma(0, T)}{C_{T_0, T} H_6 T_0}.$$

The theorem results from the following inequality for cumulants of the random variables $\zeta(0, T)$:

$$|\mathfrak{S}_k| \leqq \frac{k! H_5 C_{T_0, T}^{k-2} \sigma^2(0, T) H_6^{k-2}}{\alpha_T^{3(k-2)}}, \quad k = 3, 4, \ldots. \tag{12}$$

The proof of the inequalities (11) and (12) is involved and will not be given here.

Proof of the lemma. Without restricting the generality we set $m = 0$, $\sigma = 1$, $\Delta > 0$. On account of (1′) the series

$$K(z) = \sum_{k=2}^{\infty} \frac{\mathfrak{S}_k}{k!} z^k = \frac{z^2}{2}\left(1 + \frac{2\theta |z| H}{\Delta\left(1 - \frac{|z|}{\Delta}\right)}\right) \tag{13}$$

converges for $|z| < \Delta$, and $K(z)$ and $\tilde{\varphi}(z) = \exp\{K(z)\}$ are analytic in the circle $|z| < \Delta$. Since the moments $\mu_s = M\xi^s$ exist and can be expressed by the cumulants in closed form, we have $\mu_s = \left.\frac{d^s \hat{\varphi}(z)}{dz^s}\right|_{z=0}$ and, by (13) and Cauchy's inequality, we find that for

$$|\mu_s| \leqq \frac{s!}{\delta_1^s \Delta^s} \exp\left\{\frac{\delta_1^2 \Delta^2}{2}\left(1 + \frac{2H\delta_1}{(1-\delta_1)}\right)\right\}, \quad s = 1, 2, \ldots.$$

Therefore the Laplace transform $\varphi(z) = M e^{z\xi}$ exists and is analytic in the circle $|z| < \Delta$ and there $\ln \varphi(z) = K(z)$.

Let us start from the transformation of Esscher [*10*] and Cramér [*4*]. For arbitrary $0 \leqq h < \Delta$ we have

$$1 - F(x) = \int_x^{\infty} dF(x) = \varphi(h) \int_x^{\infty} e^{-hy} dF_h(y), \tag{14}$$

where the distribution function F_h of the random variable $\xi(h)$ is determined by the relation

$$dF_h(y) = \frac{e^{hy} dF(y)}{\varphi(h)}. \tag{15}$$

The main mass of the approximating distribution $1 - \Phi(x)$ is concentrated in the neighborhood of the point x. Therefore we should choose h in such a way that $\frac{1}{\varphi(h)} \exp\{hy\}$ takes its maximum at the point $y = x$, that is, h should be defined by the equation

$$x = \frac{d}{dh} \ln \varphi(h) = m(h). \tag{16}$$

From (15) we easily find that $m(h) = M\xi(h)$ and $\sigma^2(h) = \frac{dm(h)}{dh} = \mathfrak{D}\,\xi(h)$. If $\bar{F}_h$ is the distribution function of the normalised random variable

$$\bar{\xi}(h) = \frac{\xi(h) - m(h)}{\sigma(h)},$$

we derive from (14) and (16):

$$1 - F(x) = \varphi(h)\, e^{-hm(h)} \int_0^\infty e^{-h\sigma(h)y}\, d\bar{F}_h(y). \tag{17}$$

In the following let $0 \leqq h = \delta\varDelta$ where $0 < \delta < \delta_H$, and δ_H is determined by the condition

$$\frac{6H\delta_H}{(1-\delta_H)^3} = 1. \tag{18}$$

For the characteristic function $f_h(t) = M\exp\{it\bar{\xi}(h)\}$ of the distribution function $\bar{F}_h$ we have

$$f_h(t) \equiv e^{\frac{-itm(h)}{\sigma(h)}}\, \frac{\varphi\left(ht\frac{it}{\sigma(h)}\right)}{\varphi(h)}.$$

Expanding $\ln\varphi(z)$ in its Taylor series in the neighborhood of the point h given by $\left| h + \frac{|t|}{\sigma(h)} \right| \leqq \delta_2\varDelta$, $\delta < \delta_2 < 1$ that is, by

$$|t| \leqq \sigma(h)(\delta_2 - \delta)\varDelta, \tag{19}$$

we find

$$\begin{aligned} \ln f_h(t) &= -it\frac{m(h)}{\sigma(h)} - \ln\varphi(h) + \ln\varphi\left(h + \frac{it}{\sigma(h)}\right) = \\ &= -\frac{t^2}{2} + \frac{(it)^3}{6\sigma^3(h)} (\ln\varphi(z))'''\,{}_{z = h + \frac{i\theta t}{\sigma(n)}} = \\ &= -\frac{t^2}{2} + \theta\frac{|t|^3 H}{\sigma^3(h)(1-\delta_2)^4\varDelta}, \end{aligned} \tag{20}$$

so that

$$\begin{aligned} (\ln\varphi(z))''' &= \sum_{k=3}^\infty \frac{\mathfrak{S}_k z^{k-3}}{(k-3)!} = \theta\frac{H}{\varDelta}\sum_{k=3}^\infty (k-2)(k-1)\cdot \\ &\cdot k\left(\frac{|z|}{\varDelta}\right)^{k-3} = \frac{6\theta H}{\varDelta\left(1 - \frac{|z|}{\varDelta}\right)^4}, \quad \text{if} \quad |z| < \varDelta. \end{aligned}$$

In the same way

$$\begin{aligned} \sigma^2(z) &= (\ln\varphi(z))'' = 1 + \sum_{k=3}^\infty \frac{\mathfrak{S}_k z^{k-2}}{(k-2)!} = \\ &= 1 + \frac{\theta H|z|}{\varDelta}\sum_{k=3}^\infty (k-1)k\left(\frac{|z|}{\varDelta}\right)^{k-3} = 1 + \frac{6\theta H|z|}{\varDelta\left(1 - \frac{|z|}{\varDelta}\right)^3} \end{aligned}$$

and

$$\sigma^2(h) = 1 + \theta\varrho. \tag{21}$$

We set

$$T_0 = \frac{1}{H}\sigma^3(h)(1-\delta_2)^4 \Delta, \quad T = \varepsilon T_0, \tag{22}$$

where $0 < \varepsilon < \frac{1}{2}$ will be chosen later. Let $\delta_2 (\delta < \delta_2 < 1)$ satisfy the relation

$$\frac{\delta_2 - \delta}{(1-\delta_2)^4} = \frac{\varepsilon \sigma^2(h)}{H}. \tag{23}$$

Then, on account of (19), (22), (23) and the inequality $|e^\alpha - 1| \leqq |\alpha| e^{|\alpha|}$, for $|t| \leqq T$ we obtain

$$|f_h(t) - e^{-t^2/2}| \leqq e^{-t^2/2}(e^{|t^2/2 + \ln f_h(t)|} - 1) \leqq \frac{|t|^3}{T_0} e^{-(1/2-\varepsilon)t^2}. \tag{24}$$

Next we exploit an inequality which follows from a lemma of V. M. Zolotarev ([*13*], lemma 3): let F be some distribution function, G a function of bounded variation with the properties

$$q = \sup |G'(x)| < \infty, \quad G(-\infty) = 1 - G(\infty) = 0$$

and f and g the Fourier-Stieltjes transforms of these functions. Then for all $T > 0$ and all $\lambda > A$ the inequality

$$\sup |F(x) - G(x) \leqq 2q \frac{\lambda(s(\lambda) + Q(T))}{T(4s(\lambda) - \lambda)} \tag{25}$$

holds, where

$$s(\lambda) = \lambda \int_0^\lambda \frac{1-\cos\lambda}{\pi\lambda^2} d\lambda.$$

A is a positive solution of the inequality $4s(\lambda) = \lambda$,

$$Q(T) = \frac{T}{2\pi q}\int_0^T \left(1 - \frac{t}{T}\right) |f(t) - g(t)| \frac{dt}{t}.$$

We apply the inequality (25) for an estimate of $\sup|\bar{F}_h(y) - \Phi(y)|$. In our case $q = \frac{1}{\sqrt{2\pi}}$, $T = \varepsilon T_0$,

$$Q(T) \leqq \frac{\varepsilon T_0}{\sqrt{2\pi}} \int_0^{\varepsilon T_0} |f_h(t) - e^{-t^2/2}| \frac{dt}{t} < \frac{\varepsilon}{\sqrt{2\pi}} \int_0^\infty t^2 e^{-(1/2-\varepsilon)t^2} dt = \frac{\varepsilon}{2(1-2\varepsilon)^{\frac{3}{2}}},$$

hence for arbitrary $0 < \varepsilon < \frac{1}{2}$

$$\sup|\bar{F}_h(y) - \Phi(y)| \leqq \frac{2d(\varepsilon)}{\sqrt{2\pi}\, T_0}, \tag{26}$$

where

$$d(\varepsilon) = \min_{\lambda > A} \frac{\frac{s(\lambda)}{\varepsilon} + \frac{1}{2(1-2\varepsilon)^{\frac{3}{2}}}}{4\frac{s(\lambda)}{\lambda} - 1}. \tag{27}$$

A calculation shows that the minimum is attained for

$$\varepsilon = 0{,}282, \quad \lambda = 3{,}600$$

and equals 10,79.

Now we can approach the evaluation of the righ t-hand side of equation (17). We have

$$\begin{aligned} 1 - F(x) &= \varphi(h)\, e^{-hm(h)}\, I(h), \\ I(h)) &= I_1(h) + I_2(h), \\ I_1(h) &= \int_0^\infty e^{-h\sigma(h)y}\, d(\bar{F}_h(y) - \Phi(y)), \\ I_2(h) &= \int_0^\infty e^{-h\sigma(h)y}\, d\Phi(y). \end{aligned} \tag{28}$$

By definition, $h > 0$ and $\sigma(h) > 0$, hence the estimate (26) allows to state that

$$|I_1(h)| \leqq |F_h(0) - \Phi(0)| + \sup |F_h(y) - \Phi(y)| \leqq \frac{4d}{\sqrt{2\pi}\, T_0}. \tag{29}$$

Moreover

$$\begin{aligned} x = m(h) = h + \sum_{k=3}^{\infty} \frac{\mathfrak{S}_k}{(k-1)!} h^{k-1} &= h + \frac{\theta H h^2}{\Delta} \sum_{k=3}^{\infty} k \left(\frac{h}{\Delta}\right)^{k-3} = \\ &= h + \frac{3\theta H h^2}{\Delta \left(1 - \frac{h}{\Delta}\right)^2} = h\left(1 + \frac{\theta \varrho}{2}(1-\delta)\right), \end{aligned} \tag{30}$$

where, as before,

$$\varrho = \frac{6H\delta}{(1-\delta)^3}.$$

Similarly,

$$\begin{aligned} h\sigma^2(h) - x = \sum_{k=3}^{\infty} \mathfrak{S}_k \left(\frac{1}{(k-2)!} - \frac{1}{(k-1)!}\right) h^{k-1} &= \\ = \frac{\theta H h^2}{\Delta} \sum_{k=3}^{\infty} k(k-2) \left(\frac{h}{\Delta}\right)^{k-2} &= \frac{\theta h \varrho}{2}. \end{aligned} \tag{31}$$

The relations (30) and (31) show that

$$h\sigma(h) = x + \frac{\theta h \varrho}{2}. \tag{32}$$

In fact, in the case $\sigma(h) \leqq 1$ they imply

$$x - \frac{h\varrho}{2} \leqq h\sigma^2(h) \leqq h\sigma(h) \leqq h \leqq x + \frac{h\varrho(1-\delta)}{2},$$

and in the case $\sigma(h) \geqq 1$ we have

$$x - \frac{h\varrho(1-\delta)}{2} \leqq h \leqq h\sigma(h) \leqq h\sigma^2(h) \leqq x + \frac{h\varrho}{2},$$

which gives (32). Therefore

$$I_2(h) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-xy - y^2/2} \left(1 + \theta \sum_{k=1}^{\infty} \frac{h^k \varrho^k y^k}{2^k k!}\right) dy = e^{x^2/2}(1 - \Phi(x)) + R$$

where by the equality

$$\int_0^\infty e^{-xy} y^k\, dy = \frac{k!}{x^{k+1}}$$

and the estimate

$$\frac{h\varrho}{2x} \leqq \frac{\varrho}{2-\varrho(1-\delta)} < \frac{\varrho}{1+\delta} < 1\,,$$

derived from (30), we obtain

$$|R| \leqq \sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\,x}\left(\frac{h\varrho}{2x}\right)^k \leqq \frac{1}{\sqrt{2\pi}\,x}\,\frac{\varrho}{2(1-\varrho)+\varrho\delta}\,.$$

If $x \geqq 1$, then

$$\max\left\{\frac{1}{\sqrt{2\pi}\,x}\left(1-\frac{1}{x^2}\right), \frac{3}{4}\frac{1}{\sqrt{2\pi}\,x}\right\} \leqq (1-\Phi(x))\,e^{x^2/2} \leqq \frac{1}{\sqrt{2\pi}\,x}, \tag{33}$$

thus we finally find

$$I_2(h) = e^{x^2/2}(1-\Phi(x))\left(1+\frac{4}{3}\,\frac{\theta\varrho}{2(1-\varrho)+\varrho\delta}\right). \tag{34}$$

The relations (28), (29), (22), (23), (33) and (34) lead to

$$\frac{1-F(x)}{1-\Phi(x)} = \varphi(h)\,e^{x^2/2\,-hx}\left((1+f(\delta,H)\,\frac{x}{\Delta}\right), \tag{35}$$

where

$$|f(\delta,H)| \leqq \frac{16dH}{3(1-\varrho)^{\frac{3}{2}}\,(1-\delta_2)^4} + \frac{4\Delta}{3x}\,\frac{\varrho}{2(1-\varrho)+\varrho\delta}\,.$$

Remembering that

$$\frac{\Delta}{x} = \frac{h}{\delta x} \leqq \frac{1}{\delta\left(1-\frac{\varrho}{2}(1-\delta)\right)}, \quad \varrho = \frac{6\,H\delta}{(1-\delta)^3},$$

we can also write

$$|f(\delta,H)| \leqq \frac{16dH}{3(1-\varrho)^{\frac{3}{2}}\,(1-\delta_2)^4} + \frac{4\,H}{(1-\delta)^3\left(1-\frac{\varrho}{2}+\frac{\varrho\delta}{2}\right)\left(1-\varrho+\frac{\varrho\delta}{2}\right)}\,. \tag{36}$$

We had set $\delta = \dfrac{h}{\Delta}$. However, the relations (35), (36) remain true if we select h and δ arbitrarily, but so as to satisfy the conditions $0 < h \leqq \delta\Delta$, $\delta < \delta_H$. Since then $\sigma^2(h) = \dfrac{d}{dh}\,m(h) > 0$ by the equation (16) $x = m(h)$ there corresponds to each value x a value h. In order to fulfil the inequality $0 < h \leqq \delta\Delta$ it is necessary to consider values x which satisfy the condition

$$0 < x \leqq \bar{\delta}\Delta\,, \quad \bar{\delta} < \bar{\delta}_H\,, \tag{37}$$

where

$$\bar{\delta} = \frac{\delta}{2}(2-\varrho+\varrho\delta)\,,$$

because $x = h\left(1+\dfrac{\theta\varrho}{2}(1-\delta)\right)$, by (30).

Thus

$$f_1(\bar{\delta},H) = f(\delta,H) = \frac{16\,\theta d(\varepsilon)H}{3(1-\varrho)^{\frac{3}{2}}(1-\delta_2)^4} + \frac{4\,\theta\,H}{(1-\delta)^3\left(1-\frac{\varrho}{2}+\frac{\varrho\delta}{2}\right)\left(1-\varrho+\frac{\varrho\delta}{2}\right)}, \tag{38}$$

where δ and $\bar{\delta}$ are related by the equality $\bar{\delta} = \frac{\delta}{2}(2 - \varrho + \delta\varrho)$, δ_2 with $\delta < \delta_2 < 1$ is the positive solution of the equation

$$\frac{\delta_2 - \delta}{(1-\delta_2)^4} = \frac{\varepsilon\sigma^2(h)}{H}, \quad \sigma^2(h) \geqq 1 - \varrho, \quad 0 < \varepsilon < \tfrac{1}{2}$$

and $d(\varepsilon)$ fulfills the inequality (27).

In passing we make the following remark. If, in order to estimate $Q(T)$, we use instead of (24) the inequality

$$|f_h(t) - e^{-t^2/2}| \leqq e^{-t^2/2}(e^{|t|^3/T_0} - 1) \leqq \frac{|t|^3}{T_0} e^{-t^2/2} + \frac{|t|^6}{T_0^2} e^{-(1/2-\varepsilon)t^2},$$

then we obtain

$$d(\varepsilon) \leqq \frac{\varepsilon^{-1}s(\lambda) + \frac{1}{2} + (2\sqrt{2\pi T_0}(\frac{1}{2} - \varepsilon)^3)^{-1}}{4\lambda^{-1}s(\lambda) - 1},$$

for arbitrary $0 < \varepsilon < \frac{1}{2}$ and $\lambda > A$.

If we chose λ so as to minimize the expression

$$\frac{2s(\lambda) + \frac{1}{2}}{4\lambda^{-1}s(\lambda) - 1},$$

then $\lambda = 3{,}2467$, thus

$$d(\varepsilon) \leqq \frac{2{,}23}{\varepsilon} + 0{,}87 + \frac{0{,}35\,H}{(1-\varrho)^{\frac{3}{2}}(1-\delta_2)^4(\frac{1}{2}-\varepsilon)^3\Delta}.$$

This estimate is valuable for large Δ and relatively small x. Note that

$$|f_1(\bar{\delta}, H)| \leqq 4H\left\{1 + \frac{5{,}33\,x^2}{x^2-1}\left(1 + \min\left\{\frac{1}{2H}, \frac{1}{\sqrt[4]{2H}}\right\}\right)^4 + o(1)\right\},$$

if $\Delta \to \infty$ and $x = o(\Delta)$.

Continuing the proof, it is not difficult to verify that

$$\frac{1}{(1-\delta_2)^4} \leqq \frac{(1 + 6\varepsilon\delta + \min\{\varepsilon^{\frac{1}{4}}(1-\delta)^{\frac{3}{4}}H^{-\frac{1}{4}}, \varepsilon(1-\delta)^3H^{-1}\})^4}{(1-\delta)^4}.$$

Therefrom we obtain a more crude, but in turm clearer estimate:

$$|f_1(\bar{\delta}, H)| \leqq \frac{8H(1 + \frac{2}{3}d(\varepsilon)(\Delta + 6\varepsilon\delta + \min\{\varepsilon^{\frac{1}{4}}(1-\delta)^3H^{-1}, \varepsilon^{\frac{3}{4}}H^{-\frac{1}{4}}\})^4)}{(1-\delta)^4(1-\varrho)^{\frac{3}{2}}}. \tag{39}$$

Obviously, $\bar{\delta} > \frac{\delta(1+\delta)}{2}$, and only for $\delta = \delta_H$, that is for $\varrho = 1$, the equality $\bar{\delta}_H = \frac{\delta_H(1+\delta_H)}{2}$ is attained. In order to simplify the relation between δ and $\bar{\delta}$ in the formulation of the lemma (cf. (4)), we had set $\bar{\delta} = \frac{\delta(1+\delta)}{2}$. Therefore in the evaluation of the right-hand side of the inequality (39) we have to replace $1 - \delta$ by $1 - \bar{\delta}$. Having done so, and putting $\varepsilon = 0{,}282$, $d(\varepsilon) = 10{,}79$, we obtain (3).

It remains to consider $L(x) = \frac{x^2}{2} - hx + \ln\varphi(h)$ in (35). It follows from what we have said earlier that h can be expanded in a power series in x which converges for $|x| < \bar{\delta}_H\Delta$:

$$h = h(x) = \sum_{k=1}^{\infty} a_k x^k. \tag{40}$$

The coefficients a_k are determined by the first $k+1$ cumulants. Moreover, from Cauchy's inequality for the coefficients of a power series we find

$$|a_k| \leqq \frac{\delta_H}{\bar{\delta}_H^k \Delta^{k-1}}, \quad k = 1, 2, \ldots, \tag{41}$$

because $|h(z)|_{|z| = \bar{\delta}_H \Delta} \leqq \delta_H \Delta$ by (30). It is easy to verify that

$$\begin{aligned} a_1 &= 1, \quad a_2 = -\frac{\mathfrak{S}_3}{2}, \quad a_3 = -\frac{\mathfrak{S}_4 - 3\,\mathfrak{S}_3^2}{6}, \\ a_4 &= -\frac{\mathfrak{S}_5 - 10\,\mathfrak{S}_4 \mathfrak{S}_3 + 15\,\mathfrak{S}_3^3}{24}, \ldots. \end{aligned} \tag{42}$$

Using the expansion (40) we obtain

$$L(x) = \frac{x^2}{2} - hx + \sum_{k=2}^{\infty} \frac{\mathfrak{S}_k}{k!} h^k = \frac{x^2}{2} - \sum_{k=2}^{\infty} \frac{k-1}{k!} \mathfrak{S}_k h^k = \sum_{k=3}^{\infty} C_k x^k,$$

where

$$C_k = -\sum_{\nu=2}^{k} \frac{\nu - 1}{\nu!} \mathfrak{S}_\nu \sum_{k_1 + \cdots + k_\nu = k} a_{k_1} \ldots a_{k_\nu} = -\frac{a_{k-1}}{k}.$$

We set

$$L(x) = \frac{x^3}{\Delta} \lambda\left(\frac{x}{\Delta}\right).$$

The inequality (41) shows that the coefficients of Cramèr's series

$$\lambda(t) = \sum_{k=0}^{\infty} \lambda_k t^k$$

are subject to the estimate (5):

$$\lambda_k = -\frac{a_{k+2}}{k+3} \Delta^{k+1} = \theta \frac{\delta_H}{(k+3)\,\bar{\delta}_H^{k+2}}. \tag{43}$$

Hence

$$\varphi(h) e^{x^2/2 - hx} = \exp L(x) = \exp\left\{\frac{x^3}{\Delta} \lambda\left(\frac{x}{\Delta}\right)\right\}$$

which together with (51) concludes the proof of the relation (2) as well as the proof of the lemma, because the proof of (2′) runs analogously. The estimate (6) for $\bar{\delta}_H$ from above is verified by (4) and a simple calculation. The validity of the remark is obvious, because it folloews from the inequality (1″) that the function $\ln \varphi(z)$ is analytic in the circle $|z| < \frac{\Delta}{\sigma}$, and therefore, by Cauchy's inequalities, (1′) is satisfied.

We remark that limit theorems on deviations of type Δ for sums of independent random variables and additive functions had been considered in [8].

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Institute of Physics and Mathematics
of the Academy of Sciences of the
Lithuanian SSR and the University of
Vilnius, USSR