Z. Wahrscheinlichkeitstheorie verw. Geb. 6, 129—132 (1966)

Singularity of Measures on Linear Spaces*

R. M. Dudley

Received February 12, 1966

1. Introduction

This note contains various sufficient conditions for singularity and non-absolute-continuity of pairs of measures on a real linear space. Some of the results carry over to more general situations, e. g. to measures on a group or semigroup.

Section 2 contains some results on singularity of a measure with respect to a translate of itself, and related results on singularity of more general pairs of measures. We improve the main theorem of [*1*], Theorem 2, by weakening the hypothesis and giving a shorter proof. Then we point out some consequences close to others' work ([*3*], [*4*]). Section 3 gives some results which do not involve translations.

2. Singularity of Translates

Let S be a real linear space and $\mathscr{S}$ a σ-algebra of subsets of S. Let $S^*(\mathscr{S})$ be the linear space of all $\mathscr{S}$-measurable real linear functionals on S.

Suppose μ is a finite (nonnegative, countably additive) measure on $\mathscr{S}$. For any x in S we define the translate of μ by x:

$$\mu^x(A) = \mu(A - x)\,, \quad A - x \in \mathscr{S}\,.$$

On $S^*(\mathscr{S})$, we let $\mathscr{T}_\mu$ be the topology of convergence in μ-measure, metrized by the pseudo-metric

$$d(f,g) = \int |f(x) - g(x)|/(1 + |f(x) - g(x)|)\, d\mu(x)\,.$$

For any x in S and linear functional L on S, we let $e_x(L) = L(x)$. For two measures μ and ν, the statement that μ and ν are singular will be written $\mu \perp \nu$.

Theorem 1. *Given x in S, if e_x is not continuous on $S^*(\mathscr{S})$ for $\mathscr{T}_\mu$, then $\mu \perp \mu^x$.*

Proof. The hypotheses imply that there are f_n and f in $S^*(\mathscr{S})$ with $f_n \to f$ for $\mathscr{T}_\mu$ but $f_n(x) \not\to f(x)$. Taking subsequences, we may assume that for some neighborhood U of $f(x)$, $f_n(x) \notin U$ for all n, and that $f_n(y) \to f(y)$ for μ-almost all y in S. Letting

$$A = \{y \colon f_n(y) \to f(y)\}\,,$$

we then have $\mu(S \sim A) = 0$, while

$$\begin{aligned}\mu^x(A) &= \mu\{y \colon f_n(x+y) \to f(x+y)\}\\ &= \mu\{y \colon f_n(x) + f_n(y) \to f(x) + f(y)\}\\ &= \mu\{y \colon f_n(x) \to f(x)\} = 0\,.\end{aligned}$$

Thus $\mu \perp \mu^x$, q.e.d.

* This research was partially supported by a National Science Foundation Grant.

Theorem 1 also holds if S is a set with addition replaced by an arbitrary binary operation, and the real line as the range of the homomorphisms in $S^*(\mathscr{S})$ is replaced by any separable metric topological group G. The proof is essentially the same. This encompasses certain other natural cases, e. g. where S is a locally compact abelian group and G is the circle group. However, at present I know of no interesting applications except where S is an infinite-dimensional linear space.

For $p \geqq 1$ let

$$S_p^*(\mu) = \{L \in S^*(\mathscr{S}) : \int |L(x)|^p d\mu(x) < \infty\}$$

with the pseudo-norm

$$\|L\|_p = (\int |L|^p d\mu)^{1/p}.$$

Then $S_p^*(\mu)$ is a subspace of $S^*(\mathscr{S})$ with a topology finer than $\mathscr{T}_\mu$, so we have

Corollary 1. *If e_x is not continuous on $S_p^*(\mu)$ for $\| \|_p$, then $\mu \perp \mu^x$.*

Theorem 2 of [*1*] is the case $p = 2$ of Corollary 1. In turn, results of several other authors can be obtained, as indicated in [*1*] and later in this paper.

Corollary 2. *Suppose μ and ν are finite measures on $\mathscr{S}$ such that e_x is not continuous on $S^*(\mathscr{S})$ for $\mathscr{T}_{\mu+\nu}$. (For example, suppose $p \geqq 1$ and e_x is not continuous on $S_p^*(\mu + \nu)$.) Then $\mu \perp \nu^x$.*

Proof. $\mu + \nu \perp (\mu + \nu)^x$ by Theorem 1, so $\mu \perp \nu^x$, q.e.d.

Corollaries 1 and 2 for $p = 2$ are closely related to Theorem 4.3 of Rao and Varadarajan [*3*], which is stated in terms of certain matrices. However, translating one result into the other seems to take almost as much or more time than giving direct proofs, so we shall not go into further details.

A measure μ on a σ-algebra $\mathscr{S}$ in a linear space S will be said to have *mean x* in S if

$$\int L(y)\, d\mu(y) = L(x)$$

for all L in $S^*(\mathscr{S})$.

We turn now to a generalization of a recent result of Shepp [*4*]. Let S be the linear space of all sequences $x = \{x_n\}_{n=1}^\infty$ of real numbers (a countably infinite Cartesian product of real lines).

Theorem 2. *Let $\{p_n\}_{n=1}^\infty$ be a sequence of probability measures on the real line such that for some y in S, $M > 0$, and $\alpha > 0$,*

$$p_n((y_n - M, y_n + M)) \geqq \alpha \quad \text{for all } n.$$

Let $a \in S$ satisfy $\sum_n a_n^2 = \infty$. Let $P = \prod_{n=1}^\infty p_n$ on S. Then $P \perp P^a$.

Proof. We may assume $y_n \equiv 0$. If $\lambda_n > 0$, the transformation $\{x_n\} \to \{\lambda_n x_n\}$ of S replaces a_n for our purposes by $\lambda_n a_n$. If

$$\lambda_n = 1/\max(1, |a_n|),$$

all the hypotheses are preserved, so we can assume $|a_n| \leqq 1$ for all n. Let

$$f(x) = \begin{cases} M+1, & x > M+1 \\ x, & |x| \leqq M+1 \\ -M-1, & x < -M-1. \end{cases}$$

Let b_n be the mean of $p_n \circ f^{-1}$ and c_n the mean of

$$q_n = p_n^{a_n} \circ f^{-1}.$$

Then clearly $\alpha |a_n| \leqq |b_n - c_n| < 1$ for all n. Let $F\{x_n\} = \{f(x_n)\}$, and $Q = \prod q_n$. Now the measures $\mu = (P \circ F^{-1})^{-b}$ and $\nu = Q^{-c}$ are both products of probability measures with mean 0 and supports in $(-2M - 2, 2M + 2)$. Thus the coordinate linear functionals x_n on S belong to $S_2^*(\mu + \nu)$, and the x_n are orthogonal, with

$$\|x_n\|_2 \leqq \sqrt{2}(2M + 2) \quad \text{for all } n.$$

Thus since $\sum (b_n - c_n)^2 = \infty$, e_{c-b} is not continuous on $S_2^*(\mu + \nu)$. Hence by Corollary 2, $\mu \perp \nu^{c-b}$, i.e. $P \circ F^{-1} \perp P^a \circ F^{-1}$, so $P \perp P^a$, q.e.d.

Theorem 1 (i) of SHEPP [4] is the special case of our Theorem 2 in which the p_n are all equal to a given p.

3. Results Not Involving Translations

We begin with the following known and easily proved fact:

Theorem 3. *Let S be a set, $\mathscr{S}$ a σ-algebra of subsets of S, and μ and ν finite measures on $\mathscr{S}$. Let $\mathscr{M}(\mathscr{S})$ be the linear space of all $\mathscr{S}$-measurable real-valued functions on S. Then ν is absolutely continuous with respect to μ if and only if the identity on $\mathscr{M}(\mathscr{S})$ is continuous from $\mathscr{T}_\mu$ to $\mathscr{T}_\nu$.*

For translation of a measure, Theorem 1 differs from the "only if" part of Theorem 3 by a strengthening of the hypothesis (to discontinuity on the subspace $S^*(\mathscr{S}) \subset \mathscr{M}(\mathscr{S})$) and the conclusion (from non-absolute-continuity to singularity).

Let S be a real linear space. $\mathscr{S}$ a σ-algebra of subsets of S, and μ and ν finite measures on $\mathscr{S}$. We shall say μ is *subordinate* to ν, written $\mu \,\mathrm{s}\, \nu$, if the identity on $S^*(\mathscr{S})$ is continuous from $\mathscr{T}_\nu$ to $\mathscr{T}_\mu$. This is clearly a reflexive and transitive relation (partial ordering). We have the following immediate consequence of Theorem 3:

Corollary 3. *If μ is absolutely continuous with respect to ν, then $\mu \,\mathrm{s}\, \nu$.*

For any A and B in $\mathscr{S}$ we let $\mu_A(B) = \mu(A \cap B)$. Then clearly μ_A is a finite measure on $\mathscr{S}$ and $\mu_A \,\mathrm{s}\, \mu$. We shall say μ is *uniform* if $\mu \,\mathrm{s}\, \mu_A$ whenever $\mu(A) > 0$. We then have

Theorem 4. *If ν is uniform and is not subordinate to μ, then $\mu \perp \nu$.*

Proof. Whenever $\nu(A) > 0$, we have $\nu \,\mathrm{s}\, \nu_A$, so ν_A is not subordinate to μ. Hence by Corollary 3, ν_A is not absolutely continuous with respect to μ. Thus $\mu \perp \nu$, q. e. d.

We next show that Gaussian measures are uniform. A Borel probability measure P on the real line is called *Gaussian* if there are constants $\sigma > 0$ and m such that for each Borel set B,

$$P(B) = \int_B \exp(-(x - m)^2/2\sigma^2)\, dx/\sigma\sqrt{2\pi},$$

or if P is a point mass at m (this can be regarded as the limiting case $\sigma = 0$). A measure μ on a σ-algebra $\mathscr{S}$ in a linear space S will be called Gaussian if $S^*(\mathscr{S}) = S_2^*(\mu)$, and for each f in $S^*(\mathscr{S})$ the measure $\mu \circ f^{-1}$ on the real line (the "distribution of f") is Gaussian. (We shall not use the well-known fact that this implies Gaussian joint distributions.)

Theorem 5. *Any Gaussian probability measure μ on a linear space S is uniform.*

Proof. Suppose $\mu(A) > 0$ and $f_n \to f$ for $\mathcal{T}_{\mu_A}$. Then $f_n - f \to 0$ on A in μ-measure. $f_n - f$ has a Gaussian distribution for each n. For any $\varepsilon > 0$,

$$\mu\{x : |f_n - f|(x) < \varepsilon\} > \mu(A) - \varepsilon$$

for n large. Thus as $n \to \infty$, the mean and variance of $f_n - f$ approach 0, so $f_n - f \to 0$ for $\mathcal{T}_\mu$, and μ s μ_A, q. e. d.

For a Gaussian measure μ, the topology $\mathcal{T}_\mu$ is identical to the topology of the pseudo-norm $\| \|_p$ for any $p \geqq 1$. Of course $\| \|_2$ is generally the most convenient. It is known that μ and μ^x are mutually absolutely continuous if μ has mean zero and e_x is continuous for $\mathcal{T}_\mu$ [*1*, Theorem 3], i. e. $\mathcal{T}_\mu = \mathcal{T}_{\mu^x}$. However, there are Gaussian measures μ and ν with zero means and $\mathcal{T}_\mu = \mathcal{T}_\nu$, yet $\mu \perp \nu$. Necessary and sufficient conditions for singularity of Gaussian measures have been given by several authors, including FELDMAN [*2*].

Since the results in section 2 seem to have most of their applications using the "Hilbert" norms $\| \|_2$, it may seem natural to investigate analogues of Corollary 3 for such norms. We can say ν is *2-subordinate* to μ if the indentity is continuous from $S_2^*(\mu)$ to $S_2^*(\nu)$. However, we can have ν absolutely continuous with respect to μ but not 2-subordinate to μ, e. g. in the real line with

$$\int x^2 \, d\mu(x) < \int x^2 \, d\nu(x) = +\infty .$$

Various other unpleasant phenomena also result from the fact that a norm $\|L\|_2$ may be significantly increased by L having extremely large values on sets of very small probability. It seems advisable to eliminate such things either by hypothesis or by truncation (as in the proof of Theorem 2). The few results I have along these lines do not seem complete enough to be worth stating here.

References

[*1*] DUDLEY, R. M.: Singular translates of measures on linear spaces. Z. Wahrscheinlichkeitstheorie verw. Geb. **3**, 128–137 (1964).

[*2*] FELDMAN, J.: Equivalence and perpendicularity of Gaussian processes. Pacific J. Math. **8**, 699–708, 1295–1296 (1958).

[*3*] RAO, C. RADHAKRISHNA, and V. S. VARADARAJAN: Discrimination of Gaussian processes. Sankhya ser. A **25**, 303–330 (1963).

[*4*] SHEPP, L. A.: Distinguishing a sequence of random variables from a translate of itself. Ann. math. Statistics **36**, 1107–1112 (1965).

Department of Mathematics
University of California
Berkeley