Z. Wahrscheinlichkeitstheorie verw. Geb. 19, 117–122 (1971)

Coalescence and Spectrum of Automorphisms of a Lebesgue Space

D. Newton

Introduction

Throughout T will denote an automorphism of a Lebesgue space (M, μ). Let $\mathscr{E}(T)=\{$endomorphisms ϕ of (M,μ) such that $\phi T=T\phi\}$ and $\mathscr{A}(T)=\{$automorphisms ϕ of (M,μ) such that $\phi T=T\phi\}$, if $\mathscr{E}(T)=\mathscr{A}(T)$ then we say that T is coalescent. This concept has been used in topological dynamics, Auslander [1], and was given a measure theoretic setting by Hahn and Parry [4], where it was shown that totally ergodic automorphisms with quasi-discrete spectrum are coalescent. It is clear that the property of being coalescent is an isomorphism invariant of T, we will show that it is also an invariant of weak isomorphism but not of spectral isomorphism. The relation between coalescence and entropy is not clear. It is possible to find spectrally equivalent automorphisms with equal (zero) entropy one of which is coalescent and the other is not, however, we have been unable to construct a coalescent automorphism with positive entropy.

In this paper we look at the relation between coalescence and the spectrum of an automorphism. To this aim we introduce the notion of unitary coalescence for unitary operators on a seperable Hilbert space and give necessary and sufficient conditions for a unitary operator to be unitarily coalescent. From this we are able to deduce a sufficient condition for an automorphism T to be coalescent. In paragraph 3 we construct an invariant partition, which may be trivial, associated with the spectrum of T such that the corresponding factor transformation is coalescent and has zero entropy. Finally we consider some examples.

1. Preliminaries

For the theory of Lebesgue spaces and their endomorphisms see Rohlin [13, 14]. If ϕ is an endomorphism of the Lebesgue space (M,μ) then U_ϕ will denote the isometry of $L_2(M)$ defined by $U_\phi f=f\circ\phi$. We note that U_ϕ is unitary if and only if U_ϕ is onto, i.e. if and only if ϕ is an automorphism of (M,μ). The theorems and proofs in the following paragraph depend on the theory of spectral multiplicity and the results that we use may be found in Halmos [6], Plessner and Rohlin [12].

2. Unitary Coalescence

Let U be a unitary operator on a seperable Hilbert space $\mathscr{H}$. Denote by $\mathscr{I}(U)$ the set of isometries of $\mathscr{H}$ which commute with U and by $\mathscr{U}(U)$ the set of unitary operators of $\mathscr{H}$ which commute with U. We say that U is unitarily coalescent if $\mathscr{I}(U)=\mathscr{U}(U)$ or, equivalently, if each isometry in $\mathscr{I}(U)$ is onto.

Lemma 1. *Let A denote the U-reducing subspace of $\mathscr{H}$ consisting of all those elements whose spectral type with respect to U has uniform finite multiplicity. Then $U'(A)=A$ for each $U'\in\mathscr{I}(U)$.*

Proof. Let A_n denote the subspace of those elements of $\mathscr{H}$ whose spectral types have uniform multiplicity n, n denotes a positive integer; note that A_n may be trivial. Then $A=\bigoplus_n A_n$ and each A_n is a U-reducing subspace. If suffices to show that $U'(A_n)=A_n$. If A_n is trivial then the result is clear, otherwise, A_n may be decomposed into a direct sum of n cyclic subspaces generated by elements $x_1, \ldots, x_n$, $A_n=\bigoplus_{i=1}^{n} Z(x_i)$, where $Z(x_i)$ is the subspace generated by the set $\{U^n x_i\colon n=0, \pm, 1, \pm 2, \ldots\}$, and each x_i has the same spectral type, μ_n say. Let $U'\in\mathscr{I}(U)$, then the images $U'x_i$, $i=1, \ldots, n$, have the following properties:

1. The spectral type of $U'x_i$ is μ_n, hence $U'x_i\in A_n$.

2. The cyclic subspaces $Z(U'x_i)$. $i=1, \ldots, n$ are pairwise orthogonal and $Z(U'x_i)\subset A_n$, $i=1, \ldots, n$. It then follows that $A_n=\bigoplus_{i=1}^{n} Z(U'x_i)$, i.e., $U'(A_n)=A_n$. See [6] §§ 63, 64.

With the help of the above lemma we can give necessary and sufficient conditions for U to be unitarily coalescent.

Theorem 1. *A unitary operator U on a seperable Hilbert space $\mathscr{H}$ is unitarily coalescent if and only if there is no spectral type with uniform infinite multiplicity.*

Proof. If there is no spectral type of uniform infinite multiplicity then $\mathscr{H}=A$, where A is the subspace described in Lemma 1. It then follows from Lemma 1 that if $U'\in\mathscr{I}(U)$ then $U'(\mathscr{H})=\mathscr{H}$, i.e., U' is onto and therefore $U'\in\mathscr{U}(U)$. Hence $\mathscr{I}(U)=\mathscr{U}(U)$ and U is unitarily coalescent.

Assume now that there is a type with uniform infinite multiplicity, then $\mathscr{H}$ may be decomposed as follows: $\mathscr{H}=A\oplus B$ where $B=\bigoplus_{n\geq 1} Z(x_n)$ and the elements x_n all have the same spectral type. It follows from [6] § 60 that there exist unitary operators U_n: $Z(x_n)\to Z(x_{n+1})$ such that $U_n\, U|_{Z(x_n)}=U|_{Z(x_{n+1})}\, U_n$, where $U|_{Z(x_n)}$ means the restriction of U to the U-reducing subspace $Z(x_n)$. We now construct an element $U'\in\mathscr{I}(U)\setminus\mathscr{U}(U)$. Define U' to be the identity on A and to be equal to U_n on $Z(x_n)$ and then extend U' to an isometry of $\mathscr{H}$. It is easy to see that $U'\in\mathscr{I}(U)$, to show that $U'\notin\mathscr{U}(u)$ we note that $U'(\mathscr{H})=A\oplus\bigoplus_{n\geq 2} Z(x_n)\neq\mathscr{H}$. Hence U is not unitarily coalescent.

3. Coalescence of Automorphisms of Lebesgue Space

We have already defined what we mean by saying that T is coalescent. We first show that coalescence is an invariant of weak isomorphism.

Theorem 2. *If T is coalescent and T' is weakly isomorphic to T then T' is in fact isomorphic to T and T' is coalescent.*

Proof. Since T weakly isomorphic to T' there exist endomorphisms ϕ, ψ such that $\phi T=T'\phi$ and $\psi T'=T\psi$. Hence $\psi\phi T=T\psi\phi$, i.e. $\psi\phi\in\mathscr{E}(T)=\mathscr{A}(T)$. Since $\psi\phi$ is an automorphism it follows that both ψ and ϕ are automorphisms i.e. T'

is isomorphic to T. If $\phi' \in \mathscr{E}(T') \setminus \mathscr{A}(T')$ then $\phi^{-1}\phi'\phi \in \mathscr{E}(T) \setminus \mathscr{A}(T)$ which contradicts T coalescent, therefore T' is coalescent.

We remark that a direct consequence of this theorem is that isomorphism and weak isomorphism coincide for coalescent transformations.

A question which we have not been able to resolve is the following: if T has a factor automorphism which is not coalescent does it then follow that T itself is not coalescent? Under the restrictive assumption that the factor is a direct factor the answer is yes.

Proposition 1. *If $T \cong T_1 \times T_2$ where T_1 is not coalescent, then T is not coalescent.*

Proof. Let $\phi \in \mathscr{E}(T_1) \setminus \mathscr{A}(T_1)$, then $\phi \times I \in \mathscr{E}(T_1 \times T_2) \setminus \mathscr{A}(T_1 \times T_2)$, where I denotes the identity automorphism.

Let us say that T is unitarily coalescent if its induced unitary operator U_T on $L_2(M)$ is coalescent.

Theorem 3. *If T is unitarily coalescent then T is coalescent.*

Proof. Let $\phi \in \mathscr{E}(T)$ then $U_\phi \in \mathscr{I}(U_T) = \mathscr{U}(U_T)$. Hence U_ϕ is unitary and therefore $\phi \in \mathscr{A}(T)$ and it follows that T is coalescent.

Corollary 1. *All automorphisms with no spectral types having uniform infinite multiplicity are coalescent. In particular, ergodic automorphisms with discrete spectrum and automorphisms with simple continuous spectrum* (see Girsanov [3]).

In [17] Yuzvinskii proved that the set of automorphisms with simple continuous spectrum contains an everywhere dense G_δ-set in the set of automorphisms of (M, μ) with the weak topology, see Halmos [5].

From this and Corollary 1 we obtain

Corollary 2. *The set of coalescent automorphisms of (M, μ) contains an everywhere dense G_δ in the set of automorphisms of (M, μ) with the weak topology.*

We can construct a larger class of coalescent transformations which includes all those of Corollary 1. To do this we construct an invariant partition of (M, μ) associated with the spectrum of T.

Let $\mathscr{F}(T)$ denote the smallest U_T-reducing unitary subring of $L_2(M)$ which contains all the elements whose spectral type with respect to U_T has uniform finite multiplicity. In other words, using the notation of § 2, $\mathscr{F}(T)$ is the U_T-reducing subring generated by A. Denote the measurable partition of M corresponding to $\mathscr{F}(T)$ by $\zeta(T)$. Then $T\zeta(T) = \zeta(T)$ and so there is a factor automorphism $T_{\zeta(T)}$ corresponding to $\zeta(T)$.

Theorem 4. *The factor automorphism $T_{\zeta(T)}$ is coalescent.*

Proof. It is sufficient to consider the case $\zeta(T) = \varepsilon$, $T_{\zeta(T)} = T$ (otherwise replace M by $M/\zeta(T)$, T by $T_{\zeta(T)}$). Let $\phi \in \mathscr{E}(T)$, then $U_\phi \in \mathscr{I}(U_T)$ and by Lemma 1 $U_\phi(A) = A$. Since U_ϕ is a ring homomorphism and commutes with U_T then the smallest U_T-reducing subring containing A is also reducing for U_ϕ. But, by assumption this subring is $L_2(M)$, therefore $U_\phi \in \mathscr{U}(T)$ and hence $\phi \in \mathscr{A}(T)$. Thus $\mathscr{E}(T) = \mathscr{A}(T)$ and T is coalescent.

Theorem 5. $\zeta(T) \leqq \pi(T)$, *where $\pi(T)$ denotes the maximum partition with zero entropy, hence $h(T_{\zeta(T)}) = 0$.*

Proof. Let $L_2(\pi(T))$ denote the unitary subring corresponding to the measurable partition $\pi(T)$, since $T\pi(T)=\pi(T)$, this is a U_T-reducing subring. The unitary operator U_T has uniform infinite Lebesgue spectrum in the orthogonal complement of $L_2(\pi(T))$, Rohlin and Sinai [15], hence $A\subset L_2(\pi(T))$ and $\mathscr{F}(T)\subseteq L_2(\pi(T))$ by the definition of $\mathscr{F}(T)$. Thus $\zeta(T)\leqq\pi(T)$.

Examples. If T is an ergodic automorphism with discrete spectrum then $\zeta(T)=\pi(T)=\varepsilon$. If T is a totally ergodic automorphism with quasi-discrete but not discrete spectrum then $\zeta(T)\neq\pi(T)=\varepsilon$. In fact $T_{\zeta(T)}$ is the maximal factor automorphism of T with discrete spectrum. If T has continuous spectrum with uniform infinite multiplicity then $\zeta(T)=\nu$.

4. Normal Dynamical Systems

We first recall a few definitions and properties of normal dynamical systems.

Let η be a continuous finite measure on the unit interval I. We construct the normal dynamical system T_η as follows: let $M=\prod_{n=-\infty}^{\infty} R_n$, where R_n is the real line for each n, let T_η be the shift transformation on M, i.e. if $\omega=(\omega_n)\in M$ then $T\omega=\omega'$ where $\omega'_n=\omega_{n+1}$. Define the coordinate function x_n on M by $x_n(\omega)=\omega_n$, then we define a measure P_η on the σ algebra generated by the cylinder sets of M by the requirement that the sequence $\{x_n\}$ should form a stationary Gaussian sequence with covariance measure η. See [2], [7] for more details. If $\mathscr{X}$ denotes the closed linear span of $\{x_n\}$ in $L_2(M)$ then $L_2(M)$ is the smallest unitary subring containing $\mathscr{X}$. The maximal spectral type of U_{T_η} in the orthogonal complement of the constant functions is the class of measures equivalent to $\sum_{n=1}^{\infty}\eta^n/2^n$, where η^n denotes the n-fold convolution of η with itself. If we write $L_2(M)=C\oplus\mathscr{X}\oplus H$, where C denotes the subspace of constant functions, then the subspace $\mathscr{X}$ is a cyclic subspace, $\mathscr{X}=Z(x_0)$, and the spectral measure of the element x_0 is η. The spectral measure of U_T restricted to H is $\sum_{n=2}^{\infty}\eta^n/2^n$ and if $\eta\perp\sum_{2}^{\infty}\eta^n/2^n$ then the measure η occurs with uniform finite multiplicity 1 and hence $\mathscr{X}\subset A$ (see § 2).

Theorem 6. *If* $\eta\perp\sum_{2}^{\infty}\eta^n/2^n$ *then* $\zeta(T_\eta)=\varepsilon$ *and so* T_η *is coalescent and has zero entropy.*

Proof. We have remarked above that under this assumption $\mathscr{X}\subset A$. Since $L_2(M)$ is the smallest subring containing $\mathscr{X}$ it must also be the smallest subring containing A. Therefore $\mathscr{F}(T)=L_2(M)$ and $\zeta(T)=\varepsilon$. The remaining assertions follow from Theorems 4, 5.

We remark that a similar technique to that used in the proof of Theorem 6 enables one to prove Pinsker's result that normal dynamical systems with singular covariance measure have zero entropy.

Theorem 7 [Pinsker]. *If* η *is a singular measure then the normal dynamical system* T_η *has zero entropy.*

Proof. Since $\mathscr{X}$ is a cyclic subspace with singular spectral type it follows that $\mathscr{X} \subset L_2(\pi(T_\eta))$. (In the orthogonal complement of $L_2(\pi(T))$, T has Lebesgue type.) But the smallest unitary subring containing $\mathscr{X}$ is $L_2(M)$ therefore $L_2(M) = L_2(\pi(T_\eta))$ and hence $\pi(T_\eta) = \varepsilon$. The result is proved.

Returning to the concept of coalescence, a natural question is to ask when T_η is not coalescent.

Theorem 8. *Every normal dynamical system with positive entropy is not coalescent.*

Proof. A normal dynamical system T_η has positive entropy if and only if $\eta = \eta_1 + \eta_2$ where $0 \neq \eta_1 \ll l$ and $\eta_2 \perp l$, where l denotes Lebesgue measure on the unit interval. If $\eta = \eta_1 + \eta_2$ then $T_\eta \cong T_{\eta_1} \times T_{\eta_2}$ (this is true for any decomposition of η into pairwise orthogonal measures) and it will suffice, by Proposition 1, to show that T_{η_1} is not coalescent. Let η_1' be any measure absolutely continuous with respect to η_1 and not equivalent to η_1. Then $T_{\eta_1'}$ is a factor automorphism of T_{η_1} and can be chosen to be a proper factor in the sense that if $\phi T_{\eta_1} = T_{\eta_1'} \phi$ then ϕ can be chosen to be an endomorphism which is not an automorphism. However, by a theorem of Versik [16], all normal dynamical systems T_η, with $\eta \ll l$, are isomorphic. Hence there exists an automorphism ψ such that $\psi T_{\eta_1} = T_\eta \psi$. The endomorphism $\psi \phi$ clearly belongs to $\mathscr{E}(T_{\eta_1})$ but not to $\mathscr{A}(T_{\eta_1})$ thus T_{η_1} is not coalescent.

Note. The theorem of Versik mentioned in the proof of Theorem 7 appears without proof in [16]. A proof may be constructed from the following remarks:

1. T_l is a Bernoulli automorphism with infinite entropy and T_η, $\eta \ll l$, is a factor of T_l,
2. T_η, $\eta \ll l$, has infinite entropy [11],
3. Every factor automorphism of a Bernoulli automorphism is a Bernoulli automorphism [10],
4. Bernoulli automorphisms with infinite entropy are isomorphic [9].

5. Examples

It was stated in the introduction that coalescence was not a spectral invariant. We will give an example to show this.

In Newton and Parry [8] there is constructed an example of a normal dynamical system T_η such that $\eta \perp l$ and $\sum_2^\infty \eta^n/2^n \equiv l$. This normal dynamical system has a factor system T_η' with countable Lebesgue spectrum and zero entropy. Let $S = T_\eta' \times T_\eta' \times \cdots$ be the direct product of a countable number of copies of T_η' then S has countable Lebesgue spectrum and zero entropy. Consider the two automorphisms T_η and $T_\eta \times S$. Firstly they are spectrally isomorphic, the spectral decomposition of both in the orthogonal complement of the constants consists of the measure η with uniform multiplicity 1 and the measure l with uniform infinite multiplicity. Secondly they both have zero entropy. By Theorem 5 T_η is coalescent, however S is not coalescent, since it commutes with the one-sided shift, and therefore $T_\eta \times S$ is not coalescent. This example shows that coalescence is not a spectral invariant. The above argument shows that T_η and $T_\eta \times S$ are not weakly isomorphic and so also gives an example of two spectrally equivalent automorphisms with equal entropy which are not weakly isomorphic.

A question put to me by Parry in conversation is the following: if T has positive entropy does it follow that T is not coalescent? It is fairly easy to show that the n-shift is not coalescent. Let $X=\{0, 1, \ldots, n-1\}$, take $M=\prod_{-\infty}^{\infty} X_i$, where $X_i=X$ for each i, μ the direct product of the normalised uniform measure on X and T the shift transformation on M. An endomorphism that commutes with T is given by $\psi(\omega_n)=(\omega'_n)$ where $\omega'_n=\omega_n-\omega_{n-1}$; ψ is n to 1 and therefore belongs to $\mathscr{E}(T)$ but not $\mathscr{A}(T)$. By Theorem 2 and Sinai's Weak Isomorphism theorem it follows that any Bernoulli automorphism with entropy equal to $\log n$ is not coalescent. However we have made no progress towards proving that any other Bernoulli automorphisms are not coalescent.

References

1. Auslander, J.: Endomorphisms of minimal sets. Duke math. J. **30**, 605–614 (1963).
2. Fomin, V. S.: On dynamical systems in function space. (Russian.) Ukrain. mat. Žurn. **11**, 25–47 (1950).
3. Girsanov, I.V.: On spectra of dynamical systems generated by stationary Gaussian processes. (Russian.) Doklady Akad. Nauk SSSR **119**, 851–853 (1958).
4. Hahn, F. J., Parry, W.: Some characteristic properties of dynamical systems with quasi-discrete spectra. Math. Systems Theory **2**, 179–190 (1968).
5. Halmos, P. R.: Lectures on ergodic theory. Tokyo: Mathematical Society of Japan 1956; New York: Chelsea 1960.
6. – Introduction to Hilbert space. New York: Chelsea 1957.
7. Kakutani, S.: Spectral analysis of stationary Gaussian processes. Proc. 4th Berkeley Sympos. math. Statist. Probab. 239–247 (1960).
8. Newton, D., Parry, W.: On a factor automorphism of a normal dynamical system. Ann. math. Statistics **37**, 1528–1533 (1966).
9. Ornstein, D.: Two Bernoulli shifts with infinite entropy are isomorphic. (To appear.)
10. – Factors of Bernoulli shifts are Bernoulli shifts. (To appear.)
11. Pinsker, M. S.: Dynamical systems with completely positive and zero entropy. Doklady Akad. Nauk SSSR **133**, 1025–1026 (1960); Soviet. Math. Doklady **1**, 937–938 (1960).
12. Plessner, A. I., Rohlin, V. A.: Spectral theory of linear operators II. Uspehi mat. Nauk **1**, 71–191 (1946); Amer. math. Soc. Translat., II. Ser., **62**, 29–125 (1967).
13. Rohlin, V. A.: Fundamental ideas of measure theory. Mat. Sbornik. n. Ser. **25**, 107–150 (1949); Amer. math. Soc. Translat. I. Ser., **10**, 1–54 (1962).
14. – Selected topics from the metric theory of dynamical systems. Uspehi mat. Nauk **30**, 57–128 (1949); Amer. math. Soc. Translat. II. Ser., **49**, 171–240 (1966).
15. – Sinai, Ja. G.: Construction and properties of invariant measurable partitions. Doklady Akad. Nauk SSSR **141**, 1038–1041 (1961); Soviet. Math. Doklady **2**, 1611–1614 (1961).
16. Versik, A. M.: Spectral and metric isomorphism of some normal dynamical systems. Doklady Akad. Nauk SSSR **114**, 255–257 (1962); Soviet. Math. Doklady **3**, 693–696 (1962).
17. Yuzvinskii, S. A.: Metric automorphisms with simple spectrum. Doklady Akad. Nauk SSSR **172**, 1036–1038 (1967); Soviet. Math. Doklady **8**, 242–245 (1967).

Dr. D. Newton
School of Mathematics and Physics
University of Sussex
Falmer, Brighton
Sussex
England

(Received February 15, 1970)